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NONEXISTENCE OF CLASSICAL SOLUTIONS OF THE DIRICHLET PROBLEM FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract. Necessary conditions for the existence of classical solutions of the Dirichlet problem for fully nonlinear, nonuniformly elliptic equations are proved. The effect of the large inhomogenous term and the geometry of the domain on the solvability of the boundary value problem is shown. The above methods are applied to the equations of Monge—Ampere type.

Key words. Fully nonlinear, nonuniformly elliptic equations, Monge—Ampere type equations, comparison principle.

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INTRODUCTION AND RESULTS

This paper is concerned with the non-solvability of Dirichlet's problem for fully nonlinear elliptic equations

(1) $$F[u] \equiv F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, u = \varphi, \quad \text{on } \partial\Omega$$

in a given bounded domain Ω in R^n and arbitrarily assigned smooth boundary data. The real function $F(x, z, p, r)$ is defined on $\Gamma = \Omega \times R \times R^n \times R^{n\times n}$ ($R^{n\times n}$ denotes $n(n+1)/2$ dimentional space of real symmetric $n \times n$ matrices) and satisfies the ellipticity condition

(2) $$F_{ij}(x, z, p, r)\, \zeta^i\zeta^j > 0 \quad \text{for } (x, z, p, r) \in \Gamma,\ \zeta \in R^n\backslash 0,$$

where in (2) and further on the short notations $F_{ij} = \partial F/\partial r_{ij}$, $F_{ij.pq} = \partial^2 F/\partial r_{ij}\, \partial r_{pq}$, $F_z = \partial F/\partial z$, $u_{ij} = \partial^2 u/\partial x_i\, \partial x_j$, etc. will be used and summation convention is understood.

Suppose that $F(x, z, p, r)$ is twice differentiable and concave function of r i.e. $F_{ij.lq}(x, z, p, r)\, \eta^{ij}\eta^{lq} \leq 0$ for $(x, z, p, r) \in \Gamma$ and $\eta \in R^{n^2}\backslash 0$.

Let us define a scalar function $E(x, z, p)$ which will prove to be quite important, by

$$E(x, z, p) = F(x, z, p, p \otimes p) - F(x, z, p, 0) \qquad \text{for } (x, z, p) \in \Omega \times R \times R^n,$$

where the matrix $p \otimes p$ is $\{p_i p_j\}^n_{ij=1}$.

We will consider the effect of the large inhomogeneous term $F(x, z, p, 0)$ in comparison with the function $E(x, z, p)$ and $|p| \, SpF_{ij}(x, z, p, p \otimes p)$ ($SpF_{ij} =$ $=$ trace F_{ij}), as well as the effect of the matrix $F_{ij}(x, z, p, p \otimes p)$ on the solvability of Dirichlet's problem.

More precisely, let us suppose that

$$\text{(3)} \qquad F(x, z, p, 0) \leqq -\Psi(|p|) \, E(x, z, p) - (|p|/R) \, SpF_{ij}(x, z, p, p \otimes p),$$

for $x \in \Omega$, $z \geqq M$, $|p| \geqq L$, where M, L are positive constants, R is the radius of the largest ball contained in Ω and Ψ is a positive, continuous, monotonically increasing function satisfying the condition

$$\text{(4)} \qquad \int^{\infty} \frac{dt}{t\Psi(t)} < \infty.$$

For instance if (1) is a uniformly elliptic equation i.e.

$$\lambda |\zeta|^2 \leqq F_{ij}(x, z, p, r) \, \xi^i \xi^j \leqq \Lambda |\xi|^2,$$

where λ, Λ are positive constants, then $\lambda |p|^2 \leqq E(x, z, p) \leqq \Lambda |p|^2$ and $n\lambda |p| \leqq$ $\leqq |p| \, SpF_{ij}(x, z, p, p \otimes p) \leqq n\Lambda |p|$.

In this case (3) holds provided that

$$\text{(5)} \qquad F(x, z, p, 0) \leqq -C |p|^2 \, ln^{1+\varepsilon} |p|,$$

for some positive constants C and ε.

The following theorem relates the behaviour of the large inhomogeneous term with the non-solvability of Dirichlet's problem.

Theorem 1. *Let Ω be a bounded domain in R^n, whose boundary is of class $C^{1,1}$ and F be a real, smooth and concave function of r satisfying the conditions (2) and (3).*

Moreover, let F be non-increasing in z for each $(x, p, r) \in \Omega \times R^n \times R^{n \times n}$.

Then there exists C^∞ boundary data such that the Dirichlet problem (1) has no solution:

Remark 1. Theorem 1 shows that the assumption

$$|F(x, z, p, 0)| \leqq C(1 + |p|^2),$$

in theorem 8.2 in [2] can not be weakened. If we suppose that

$$F(x, z, p, 0) \leqq -C(1 + |p|^{2+\varepsilon}).$$

for some positive constants C and ε i.e.

$$|F(x, z, p, 0)| \geqq C(1 + |p|^{2+\varepsilon}),$$

then the Dirichlet problem (1) is not generally solvable even in uniformly convex domain Ω.

Let us now consider a second type of non-solvability, which is due to the structure of the matrix $F_{ij}(x, z, p, p \otimes p)$, rather than to the inhomogeneous term $F(x, z, p, 0)$. More precisely, we suppose that

$$\Psi(|p|)\, E(x, z, p) \leqq |p|\, SpF_{ij}(x, z, p, p \otimes p), \tag{6}$$

for $|p| \geqq L$, $z \geqq M$ and $x \in \Omega$, where M, L are positive constants and Ψ is the same function as in (3). In this case we will demonstrate the need for the geometric restrictions on the domain Ω in order for the Dirichlet problem to be generally solvable.

Suppose the following conditions on the asymptotic behaviour of the coefficients $F_{ij}(x, z, p, p \otimes p)$, $F(x, z, p, 0)$ for large values of $|p|$

$$\begin{aligned} F_{ij}(x, z, p, p \otimes p)/SpF_{ij}(x, z, p, p \otimes p) &= f^{ij}(x, \sigma) + 0(1), \\ F(x, z, p, 0)/(|p|\, SpF_{ij}(x, z, p, p \otimes p)) &= f(x, \sigma) + 0(1), \end{aligned} \tag{7}$$

as $|p| \to \infty$, $\sigma = p/|p|$ hold. Here $f^{ij}(x, \sigma)$, $f(x, \sigma)$ are continuous functions of their arguments. Using the matrix $\mathscr{F}(x, \sigma) = \{f^{ij}(x, \sigma)\}_{ij=1}^{n}$ we introduce a generalized notion of mean curvature. Let y be a point of $\partial\Omega$ and ν denote the unit outer normal to $\partial\Omega$ at y. Also let $k_1, k_2, \ldots, k_{n-1}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be respectively the principal curvatures and principal directions of $\partial\Omega$ at y. We then put

$$\mathscr{H}(y, \nu) = \sum_{i=1}^{n-1} \lambda_i \mathscr{F}(y, \nu)\, \lambda_i k_i + \nu \mathscr{F}(y, \nu)\, \nu \,.\, H, \tag{8}$$

where H is the ordinary mean curvature of $\partial\Omega$ at y.

Example 1. Equations of Monge–Ampere type

$$\det D^2 u = g(x, u, Du). \tag{9}$$

This equation is elliptic only when the Hessian matrix $D^2 u$ is positive (or negative) and we consider convex solutions u and positive functions g. For the equation (9) simple computations give us $E(x, z, p) = 0$ and $SpF_{ij}(x, z, p, p \otimes p) = 0)$ for $n > 2$ and $|p|^2$ for $n = 2$, and hence (6) holds. If we write equation (9) in the form

$$F(D^2 u) = \log \det D^2 u = \log g(x, u, Du), \tag{10}$$

we then have that the function F is concave on the cone of nonnegative matrices

$R^{n\times n}$. In this case for the equation (10) we obtain $E(x, z, p) = -\infty$, $SpF_{ij}(x, z, p, p\otimes p) \geqq 0$ for $n > 2$ and $+\infty$ for $n = 2$, and again (6) holds. Moreover, when $n = 2$

$$\mathscr{F}(y, v) = \begin{pmatrix} v_2^2 & -v_1 v_2 \\ -v_1 v_2 & v_1^2 \end{pmatrix}, \qquad f(y, v) = 0$$

and

$$\mathscr{K}(y, v) = (v_1^4 + v_2^4)\, k + 2v_1^2 v_2^2 k = k(y),$$

i.e. the generalized mean curvature at y coincides with the ordinary curvature at y.

Let us go back to equation (1) and formulate the following nonexistence result depending on the generalized mean curvature.

Theorem 2. *Let Ω be a bounded domain in R^n, whose boundary is of class $C^{1,1}$ and F be a real, smooth and concave function of r satisfying the conditions* (2), (6), (7). *Moreover, let F be nonincreasing in z for $(x, p, r) \in \Omega \times R^n \times R^{n\times n}$ and $F(x, z, p, 0) \leqq 0$ for $z \geqq M$, $|p| \geqq L$. If the geometric condition*

$$\mathscr{K}(y, v) \geqq -f(y, v) \tag{11}$$

fails at a single point y of the boundary surface, then there exists smooth boundary data for which no solution of the Dirichlet problem is possible.

For instance, the condition (2) for the equation of Monge–Ampere type (10) in R^2 is $k(y) \geqq 0$, where $k(y)$ is the ordinary curvature at $y \in \partial\Omega$ (see example 1). Hence the Dirichlet problem for (10) in $\Omega \subset R^2$ is not solvable for arbitrary boundary data when the domain is nonconvex, which is a well-known result.

For convenience, we will directly prove the following nonexistence theorem for equations of Monge–Ampere type.

Theorem 3. *Let the positive function $g(x, z, p)$ satisfy the condition*

$$g(x, z, p) \geqq \Psi(|p|)\, |p|^{n+1} \tag{12}$$

for $z \geqq M$, $|p| \geqq L$, $x \in \Omega_y$ (Ω_y is a neighbourhood of some point $y \in \delta\Omega$), where M, L are positive constants, Ψ is the same function as in (3), (4) *and Ω be a uniformly convex $C^{1,1}$ domain in R^n. Then there exists C^∞ boundary data such that the Dirichlet problem for the equation of Monge–Ampere type* (9) *in Ω has no convex solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$.*

For example, the condition (12) holds when

$$g(x, z, p) \geqq C\, |p|^{n+1}\, ln^{1+\varepsilon} |p| \qquad \text{for } x \in \Omega_y,\ z \geqq M,\ |p| \geqq L,$$

where C, M, L, ε are positive constants.

The result in theorem 3 is the best possible one. This follows from th. 2 in [8], where under the assumption:

$$g(x, \varphi(x), p) \leqq |p|^{n+1} \Psi(|p|)$$

for all x in some neighbourhood of $\partial\Omega$, $|p| \geqq L$, where the continuous non-decreasing, positive function Ψ satisfies the condition

$$\int^{\infty} dt/(t\Psi(t)) = \infty,$$

the Dirichlet problem for the equation of Monge–Ampere type (9) is generally solvable.

As for the results in theorems 1, 2, they are in many ways the best possible, too, which follows from th. 1, 2 in [9]. Moreover, when $F(x, z, p, r)$ is a linear function of r, the results in theorems 1, 2 remain the best possible ones, as it follows from [11].

Let us recall that in the quasilinear case, Serrin, [11], introduces the classes of irregularly and singularly elliptic equations and proves the corresponding non-existence theorems. In this paper we introduce the classes (3) and (6) of fully, nonlinear, nonuniformly elliptic equations which contain the equations considered by Serrin when $F(x, z, p, r)$ is a linear function of r. Thus we extend Serrin's work for quasilinear equations to fully, nonlinear equations.

Finally we would like to thank Prof. Trudinger who called our attention to the recently published work of Trudinger and Urbas [13], in which results similar to those in theorem 3 are proved. More precisely, from theorem 1.3 in [13] it follows that the Dirichlet problem for the equation of Monge–Ampere type (9) is not generally solvable when

$$g(x, z, p) \geqq C(1 + |p|^2)^{\frac{\alpha}{2}} \qquad \text{for all} \qquad xN_y, z \in R, p \in R^n$$

and $\alpha > n + 1$, where N_y is a neighbourhood of some point $y \in \partial\Omega$.

The paper is divided into two paragraphs. In the first one we prove theorems 1, 2.

Paragraph 2 deals with the nonsolvability of Dirichlet's problem for the equations of Monge–Ampere type and theorem 3 is proved.

1. PROOFS OF THEOREMS 1 AND 2

The main tool for our treatment of nonexistence results is the following variant of comparison principle (see th. 17.1 in [4], p. 443).

Theorem 4. *Let Ω be a bounded domain in R^n and γ-a relatively open C^1 portion of $\partial\Omega$. Let $u \in C(\overline{\Omega}) \cap C^2(\Omega \cup \gamma)$, $v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $F[u] \geqq F[v]$ in Ω, $u \leqq v$ on $\partial\Omega \setminus \gamma$, $\partial v/\partial\tau = -\infty$ on γ (τ denotes the unit normal direction into Ω). Moreover*

(i) *the function F is continuously differentiable with respect to z, p, r variables;*

(ii) *the operator F is elliptic on all functions of the form* $tu + (1 - t)v$, $0 \leqq t \leqq 1$;

(iii) *the function F is non-increasing in z for each* $(x, p, r) \in \Omega \times R^n \times R^{n\times n}$.

It then follows that $u \leqq v$ in Ω.

Proof. Suppose that the conclusion of theorem 4 is not true. Since $u \leqq v$ on $\partial\Omega\backslash\gamma$, from theorem 17.1 in [4] it follows that the function $u - v$ attains its maximum at some point $P \in \gamma$. This is impossible because $\partial(u - v)/\partial\tau = +\infty$ at P.

Further we need the following simple lemmas.

Lemma 1. *Let the continuous, positive, monotonically increasing function Ψ satisfy (4). Then for any positive constants a, b, l_0, δ_0 there exists a constant $\delta(a, b, l_0, \delta_0)$, $0 < \delta < \delta_0$ and a nonnegative function $v \in C[a, b] \cap C^2(a, b)$, satisfying the conditions: $v' \leqq -l_0$, $v'(a) = -\infty$ and $v'' - v'/t = 2\delta(v')^2\,\Psi(a\,|\,v'\,|/t)$ for $t \in (a, b)$.*

Proof. Let the constant $l \geqq l_0$ satisfy the inequality

$$a\int_l^\infty \frac{dt}{t^2\Psi(t)} < \delta_0(b^2 - a^2) \qquad \text{and} \qquad \delta = \frac{a}{b^2 - a^2}\int_l^\infty \frac{dt}{t^2\Psi(t)}.$$

Let us consider the function v given by

$$v(t) = \int_b^t sG^{-1}(\delta(s^2 - a^2))\,ds, \qquad \text{where} \qquad G(s) = \int_{-\infty}^s \frac{dt}{t^2\Psi(a\,|\,t\,|)}.$$

It is evident that $G(s)$ is monotonically increasing for $s \leqq -l/a$ and thus there exists a well defined inverse function G^{-1}: $(0, G(-l/a)) \to (-\infty, -l/a)$. One can easily check that $v \geqq 0$, $v' = tG^{-1}(\delta(t^2 - a^2)) \leqq -l \leqq -l_0$, $v'(a) = -\infty$ and $v'' = G^{-1}(\delta(t^2 - a^2)) + 2\delta t^2/G'(G^{-1}(\delta(t^2 - a^2))) = v'/t + 2\delta(v')^2\;\Psi(a\,|\,v'\,|/t)$. Moreover $v \leqq \dfrac{1}{2\delta}\displaystyle\int_{l_0}^\infty \frac{dt}{t\Psi(t)}$ and $v \in C[a, b] \cap C^2(a, b)$.

Lemma 2. *Let the continuous, positive, monotonically increasing function Ψ satisfy (4). Then for any positive constants a, b, l_0, δ_0 there exists a constant $\delta(b - a, l_0, \delta_0)$, $0 < \delta < \delta_0$ and a nonnegative function $h_+/h_-/$, $h_\pm \in C[a, b] \cap C^2(a, b)$ satisfying the conditions: $h'_+ \geqq l_0(h'_- \leqq -l_0)$, $h'_+(b) = +\infty(h'_-(a) = -\infty)$, $h''_\pm = \delta(h'_\pm)^2\,\Psi(|\,h'_\pm\,|)$ for $t \in (a, b)$.*

Proof. Let the constant $l \geqq l_0$ satisfy the inequality

$$\int_l^\infty \frac{dt}{t^2\Psi(t)} < \delta_0(b - a) \qquad \text{and} \qquad \delta = \frac{1}{b - a}\int_l^\infty \frac{dt}{t^2\Psi(t)}.$$

Let us consider the function $h_+(h_-)$ given by

$$h_+ = \int_a^t H_+^{-1}(\delta(b-s))\,\mathrm{d}s (h_- = \int_b^t H_-^{-1}(\delta(s-a))\,\mathrm{d}s),$$

where

$$H_+(s) = \int_s^\infty \frac{\mathrm{d}t}{t^2 \Psi(t)} \left(H_-(s) = \int_{-\infty}^s \frac{\mathrm{d}t}{t^2 \Psi(|t|)} \right).$$

It is evident that $H_+(H_-)$ is monotonically decreasing (increasing) for $s \geqq l (s \leqq -l)$ and thus there exists a well defined inverse function $H_+^{-1} : (0, H_+(l)) \to (l, \infty)$ $(H_-^{-1} : (0, H_-(-l)) \to (-\infty, -l))$. One easily checks that $h_\pm \geqq 0$, $h'_+ = H_+^{-1}(\delta(b-t)) \geqq l \geqq l_0$, $h'_+(b) = \infty (h'_- = H_-^{-1}(\delta(t-a)) \leqq -l \leqq -l_0$, $h'_-(a) = -\infty)$ and $h''_\pm = \mp\delta/H'_\pm(H_\pm^{-1}) = \delta(h'_\pm)^2\, \Psi(|h'_\pm|)$. Moreover $h_\pm \leqq \frac{1}{\delta}\int_{l_0}^\infty \frac{\mathrm{d}t}{t\Psi(t)}$ and $h_\pm \in C[a,b] \cap C^2(a,b)$.

Lemmas 1, 2 and theorem 4 can be used now to derive theorem 1.

Proof of theorem 1: We suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is a solution of the Dirichlet problem (1). Let K be the ball with the largest radius $2R$ contained in Ω and P be a fixed boundary point of $\partial\Omega$ at which K is internally touching. Let r denote the distance from P. In the domain $\Omega_1 = \{x \in \Omega;\ R < r < \operatorname{diam} \Omega\}$, we consider the function $h(r) = v(r) + M_1$, $M_1 = \max(M, \sup_{\partial\Omega \cap \{r \geqq R\}} u)$ where $v(r)$ is the function defined in lemma 1, when $a = R$, $b = \operatorname{diam}\Omega$, $l_0 = \max(L, \Psi^{-1}(1))$, $\delta_0 = 1/2$ and $\delta = \delta_1$.

One easy calculation yields

$$\begin{aligned} F[h] &= F(x, h, h'Dr, (h'' - h'/r)\, Dr \otimes Dr + (h'/r)\, \delta^{ij}) \leqq \\ &\leqq F(x, h, h'Dr, h'^2 Dr \otimes Dr) + (h'' - h'/r - h'^2)\, F_{ij}(x, h, h'Dr, h'^2 Dr \otimes \\ &\otimes Dr)\, r_i r_j + (h'/r)\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr) \leqq E(x, h, h'Dr) + \\ &+ F(x, h, h'Rr, 0) + (h'/r)\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr) + \\ &+ (h'' - h'/r\, h'^2)\, F_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr)\, r_i r_j. \end{aligned}$$

When $h'' - h'/r - h'^2 \leqq 0$, from (3) we have the estimate

$$\begin{aligned} F[h] &\leqq E(x, h, h'Dr) + F(x, h, h'Dr, 0) \leqq \Psi(|h'|)\, E(x, h, h'Dr) + \\ &+ F(x, h, h'Dr, 0) \leqq 0. \end{aligned}$$

If $h'' - h'/r - h'^2 > 0$ it follows that

$$\begin{aligned} F[h] &\leqq ((h'' - h'/r)/h'^2)\, E(x, h, h'Dr) + F(x, h, h'Dr, 0) \leqq \\ &\leqq 2\delta_0 \Psi(a\,|h'|/r)\, E(x, h, h'Dr) + F(x, h, h'Dr, 0) \leqq \\ &\leqq \Psi(|h'|)\, E(x, h, h'Dr) + F(x, h, h'Dr, 0) \leqq 0, \end{aligned}$$

We may now apply the comparison principle, theorem 4 to the domain Ω_1 and the open boundary set $\gamma = \Omega \cap \{r = R\}$. Since $u \leqq h$ on $\partial\Omega \cap \{r \geqq R\}$ and

$\partial h/\partial\tau(R) = v'(R) = -\infty$ (τ denotes the unit normal direction into Ω), it follows that

$$u \leqq h \leqq \max(M, \sup_{\partial\Omega\cap\{r\geqq R\}} u) + \frac{1}{\delta_1}\int_{l_0}^{\infty}\frac{ds}{s\Psi(s)} = M_2.$$

Let now ε be an arbitrary real number between 0 and $R/2$ and ϱ denote the distance from the centre of K. In the domain $\Omega_2 = \{x \in \Omega;\ r < R \text{ and } R < \varrho < 2R - \varepsilon\}$ we consider the function $h(\varrho) = M_2 + h_+(\varrho)$ where h_+ is defined in lemma 2 when $a = R$, $b = 2R - \varepsilon$, $l_0 = \max(L, \Psi^{-1}(1))$, $\delta_0 = 1$ and $\delta = \delta_2 = \frac{1}{R-\varepsilon}\int_{l_2}^{\infty}\frac{dt}{t^2\Psi(t)} \geqq \frac{2}{R}\int_{l_2}^{\infty}\frac{dt}{t^2\Psi(t)}$. Here the constant $l_2 \geqq l_0$ satisfies the inequality $\int_{l_2}^{\infty}\frac{dt}{t^2\psi(t)} \leqq R/2 < R - \varepsilon$.

An easy calculation now gives

$$\begin{aligned} F[h] &= F(x, h, h'D\varrho, (h'' - h'/\varrho)\, D\varrho \otimes D\varrho + (h'/\varrho)\,\delta^{ij} \leqq E(x, h, h'D\varrho) + \\ &\quad + F(x, h, h'D\varrho, 0) + (h'/\varrho)\, SpF_{ij}(x, h, h'D\varrho, h'^2 D\varrho \otimes D\varrho) + \\ &\quad + (h'' - h'/\varrho - h'^2)\, F_{ij}(x, h, h'D\varrho, h'^2 D\varrho \otimes D\varrho)\,\varrho_i\varrho_j. \end{aligned}$$

When $h'' - h'/\varrho - h'^2 \leqq 0$ from (3) we have the estimate

$$\begin{aligned} F[h] &\leqq \Psi(|h'|)\, E(x, h, h'D\varrho) + F(x, h, h'D\varrho, 0) + \\ &\quad + (h'/\varrho)\, SpF_{ij}(x, h, h'D\varrho, h'^2 D\varrho \otimes D\varrho) \leqq 0. \end{aligned}$$

If $h'' - h''/\varrho - h'^2 > 0$ it follows that

$$\begin{aligned} F[h] &\leqq ((h'' - h'/\varrho)/h'^2)\, E(x, h, h'D\varrho) + F(x, h, h'D\varrho, 0) + \\ &\quad + (h'/R)\, SpF_{ij}(x, h, h'D\varrho, h'^2 D\varrho \otimes D\varrho) \leqq \delta_2\Psi(|h'|)\, E(x, h, h'D\varrho) + \\ &\quad + F(x, h, h'D'D\varrho, 0) + (h'/\varrho)\, SpF_{ij}(x, h, h'D\varrho, h'^2 D\varrho \otimes D\varrho) \leqq 0. \end{aligned}$$

Consequently $F[h] \leqq 0$. Let us apply the comparison principle, theorem 4, to the domain Ω_2 and the boundary set $\gamma = \{\varrho = 2R - \varepsilon\} \cap \{x \in \Omega,\ r < R\}$. Since $u \leqq h$ on $\{x \in \Omega;\ r = R\}$ and $\partial h/\partial\tau(2R - \varepsilon) = -h'_+(2R - \varepsilon) = -\infty$ (τ denotes the unit normal direction into Ω_2) it follows that

$$\begin{aligned} u \leqq h \leqq M_2 + \frac{1}{\delta_2}\int_{l_0}^{\infty}\frac{ds}{s\Psi(s)} &\leqq \max(M, \sup_{\partial\Omega\cap\{r\geqq R\}} u) + \\ &+ \left[\frac{1}{\delta_1} + \frac{R}{2}\left(\int_{l_2}^{\infty}\frac{dt}{t^2\Psi(t)}\right)^{-1}\int_{l_0}^{\infty}\frac{dt}{t\psi(t)}\right]. \end{aligned}$$

Letting $\varepsilon \to 0$ and using the continuity of u yields the same inequality at P, i.e.

$$u(P) \leqq \max(M, \sup_{\partial\Omega\cap(r\geqq R)} u) + \left[\frac{1}{\delta_1} + \frac{R}{2}\left(\int_{l_2}^{\infty}\frac{dt}{t^2\Psi(t)}\right)^{-1}\right]\int_{l_0}^{\infty}\frac{dt}{t\Psi(t)}.$$

Thus in order to prove theorem 1 it then suffices to choose the boundary values φ such that $\varphi = 0$ on $\partial\Omega \cap \{r \geqq R\}$ and

$$\varphi(P) > M + \left[\frac{1}{\delta_1} + \frac{R}{2}\left(\int_{l_2}^{\infty} \frac{dt}{t^2\Psi(t)}\right)^{-1}\right] \int_{l_0}^{\infty} \frac{dt}{t\Psi(t)}.$$

Proof of theorem 2. We suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is a solution of the Dirichlet problem (1) and let $P(y) \in \partial\Omega$ be a point where the geometric condition (11) fails, i.e. $\mathcal{H}(y, \nu) + f(y, \nu) < -3\eta$ at $P(y)$ for some positive constant $\eta < 1$. Then there exists a quadric surface S tangent to $\partial\Omega$ at P such that (see [11], p. 465):

(i) S has a unique parallel projection onto the tangent plane at P;

(ii) the point set $S \cap \{r < R\}$ (r denotes the distance from P) is contained in the closure of Ω for all sufficiently small values of R and

(iii) the generalized mean curvature $\mathcal{H}_s$ of the surface S (see (8)) satisfy the condition $\mathcal{H}_s \leqq \mathcal{H} + \eta$ for the positive constant η. Consequently $\mathcal{H}_s(y, \nu) + f(y, \nu) < -2\eta$ at $P(y)$.

As in the proof of theorem 1 we consider the domain $\Omega_1 = \{x \in \Omega;\ R < r < \operatorname{diam} \Omega\}$ and the function $h(r) = v(r) + M_1$ $M_1 = \max(M, \sup_{\partial\Omega \cap \{r \geqq R\}} u)$ where $v(r)$ is the function defined in lemma 1 when $a = R$, $b = \operatorname{diam} \Omega$, $l_0 = \max(L, \Psi^{-1} \cdot (\operatorname{diam} \Omega)) = l_3\delta_0 = (2 \operatorname{diam} \Omega)^{-1}$, $\delta = \delta_3$.

An easy calculation yields

$$F[h] \leqq E(x, h, h'Dr) + F(x, h, h'Dr, 0) + (h'/r)\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \times Dr) + (h'' - h'/r - h'^2)\, F_{ij}(x, h, h'Dr, h'^2 Dr \times Dr)\, r_i r_j.$$

When $h'' - h'/r - h'^2 \leqq 0$ we have from (6) the estimate

$$F[h] \leqq (1/\operatorname{diam} \Omega)\,(\Psi(|h'|)\, E(x, h, hDr) - |h'|\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr)) \leqq 0.$$

If $h'' - h'/r - h'^2 > 0$ it follows that

$$F[h] \leqq ((h'' - h'/r)/h'^2)\, E(x, h, h'Dr) - (|h'|/\operatorname{diam} \Omega)\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr) \leqq (1/\operatorname{diam} \Omega)\,(\delta_0 \operatorname{diam} \Omega \,.\, E(x, h, h'Dr) - |h'|\, SpF_{ij}(x, h, h'Dr, h'^2 Dr \otimes Dr)) \leqq 0.$$

Consequently $F[h] \leqq 0$, $\partial h/\partial\tau(R) = h'(R) = v'(R) = -\infty$ and $h \geqq u$ on $\partial\Omega \cap \{r \geqq R\}$. From the comparison principle, theorem 4, we have the estimate

$$u \leqq h \leqq \max(M, \sup_{\partial\Omega \cap \{r \geqq R\}} u) + \frac{1}{\delta_3} \int_{l_3}^{\infty} \frac{ds}{s\Psi(s)} \equiv M_3.$$

Further on we will need the following simple lemma.

Lemma 3. *Let the assumptions of theorem 2 hold. Then* $f^{ij}(x, \sigma)\, \sigma^i \sigma^j = 0$ *for* $x \in \Omega$ *and* σ *is a unit vector in* R^n, *where* f^{ij} *are defined in* (7).

Proof. From the definition of f^{ij} and (6) it follows that

$$f^{ij}(x, \sigma)\,\sigma^i\sigma^j = F_{ij}(x, z, t\sigma, t^2\sigma \otimes \sigma)\,\sigma^i\sigma^j/SpF_{ij}(x, z, t\sigma, t^2\sigma \otimes \sigma) + 0(1) \leqq$$
$$\leqq E(x, z, t\sigma)/(t^2 SpF_{ij}(x, z, t\sigma, t^2\sigma \otimes \sigma)) + 0(1) \leqq$$
$$\leqq t/(t^2\Psi(|\,t\,|)) + 0(1) \to 0 \quad \text{as} \quad t \to \infty.$$

Let us go back to the proof of theorem 2 and define the function $d(x)$ as the distance from x to the surface S. Let ε be an arbitrary real number between 0 and $R/2$. In the domain $\Omega_3 = \{x \in \Omega;\, r < R \text{ and } \varepsilon < d(x)\}$, where r is the distance from $P(y)$, the function $d(x)$ is of class $C^{1,1}$, when R is sufficiently small (see [11], p. 421, or [4] p. 381).

Consider the function $h(d) = h_-(d) + M_3$, where $h_-(d)$ is defined in lemma 2 when $a = \varepsilon$, $b = R$, $l_0 = \max(L, \Psi^{-1}(1/\eta))$, $\delta_0 = \eta$ and

$$\delta \equiv \delta_4 = \frac{1}{R - \varepsilon} \int\limits_{l_4}^{\infty} \frac{dt}{t^2\Psi(t)} \geqq \frac{2}{R} \int\limits_{l_4}^{\infty} \frac{dt}{t^2\Psi(t)}.$$

Here the constant $l_4 \geqq \max(L, \Psi^{-1}(1/\eta))$ satisfies the inequality

$$\int\limits_{l_4}^{\infty} \frac{dt}{t^2\Psi(t)} \leqq \frac{\eta R}{2} < \eta(R - \varepsilon).$$

In the following calculation we will use an important identity to calculate $F_{ij}d_{ij}$ (see [11], p. 422, lemma 1). Thus we obtain the inequalities

$$F[h] = F(x, h, h'Dd, h''Dd \otimes Dd + h'D^2d) \leqq F(x\,h, h'Dd, h'^2Dd \otimes Dd) +$$
$$+ F_{ij}(x, h, h'Dd, h'^2Dd \otimes Dd)\,((h'' - h'^2)\,d_id_j + h'd_{ij}) = E(x, h, h'Dd) +$$
$$+ F(x, h, h'Dd, 0) + (h'' - h'^2)\,F_{ij}(x, h, h'Dd, h'^2Dd \otimes Dd)\,d_id_j -$$
$$- h' \sum_{i=1}^{n-1} \frac{k_i}{1 - k_id} \cdot \lambda_i F_{kl}(x, h, h'Dd, h'^2Dd \otimes Dd)\,\lambda_l.$$

Considering $h'' - h'^2 \leqq 0$ and $h'' - h'^2 > 0$ we obtain in both cases the following inequalities

$$F[h] \leqq \eta\Psi(|\,h'\,|)\,E(x, h, h'Dd) + F(x, h, h'Dd, 0)$$
$$- h' \sum_{i=1}^{n-1} \frac{k_i}{1 - k_id} \cdot \lambda_i F_{kl}(x, h, h'Dd, h'^2Dd \otimes Dd)\,\lambda_l \leqq$$
$$\leqq \eta\Psi(|\,h'\,|)\,E(x, h, h'Dd) + |\,h'\,|\,SpF_{kl}(x, h, h'Dd, h'^2Dd \otimes Dd)\,(\sum_{i=1}^{n-1} \lambda_i \mathscr{F}(x, \nu)\,\lambda_i k_i +$$
$$+ f(x, \nu) + 0(1) + \mathcal{O}(R)) \quad \text{as} \quad |\,h'\,| \to \infty,$$

where the normal vector ν and the principal directions and principal curvatures are calculated at the point $z(x)$ on S nearest to x.

Since $\mathscr{H}_s(x, \nu) + f(x, \nu) \leqq \mathscr{H}_s(y, \nu) + f(y, \nu) + \mathcal{O}(R)$ it follows that

$$F[h] \leqq \eta\Psi(|h'|)\,E(x, h, h'Dd) + |h'|\,SpF_{kl}(x, h, h'Dd, h'^2Dd \otimes Dd)\,(\mathscr{K}_s(y, v) + f(y, v) + 0(1) + \mathcal{O}(R)) \leqq \eta\Psi(|h'|)\,E(x, h, h'Dd) - \eta\,|h'|\,SpF_{kl}(x, h, h'Dd, h'^2Dd \otimes Dd) \leqq 0 \quad \text{as} \quad |h'| \to \infty \quad \text{and} \quad R$$

is sufficiently small. Consequently $F[h] \leqq 0$, $\partial h/\partial\tau(\varepsilon) = h'_-(\varepsilon) = -\infty$, $h \geqq u$ on $\Omega \cap \{r = R\}$ and from the comparison principle, theorem 4, we have the estimate

$$u \leqq h \leqq \max(M, \sup_{\partial\Omega \cap \{r \geqq R\}} u) + \frac{1}{\delta_3}\int_{l_3}^{\infty}\frac{ds}{s\Psi(s)} + \frac{R}{2}\left(\int_{l_4}^{\infty}\frac{dt}{t^2\Psi(t)}\right)^{-1}\int_{l_4}^{\infty}\frac{dt}{t\Psi(t)},$$

when R is sufficiently small.

Letting $\varepsilon \to 0$ and using the continuity of u yields the same inequality at P.

Thus to prove theorem 2 it then suffices to choose the boundary values φ such that $\varphi = 0$ on $\partial\Omega \cap \{r \geqq R\}$ and

$$\varphi(P) > M + \frac{1}{\delta_3}\int_{l_3}^{\infty}\frac{dt}{t\Psi(t)} + \frac{R}{2}\left(\int_{l_4}^{\infty}\frac{dt}{t^2\Psi(t)}\right)^{-1}\int_{l_4}^{\infty}\frac{dt}{t\Psi(t)}.$$

§ 2. EQUATIONS OF MONGE-AMPERE TYPE

In this paragraph we will prove theorem 3.

Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a convex solution of (9), K be an internally touching ball to $\partial\Omega$ at y with radius $3R$ and r denote the distance from the centre of K. Let $K_1(y, 2R)$ be a ball and R be sufficiently small so that $K \cap K_1 \subset \Omega_y$. In the domain $\Omega_4 = K_1 \cap \{x \in \Omega;\ r < 3R - \varepsilon \text{ where } 0 < \varepsilon < R\}$, we consider the function $h(r) = h_+(r) + M_4$, where $M_4 = \max(M, \sup_{\partial\Omega\setminus K_1} u)$ and h_+ is defined in lemma 2 with $a = R$, $b = 3R - \varepsilon$, $l_0 = L$, $\delta_0 = \min(1, R^{n-1})$ and

$$\delta = \delta_5 = \frac{1}{2R - \varepsilon}\int_{l_5}^{\infty}\frac{dt}{t^2\Psi(t)} \geqq \frac{1}{R}\int_{L}^{\infty}\frac{dt}{t^2\Psi(t)}.$$

Here the constant $l_5 \geqq L$ satisfies the inequality

$$\int_{l_5}^{\infty}\frac{dt}{t^2\Psi(t)} < \delta_0 R < \delta_0(2R - \varepsilon).$$

One easy calculation gives that the Hessian matrix $D^2h = ((h'' - h'/r)\,r_i r_j + (h'/r)\,\delta^{ij})$ is positive and we may apply the comparison principle, theorem 4 (see also lemma 3.1 in [13]) to the functions $h(r)$ and u. Thus we have the estimates $\det D^2h = (h'_+/r)^n + (h''_+ - h'_+/r)(h'_+/r)^{n-1} \leqq \delta_5(h'_+)^{n+1}\,\Psi(h'_+)/R^{n-1} \leqq |Dh|^{n+1}\,.\,\Psi(|Dh|) \leqq g(x, h, Dh)$ in Ω_4.

Since u is convex we have $\sup_{\Omega\setminus K_1} u \leqq \sup_{\partial\Omega\setminus K_1} u$ and $u \leqq h$ on $\Omega \cap \partial K_1$. Moreover $h'(3R - \varepsilon) = h'_+(3R - \varepsilon) = \infty$ and from the comparison principle it follows that

$$u \leqq h \leqq \max(M, \sup_{\partial\Omega\setminus K_1} u) + \left(\frac{1}{R} \int_L \frac{dt}{t^2\Psi(t)}\right)^{-1} \int_L^{\infty} \frac{dt}{t\Psi(t)}.$$

Letting $\varepsilon \to 0$ and using the continuity of u yields the same inequality at y. Thus in order to prove theorem 3 it then sufficies to choose the boundary values φ such that $\varphi = 0$ on $\partial\Omega\setminus K_1$ and

$$\varphi(y) > M + \left(\frac{1}{R} \int_L^{\infty} \frac{dt}{t^2\Psi(t)}\right)^{-1} \int_L^{\infty} \frac{dt}{t\Psi(t)}.$$

REFERENCES

[1] Bernstein S., *Conditions necessaires et suffisantes pour la possibilite du probleme de Dirichlet*, C. R. Acad. Sci. Paris, 150, (1910), 514–515.

[2] Bernstein S., *Sur les equations du calcul des variations*, Ann. Sci. Ecole Norm. Sup., 29, (1912), 431–485.

[3] Evans L. C., *Classical solutions of fully, nonlinear convex, second order elliptic equations*, Comm. Pure Appl. Math. 25, (1982), 333–363.

[4] Gilbarg D., Trudinger N. S., *Elliptic partial differential equations of second order*, Springer Verlag, New York, 1983.

[5] Ivanov A. V., *Quasilinear degerate and nonuniformly elliptic and parabolic equations of second order*, Trudy Mat. Inst. Steklov (in Russian).

[6] Krylov N. V., Safonof M. V., *Certain properties of solutions of parabolic equations with measurable coefficients*, Izvestia Acad. Nauk SSSR, 40, (1980), 161–175.

[7] Krylov N. V., *On degenerate nonlinear elliptic equations I*, Mat. Sb. 120 (162), (1983) 311–330, II. Mat. Sb. 121(163), (1983), 211–232.

[8] Kutev N., *Gradient estimates for equation of Monge-Ampere type*, to appear.

[9] Kutev N., *Existence and nonexistence of classical solutions of the Dirichlet problem for a class of fully nonlinear, nonuniformly elliptic equations*, to appear.

[10] Ladyzhenskaya O. A., Ural'tseva N. N., *Linear and quasilinear elliptic equations*. Acad. Press, New York, 1968.

[11] Serrin J., *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc., London, Ser. A 264, (1969), 413–496.

[12] Trudinger N. S., *Fully nonlinear, uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc. 287, (2), (1983), 751–769.

[13] Trudinger N. S., Urbas J. I. E., *The Dirichlet problem for the equation of prescribed Gaus curvature*, Bull. Austral. Math. Soc. 28, (1983), 217–231.

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