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POINTWISE TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper all pointwise transformations converting every linear system of differential equations of the first order into a system of the same kind or into a single scalar equation of the first order are found.

Key words. Linear differential system, pointwise transformation, automorphism.

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1. INTRODUCTION

Let us consider matrix linear differential equations

$$Y' = P(x)\,Y, \tag{1}$$

where P is a continuous function on an open interval with values in the set of all $n \times n$-matrices and Y is an $n \times n$-matrix. The aim of this paper is to describe all pointwise transformations

$$\begin{aligned} t &= T_1(x, Y), \\ Z &= T_2(x, Y), \end{aligned} \tag{2}$$

converting every equation of the form (1) in variables x, Y into an equation of the same type in variables t and Z. Let us remember some known examples of such transformations $(x, Y) \to (t, Z)$.

The transformation $(x, Y) \to (x, (Y^{-1})^*)$, defined for all regular matrices (* stands for the transpose), converts the equation (1) into the adjoint equation

$$Z' = -P^*(t)\,Z.$$

The transformation $(x, Y) \to (x, (\det Y)^{\lambda}\, Y)$, defined for all matrices with positive determinant and $\lambda \in \mathbf{R}$, converts the equation (1) into the equation

$$Z' = (\lambda \operatorname{tr} P(t) + P(t))\,Z,$$

where tr denotes the trace of a matrix.

Further examples are the transformations $(x, Y) \to (k(x), M(x)\,Y)$ and $(x, Y) \to \to (x, YC)$, where k is a real function and M is an $n \times n$-matrix function with some addititional properties, C is a regular matrix.

The main result, the exact formulation of which is given in the following section, states that every pointwise transformation that converts every equation (1) with a coefficient $P(x)$ of the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ p_0(x) & p_1(x) & p_2(x) & \dots & p_{n-1}(x) \end{pmatrix} \tag{3}$$

into an equation (1) arises only as a composition of the four types of transformations mentioned above. The third section contains several lemmas the proof of the main theorem is based on. The proof itself is given in the next section. With the aid of Lemma 1 the problem to describe all pointwise transformations is converted into the task to find all monomorphisms of the group $\mathbf{GL}_n^+(\mathbf{R})$. In section 5 the same method is applied to describe all pointwise transformations of the equations (1), where the solution is considered in the form of column vector. In the last section a further application of these methods yields the description of all pointwise transformations converting every equation (1) into a single (scalar) linear differential equation of the first order.

2. NOTATION AND MAIN RESULT

Let $n \geqq 1$ be an integer, let $r \geqq 0$ be an integer or ∞ and let I and J bè open intervals. The set of all real $n \times n$-matrices will be denoted as $\mathbf{M}_n$. The symbol $\mathbf{L}_n$ will be used for the subset of $\mathbf{M}_n$ consisting of all matrices of the form (3). The symbol $\mathbf{GL}_n$ will stand for the group of real regular $n \times n$-matrices with usual multiplication. However, sometimes $\mathbf{GL}_n$ will be considered only as the set of regular matrices. The same holds for its subgroups $\mathbf{GL}_n^+$ of all real $n \times n$-matrices with positive determinants and SL_n of all real $n \times n$-matrices with determinants 1. The set of all real $n \times n$-matrices with negative determinants will be denoted as $\mathbf{GL}_n^-$. The symbols E, tr, det, $^{-1}$, *, $\mathbf{R}^+$ will stand for the unit matrix, the trace, the determinant, the inverse, the transpose and the set of real numbers, respectively. For our purpose the norm of a matrix $A = (A_{ij}) \in \mathbf{M}_n$, $i, j = 1, 2, \dots, n$, is defined as $\| A \| = \max \{| A_{ij} |,\ i, j = 1, 2, \dots, n\}$.

The sets of all continuous and r-times continuously differentiable functions defined on M with values in N will be denoted as $C(M, N)$ and $C^r(M, N)$, respectively. Further, we shall write $E(P, I)$ for the equation (1) with $P \in C(I, \mathbf{M}_n)$. We shall take into account only equations with continuous coefficients and their classical solutions, however, using the same method, similar results can be obtained in Caratheodory's theory of differential equations as well.

Equations $E(P, I)$ with $P \in C(I, \mathbf{L}_n)$ correspond to linear homogeneous differential equations of the n-th order. For every vector function $y = (y_1, y_2, \ldots, y_n) \in \in C^n(I, \mathbf{R}^n)$ the symbol $W[y]$ will denote the function of I into $\mathbf{M}_n$ given by the Wronski matrix

$$W[y](x) = \begin{pmatrix} y_1(x), & y_2(x), & \ldots, & y_n(x) \\ y_1'(x), & y_2'(x), & \ldots, & y_n'(x) \\ \multicolumn{4}{c}{\dotfill} \\ y_1^{(n-1)}(x), & y_2^{(n-1)}(x), & \ldots, & y_n^{(n-1)}(x) \end{pmatrix}.$$

A mapping $T = (T_1, T_2) : I \times \mathbf{GL}_n^+ \to J \times \mathbf{M}_n$ will be considered as a pointwise transformation if the following assumptions are satisfied:

(A1) *T is a homeomorphism of $I \times \mathbf{GL}_n^+$ into $J \times \mathbf{M}_n$.*

(B1) *For every equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ there is a an equation $E(R, J)$ with $R \in C(J, \mathbf{M}_n)$ such that for every solution $Y \in C^1(I, \mathbf{GL}_n^+)$ of the equation $E(P, I)$ the couples (t, Z)*

$$t = T_1(x, Y(x)), \qquad Z = T_2(x, Y(x)),$$

for $x \in I$ form a graph of a function $Z(t)$ representing a solution of $E(R, J)$ on J.

Now we are in a position to formulate the main result.

Theorem 1. *A mapping $T = (T_1, T_2)$ satisfies the conditions* **(A1)** *and* **(B1)** *if and only if*

$$T_1(x, Y) = k(x) \tag{4}$$

and either

$$T_2(x, Y) = M(x)(\det Y)^\lambda YC, \qquad \lambda \neq -1/n, \tag{5a}$$

or

$$T_2(x, Y) = M(x)(\det Y)^\lambda (Y^{-1})^* C, \qquad \lambda \neq 1/n, \tag{5b}$$

where $\lambda \in \mathbf{R}$, $C \in \mathbf{GL}_n$, k is a homeomorphism of I onto J having the inverse $h \in \in C^1(J, I)$, $M \in C(I, \mathbf{GL}_n)$ and $M \circ h \in C^1(J, \mathbf{GL}_n)$.

A mapping $T = (T_1, T_2)$ will be called a pointwise transformation of the first order systems if it satisfies **(A1)** and **(B1)**, where the condition $P \in C^r(I, \mathbf{L}_n)$ is replaced by the condition $P \in C(I, \mathbf{M}_n)$. Theorem 1 implies immediately the following

Consequence. *A mapping $\hat{T} = (T_1, T_2)$ is a pointwise transformation of the first order systems if and only if it has the form* (4), (5a) *or* (5b), *where $\lambda \in \mathbf{R}$, $C \in \mathbf{GL}_n$, k is a homeomorphism of I onto J having the inverse $h \in C^1(J, I)$, $M \in C(I, \mathbf{GL}_n)$ and $M \circ h \in C^1(J, \mathbf{GL}_n)$.*

Remark. The pointwise transformation (4), (5) described in Theorem 1 converts every equation $E(P, I)$ into the equation

$$\begin{aligned} Z' = \big[&N'(t)\, N^{-1}(t) + \lambda h'(t) \operatorname{tr} P(h(t))\, E + \\ &+ h'(t)\, N(t)\, P(h(t))\, N^{-1}(t)\big]\, Z \end{aligned} \tag{6a}$$

in case (5a), or into the equation

$$\begin{aligned} Z' = \big[&N'(t)\, N^{-1}(t) + \lambda h'(t) \operatorname{tr} P(h(t))\, E - \\ &- h'(t)\, N(t)\, P^*(h(t))\, N^{-1}(t)\big]\, Z \end{aligned} \tag{6b}$$

in case (5b), where $N = M \circ h$. It can be easily checked from the fact that the pointwise transformation converts every solution $Y \in C^1(I, \mathbf{GL}_n)$ either into the function

$$Z(t) = N(t) \left[\det Y(h(t))\right]^\lambda Y(h(t))\, C$$

or into the function

$$Z(t) = N(t) \left[\det Y(h(t))\right]^\lambda \left[Y^{-1}(h(t))\right]^* C.$$

3. AUXILIARY LEMMAS

In this section we shall give several lemmas needed in the proof of Theorem 1 and in the following sections.

Lemma 1. *Let $a < b$ be real numbers, let $A, B \in \mathbf{GL}_n^+$ and let $A^i, B^i \in R^n$ for all $i \geqq n$. There is a function $y \in C^\infty([a, b], \mathbf{R}^n)$ such that*

(a) *$W[y](a) = A$, $W[y](b) = B$,*
(b) *$W[y](x) \in \mathbf{GL}_n^+$ for every $x \in [a, b]$,*
(c) *$y^{(i)}(a) = A^i$, $y^{(i)}(b) = B^i$ for every $i \geqq n$.*

The idea of the proof belongs to V. Šverák (personal communication). We shall sketch it only since the detailed proof would be too long. For details see [2].

Without loss of generality we can suppose $a = 0$ and $b = 1$. First, we shall prove that there is a function $v \in C^\infty([0, 1], \mathbf{R}^n)$ satisfying (a) and (b). This fact will be written as $A \xrightarrow{v} B$. For every matrix $A \in \mathbf{GL}_n^+$ define

$$\mathbf{G}_A = \{B \in \mathbf{GL}_n^+, \text{ there is a } v \in C^\infty([0, 1], \mathbf{R}^n), A \xrightarrow{v} B\}.$$

We would wish to show that $\mathbf{G}_A = \mathbf{GL}_n^+$ for every $A \in \mathbf{GL}_n^+$. $\mathbf{GL}_n^+$ is connected, hence, it is sufficient to prove that $\mathbf{G}_A$ is nonempty, open and closed. We can do it in the following steps:

(i) *$A \in \mathbf{G}_A$ for every $A \in \mathbf{GL}_n^+$.*

The function v with the property $A \xrightarrow{v} A$ is taken as a fundamental solution of a suitable linear homogeneous differential equation of the n-th order with constant coefficients. This fundamental solution has the period 1 and satisfies the initial condition $W[v](0) = A$.

(ii) *$\mathbf{G}_A$ is open.*

Let $A \xrightarrow{v} B$. There is a $\delta > 0$ such that for every $C \in \mathbf{GL}_n^+$ with $\| C - B \| < \delta$ there is a function $z \in C^\infty([0, 1], \mathbf{M}_n)$, $W[z](0) = 0$, $W[z](1) = C - B$ with the norm $\| z \| = \sup_{x \in [0,1]} \| W[z](x) \|$ small enough for $W[v + z](x)$ to be an element of the set $\mathbf{GL}_n^+$ for every $x \in [0, 1]$. Then $A \xrightarrow{v+z} C$.

(iii) *Let $A, B, C \in \mathbf{GL}_n^+$. If $B \in \mathbf{G}_A$ and $C \in \mathbf{G}_B$ then $C \in \mathbf{G}_A$.*

(iv) *For every $B \in \mathbf{GL}_n^+$ there is a $\Delta > 0$ such that $B \in \mathbf{G}_C$ whenever $C \in \mathbf{GL}_n^+$ and $\| C - B \| < \Delta$.*

Step (i) and considerations similar to those in (ii) are used in the proof of this statement.

(v) *G_A is closed.*

Let $B \in \mathbf{GL}_n^+$, $C_j \in \mathbf{G}_A$ and $\lim_{j \to \infty} C_j = B$. Choose $\Delta > 0$ according to (iv). There is a subscript j such that $\| C_j - B \| < \Delta$. Then (iv) implies $B \in \mathbf{G}_{Cj}$, and according to (iii), we get $B \in \mathbf{G}_A$.

We have proved the existence of a function $v \in C^\infty([0, 1], \mathbf{R}^n)$, which satisfies (a) and (b) of Lemma 1. One can find a function $z \in C^\infty([0, 1], \mathbf{R}^n)$ such that

$$\begin{aligned} z^{(i)}(0) &= z^{(i)}(1) = 0 && \text{for } 0 \leqq i \leqq n - 1, \\ z^{(i)}(0) &= A^i - v^{(i)}(0) && \text{for } i \geqq n, \\ z^{(i)}(1) &= B^i - v^{(i)}(1) && \text{for } i \geqq n, \end{aligned}$$

with the norm $\| z \| = \sup \{\| W[z](x) \|, x \in [0, 1]\}$ sufficiently small for $W[v + z](x)$ to be an element of $\mathbf{GL}_n^+$ for every $x \in [0, 1]$. If we put $y = v + z$, we get a required function satisfying (a), (b) and (c).

Lemma 2. *Every continuous homomorphism $\chi : \mathbf{GL}_n^+ \to \mathbf{GL}_1^+$ has the form*

$$\chi(A) = (\det A)^\lambda, \tag{7}$$

where $\lambda \in \mathbf{R}$.

Proof. In [1] it is derived that every continuous homomorphism of $\mathbf{GL}_n$ into $\mathbf{GL}_1$ is of the form (7). Similarly one proves that every continuous homomorphism χ: $\mathbf{GL}_n^+ \to \mathbf{GL}_1^+$ can be extended to a continuous homomorphism $\bar{\chi}$: $\mathbf{GL}_n \to \mathbf{GL}_1$, see [2]. So the statement of Lemma 2 follows directly from the description of all continuous homomorphisms of $\mathbf{GL}_n$ into $\mathbf{GL}_1$.

Lemma 3. *Every continuous monomorphism G: $\mathbf{GL}_n^+ \to \mathbf{GL}_n^+$ is either of the form*

$$G(A) = (\det A)^{\lambda}\, C^{-1}AC, \qquad \lambda \neq -\frac{1}{n}, \tag{8a}$$

or of the form

$$G(A) = (\det A)^{\lambda}\, C^{-1}(A^{-1})^{*}C, \qquad \lambda \neq \frac{1}{n}, \tag{8b}$$

where $\lambda \in \mathbf{R}$, $C \in \mathbf{GL}_n$.

Proof. First, we prove that every continuous monomorphism G on $\mathbf{GL}_n^+$ is already an automorphism. Let U be a compact neighbourhood of the unit matrix in $\mathbf{GL}_n^+$. G is a homeomorphism of U on $G(U)$ and that is why G is a homeomorphism on the interior int U of U as well. From the properties of homeomorphisms on open sets in the Euclidian spaces it follows that $G(\text{int } U)$ is an open neighbourhood of the unit matrix in $\mathbf{GL}_n^+$. Consequently, $G(\mathbf{GL}_n^+)$ is a subgroup of $\mathbf{GL}_n^+$ containing a neighbourhood of the unit matrix. Since $\mathbf{GL}_n^+$ is connected topological group, we have $G(\mathbf{GL}_n^+) = \mathbf{GL}_n^+$. (See [3].)

Further, $\det G$ is a continuous homomorphism of $\mathbf{GL}_n^+$ on $\mathbf{R}^+ = \mathbf{GL}_1^+$ and, according to Lemma 2, it has the form (7) with $\lambda \neq 0$. Hence, $G(\mathbf{SL}_n) = \mathbf{SL}_n$.

For every $A \in \mathbf{GL}_n^+$ there is a unique $a \in \mathbf{R}^+$ and $B \in \mathbf{SL}_n$ such that

$$A = aB, \tag{9}$$

where $a = (\det A)^{1/n}$. The automorphism G maps the centre of $\mathbf{GL}_n^+$ onto itself that is why

$$G(aE) = \chi(a)\, E,$$

where χ is a continuous real multiplicative function on $\mathbf{R}^+$. Hence we get

$$\chi(a) = a^{\nu},$$

where $\nu \neq 0$, because G is an automorphism. According to [4] all automorphisms on $\mathbf{SL}_n$ have the form $B \to C^{-1}BC$ or $B \to C^{-1}(B^{-1})^{*}C$ with $C \in \mathbf{GL}_n$. From this fact and from (9) we get

$$G(A) = G(aB) = G(aE)\, G(B) = \chi(a)\, G(B)$$

and further either

$$G(A) = a^{\nu}C^{-1}BC = a^{\nu-1}C^{-1}AC = (\det A)^{\lambda}\, C^{-1}AC,$$

where $\lambda = \dfrac{\nu - 1}{n} \mp \dfrac{1}{n}$, or

$$G(A) = a^{\nu}C^{-1}(B^{-1})^{*}C = a^{\nu+1}C^{-1}(A^{-1})^{*}C = (\det A)^{\lambda}\, C^{-1}(A^{-1})^{*}C,$$

where $\lambda = \dfrac{\nu + 1}{n} \mp \dfrac{1}{n}$.

4. PROOF OF THEOREM 1

One can easily find out that the transformations described by the formulae (4) and (5a) or (5b) satisfy the assumptions **(A1)** and **(B1)**. The proof of the converse statement will be given in several steps.

I. We shall show that T_1 is independent of the variable Y. For $t \in J$ denote $S_t = (\{t\} \times \mathbf{M}_n) \cap T(I \times \mathbf{GL}_n^+)$. According to **(B1)**, $S_t \neq 0$ for every $t \in J$.

First, we shall prove that for every $t \in J$ there is a unique $x = h(t) \in I$ such that $T^{-1}(S_t) \subset \{x\} \times \mathbf{GL}_n^+$. On the contrary, suppose there are $(x_1, Y_1), (x_2, Y_2) \in \in I \times \mathbf{GL}_n^+$, $x_1 \neq x_2$, such that $T_1(x_i, Y_i) = t$ for $i = 1, 2$. According to Lemma 1 there is a function $y \in C^{\infty}(I, \mathbf{R}^n)$ such that $W[y]$ is a solution of some equation $E(P, I)$ with $P \in C^{\infty}(I, \mathbf{L}_n)$ and $W[y]\,(x_i) = Y_i$ for $i = 1, 2$. The transformation T converts the function $W[y]$ into a function, which is contradictory to the fact that $T(x_i, Y_i) \in S_t$ for $i = 1, 2$.

Now it is sufficient to prove that h is injective. Let $Y \in C^1(I, \mathbf{GL}_n^+)$ be a solution of some equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$. T transforms the function Y in some function Z on the interval J. If $t_1, t_2 \in J$, $t_1 \neq t_2$, there are $x_1, x_2 \in I$, $x_1 \neq x_2$, such that $T(x_i, Y(x_i)) = (t_i, Z(t_i))$ for $i = 1, 2$. Hence $h(t_1) = x_1 \neq x_2 = h(t_2)$.

Consequently,

$$T_1(x, Y) = k(x),$$

where k is a homeomorphism of J onto I with the inverse function h. Put

$$F(t, Y) = T_2(h(t), Y).$$

Then the mapping T converts every solution $Y \in C^1(I, \mathbf{GL}_n^+)$ of some equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ into the function

$$Z(t) = F(t, Y(h(t))). \tag{10}$$

II. We shall show that the transformation described by (10) has the following properties:

(a) *Let* $B \in \mathbf{GL}_n^+$, $D \in \mathbf{M}_n$ *be fixed. If for some solution* $Y \in C^1(I, \mathbf{GL}_n^+)$ *of some equation* $E(P, I)$ *with* $P \in C^r(I, \mathbf{L}_n)$ *there holds*

$$F(t, Y(h(t))\, B) = F(t, Y(h(t)))\, D \quad \text{on } J, \tag{11}$$

then (11) *is fulfilled for every equation* $E(P, I)$ *with* $P \in C^r(I, \mathbf{L}_n)$ *and every its solution* $Y \in C^1(I, \mathbf{GL}_n^+)$.

(b) *Let* $x \in \{+, -\}$ *be fixed and let* $F(t, Y(h(t))) \in C^1(J, \mathbf{GL}_n^\sigma)$ *for some solution* $Y \in C^1(I, \mathbf{GL}_n^+)$ *of some equation* $E(P, I)$ *with* $P \in C^r(I, \mathbf{L}_n)$. *Then the same holds for every equation* $E(P, I)$ *with* $P \in C^r(I, \mathbf{L}_n)$ *and for every its solution* $Y \in C^1(I, \mathbf{GL}_n^+)$.

Let us prove (a). Let $Y_1 \in C^1(I, \mathbf{GL}_n^+)$ be a solution of some equation with a coefficient in $C^r(I, \mathbf{L}_n)$ satisfying (11). Let Y_2 be an arbitrary solution of some equation with a coefficient in $C^r(I, \mathbf{L}_n)$. Choose $a, b \in I$, $a < b$. Then $I = I_1 \cup [a, b] \cup I_2$, where I_1, I_2 are open intervals, $I_i \cap [a, b] = \emptyset$ for $i = 1, 2$. Lemma 1 implies the existence of a function $Y \in C^1(I, \mathbf{GL}_n^+)$ which is a solution of some equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ and $Y = Y_i$ for $i = 1, 2$. YB is a solution of this equation as well. The transformation T converts this equation into an equation $E(R, J)$ with solutions $F(t, Y(h(t)))$ and $F(t, Y(h(t))\, B)$. Since (11) is fulfilled for Y on the interval $h^{-1}(I_1) \subset J$, it is also satisfied for Y on the whole interval as well as for Y_2 on the interval $h^{-1}(I_2)$. Hence, (11) is satisfied for Y_2 on J.

The proof of (b) is analogous to that of (a).

III. Let $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ be a fixed equation. The transformation T converts it into an equation $E(R, J)$, $R \in C(J, \mathbf{M}_n)$. Consider their solutions $Y \in C^1(I, \mathbf{GL}_n^+)$ and $Z \in C^1(J, \mathbf{GL}_n)$, respectively. Define the mapping $G\colon \mathbf{GL}_n^+ \to \mathbf{M}_n$ in the following way

$$G(B) = D \quad \text{iff} \quad F(t, Y(h(t))\, B) = Z(t)\, D.$$

The mapping G has these properties:

(a) G *is a homeomorphism of* $\mathbf{GL}_n^+$ *into* $\mathbf{GL}_n^\sigma$, *where* $\sigma \in \{+, -\}$.

(b) *Let* $B \in \mathbf{GL}_n^+$, $D \in \mathbf{M}_n$ *be fixed. If for some* $A \in \mathbf{GL}_n^+$ *there holds*

$$G(AB) = G(A)\, D, \tag{12}$$

then (12) *is satisfied for every* $A \in \mathbf{GL}_n^+$.

We are going to prove that G is continuous. Let $B_j \in \mathbf{GL}_n^+$ for $j = 1, 2, \ldots$ and $\lim_{j\to\infty} B_j = B \in \mathbf{GL}_n^+$. Put $G(B_j) = D_j$, $G(B) = D$. Then we get

$$\lim_{j\to\infty} Z(t)\, D_j = \lim_{j\to\infty} F(t, Y(h(t))\, B_j) = F(t, Y(h(t))\, B) = Z(t)\, D.$$

Since $Z(t) \in \mathbf{GL}_n$, we have $\lim_{j\to\infty} D_j = D$ and G is continuous.

Let $B_1, B_2 \in \mathbf{GL}_n^+$, $B_1 \neq B_2$. Foi every $x \in I$ we have $Y(x)\,B_1 \neq Y(x)\,B_2$, T is a homeomorphism hence $Z(t)\,G(B_1) \neq Z(t)\,G(B_2)$ and we get $G(B_1) \neq G(B_2)$.

G is a continuous injective mapping defined on the open subset of the Euclidian space. Local compactness and considerations similar to those in the proof of Lemma 3 imply that G maps open sets on open sets and is a homeomorphism. Since $\mathbf{GL}_n^+ \cup \mathbf{GL}_n^-$ is dense in $\mathbf{M}_n$, there is a $B \in \mathbf{GL}_n^+$ such that $G(B) \in \mathbf{GL}_n^\sigma$ for some $\sigma \in \{+, -\}$. According to (b) in **II**, $G(\mathbf{GL}_n^+) \in \mathbf{GL}_n^\sigma$.

(b) is a consequence of (a) in **II**.

IV. Let Y be the function appearing in the definition of G. According to (a) of the previous step, $F(t, Y(h(t))) \in C^1(J, \mathbf{GL}_n^\sigma)$ for fixed $\sigma \in \{+, -\}$. Therefore, let us specify the mapping G so that in its definition we choose $Z(t) = F(t, Y(h(t)))$. Then G defined in this way satisfies again (a) an (b) in **III**, and moreover, we get $G(E) = E$ and $G(\mathbf{GL}_n^+) \subset \mathbf{GL}_n^+$. Now we shall show that this G is a homomorphism of $\mathbf{GL}_n^+$ into $\mathbf{GL}_n^+$.

Let $A \in \mathbf{GL}_n^+$ be fixed. Then, $G(A) \in \mathbf{GL}_n^+$ and for every $B \in \mathbf{GL}_n^+$ there is a matrix D such that

$$G(AB) = G(A)\,D.$$

According to (b) is **III** this formula is satisfied for all matrices from $\mathbf{GL}_n^+$ on the place of A. The choice $A = E$ gives

$$G(B) = D.$$

Hence

$$G(AB) = G(A)\,G(B)$$

and G is a continuous monomorphism of $\mathbf{GL}_n^+$ into $\mathbf{GL}_n^+$.

V. According to (a) in **II** and the definition of G, we get for all the equations $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ and all their solutions $Y \in C^1(I, \mathbf{GL}_n^+)$ the relation

$$F(t, Y(h(t))\,B) = F(t, Y(h(t))\,G(B), \tag{13}$$

where B is an arbitrary matrix from $\mathbf{GL}_n^+$. For any $t \in J$ we can find a solution $Y \in C^1(I, \mathbf{GL}_n^+)$ of some equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ such that $Y(h(t)) = E$. Then (13) yields

$$F(t, B) = F(t, E)\,G(B).$$

Here G is a continuous monomorphism of $\mathbf{GL}_n^+$ into $\mathbf{GL}_n^+$ and hence, according to Lemma 3, we get either

$$T_2(x, Y) = M(x)\,(\det Y)^\lambda\, YC, \qquad \lambda \neq -\frac{1}{n},$$

or

$$T_2(x, Y) = M(x)(\det Y)^\lambda (Y^{-1})^* C, \qquad \lambda \neq \frac{1}{n},$$

where $M(x) = F(k(x), E)\, C^{-1}$, $\lambda \in \mathbf{R}$ and $C \in \mathbf{GL}_n$.

VI. The transformation T converts every function $Y \in C^1(I, \mathbf{GL}_n^+)$ into either

$$Z(t) = N(t)\,[\det Y(h(t))]^\lambda\, Y(h(t))\, C, \qquad \lambda \neq -\frac{1}{n}, \tag{14a}$$

or

$$Z(t) = N(t)\,[\det Y(h(t))]^\lambda\, [Y^{-1}(h(t))]^* C, \qquad \lambda \neq \frac{1}{n}, \tag{14b}$$

where $N = M \circ h$. It remains to be proved $h \in C^1(J, I)$ and $N \in C^1(J, \mathbf{GL}_n)$.

Let $Y_1, Y_2 \in C^1(I, \mathbf{GL}_n^+)$ be solutions of the equations $E(P_1, I)$ and $E(P_2, I)$ with $P_1, P_2 \in C^r(I, \mathbf{L}_n)$, respectively. Moreover, let us choose P_1 and P_2 such that $P_1(x) \neq P_2(x)$ for every $x \in I$ and $\operatorname{tr} P_1 \equiv \operatorname{tr} P_2 \equiv 0$ on the whole interval I. Then det Y_2 are constants and

$$\begin{aligned}(Y_1^{-1}Y_2)'(x) &= (Y_1^{-1})'(x)\, Y_2(x) + Y_1^{-1}(x)\, Y_2'(x) = \\ &= -Y_1^{-1}(x)\, P_1(x)\, Y_2(x) + Y_1^{-1}(x)\, P_2(x)\, Y_2(x) = \\ &= Y_1^{-1}(x)\,[P_2(x) - P_1(x)]\, Y_2(x) \neq 0.\end{aligned}$$

The functions Y_1 and Y_2 are converted into the functions Z_1 and Z_2 by the transformation T. In the case (14a) we obtain

$$C(Z_1^{-1}Z_2)(t)\, C^{-1} = a(Y_1^{-1}Y_2)(h(t))$$

and, similarly, in case (14b)

$$C(Z_2^{-1}Z_1)(t)\, C^{-1} = b(Y_1^{-1}Y_2)^*(h(t)),$$

where a, b are nonzero constants. The left-hand sides of both equalities are continuously differentiable functions and the right-hand sides are matrix functions of the form $V \circ h$, where $V \in C^1(I, \mathbf{GL}_n^+)$ and $h \in C(J, I)$. Since $V'(x) \neq 0$ for all $x \in I$, we get $h \in C^1(J, I)$. Now, the condition $N \in C^1(I, \mathbf{GL}_n)$ follows immediately from (14). The proof of Theorem 1 is completed.

5. APPLICATION

In this section we shall deal again with equations (1), where $P \in C^r(I, \mathbf{L}_n)$ or $P \in C(I, \mathbf{M}_n)$ but solutions are now considered to be column vectors. In this case a mapping $\tau: I \times \mathbf{R}^n \to J \times \mathbf{R}^n$

$$\begin{aligned} t &= \tau_1(x, y), \\ z &= \tau_2(x, y), \end{aligned} \tag{15}$$

is usually understood as pointwise transformation. Under the condition that τ is a C^n-diffeomorphism in [5] Wilczynski found the most general form of the transformation (15) converting every equation (1) in variables x, y with $P \in C^{n-1}(I, \mathbf{M}_n)$ into an equation of the same kind in variables t and z. We shall show that the transformation (15) can be considered as a special case of transformations examined in the previous sections. Using Theorem 1 we shall derive the general form of such transformations without assumptions of differentiability.

We shall take into account transformations (15) with the following properties:

(A2) *$\tau = (\tau_1, \tau_2)$ is a homeomorphism of $I \times \mathbf{R}^n$ into $J \times \mathbf{R}^n$.*

(B2) *For every equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ (or $P \in C(I, \mathbf{M}_n)$) there is an equation $E(R, J)$ with $R \in C(J, \mathbf{M}_n)$ such that for every solution $y \in C^1(I, \mathbf{R}^n)$ of the equation $E(P, I)$ the couples (t, z)*

$$t = \tau_1(x, y(x)), \qquad z = \tau_2(x, y(x))$$

for $x \in I$ form a graph of a function $z(t)$ representing a solution of the equation $E(R, J)$.

Theorem 2. *Let $n \geqq 2$. Every transformation satisfying* **(A2)** *and* **(B2)** *is of the form*

$$\tau_1(x, y) = k(x), \qquad \tau_2(x, y) = M(x)\, y, \tag{16}$$

where k is a homeomorphism between I and J with the inverse function $h \in C^1(J, I)$ and $M \in C(I, \mathbf{GL}_n)$, $M \circ h \in C^1(J, \mathbf{GL}_n)$.

Proof. First, we prove that τ_1 depends only on the variable x. For $t \in J$ put $S_t = (\{t\} \times \mathbf{R}^n) \cap \tau(I \times \mathbf{R}^n)$. According to **(B2)**, $S_t \neq 0$ for every $t \in J$. Suppose there is a $t \in J$ and $(x_1, y_1), (x_2, y_2) \in I \times \mathbf{R}^n$, $x_1 \neq x_2$, such that $\tau_1(x_i, y_i) = t$ for $i = 1, 2$. According to Lemma 1 for $y_1 \neq 0$ and $y_2 \neq 0$ or for $y_1 = y_2 = 0$, there is a solution $y \in C^1(I, \mathbf{R}^n)$ of some equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$, $y(x_i) = y_i$ for $i = 1, 2$. The transformation τ converts the function y into a function, which contradicts to $\tau_1(x_i, y(x_i)) = t$ for $i = 1, 2$.

Since τ is a homeomorphism, $\tau^{-1}(S_t)$ is an n-dimensional manifold with the following properties:

(i) If $(x_i, y_i) \in \tau^{-1}(S_t)$ and $y_i \neq 0$ for $i = 1, 2$, then $x_1 = x_2$.

(ii) There is at most one $x \in I$ such that $(x, 0) \in \tau^{-1}(S_t)$.

Hence for every $t \in J$ there is a unique $x = h(t)$ such that $\tau^{-1}(S_t) = \{x\} \times \mathbf{R}^n$. The proof that h is a homeomorphism of J on I can be performed in the same way as in Section 4.

Denote by k the inverse function of h and for every matrix $Y \in \mathbf{GL}_n^+$ with columns $Y_i \in \mathbf{R}^n$, $1 \leqq i \leqq n$, define

$$T_1(x, Y) = k(x), \qquad T_2(x, Y) = (\tau_2(x, Y_1), \tau_2(x, Y_2), \ldots, \tau_2(x, Y_n)). \tag{17}$$

The transformation T satisfies the assumptions (**A1**) and (**B1**) and, therefore, according to Theorem 1 we have

$$T_2(x, Y) = M(x)\, G(Y),$$

where $M \in C(I, \mathbf{GL}_n)$ and G is a continuous monomorphism of $\mathbf{GL}_n^+$ into itself. Because of T_2 being defined by (17) there exists a mapping $g : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n$ such that

$$G(Y) = (g(Y_1), g(Y_2), \ldots, g(Y_n)). \tag{18}$$

Taking into account (18) one can show with the aid of Lemma 3 that

$$G(Y) = Y$$

and, consequently,

$$\tau_2(x, y) = M(x)\, y$$

for every $y \in \mathbf{R}^n \setminus \{0\}$. From the continuity of τ_2 it is valid for $y = 0$ as well. The fact that $h \in C^1(J, I)$ and $M \circ h \in C^1(J, \mathbf{GL}_n)$, follows from Theorem 1.

6. POINTWISE TRANSFORMATIONS OF SYSTEMS (1) INTO SINGLE SCALAR EQUATIONS

First, we shall describe all pointwise transformations $T = (T_1, T_2): I \times \mathbf{GL}_n^+ \to J \times \mathbf{R}$

$$t = T_1(x, Y),$$
$$z = T_2(x, Y),$$

converting every system of equations (1) with $P \in C^r(I, \mathbf{L}_n)$ or $P \in C(I, \mathbf{M}_n)$ into a single scalar equation

$$z' = p(t)\, z,$$

with $p \in C(J, \mathbf{A})$ on the interval J.

Let T satisfy the following conditions

(**A3**) *$T : I \times \mathbf{GL}_n^+ \to J \times \mathbf{R}$ is a continuous mapping and T_2 is not identically zero.*

(**B3**) *For every equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ (or $P \in C(I, \mathbf{M}_n)$) there is an equation $E(p, J)$ with $p \in C(J, \mathbf{A})$ such that if $Y \in C^1(I, \mathbf{GL}_n^+)$ is a solution of $E(P, I)$ then T is injective on the graph of the function Y and for $x \in I$ the couples (t, z)*

$$t = T_1(x, Y(x)), \qquad z = T_2(x, Y(x)).$$

form a graph of a function $z(t)$ representing a solution of $E(p, J)$.

Theorem 3. *A mapping $T = (T_1, T_2)$ satisfies* (**A3**) *and* (**B3**) *if and only if*

$$\begin{aligned} T_1(x, Y) &= k(x), \\ T_2(x, Y) &= v(x)\, (\det Y)^\lambda, \end{aligned} \tag{19}$$

where $\lambda \in \mathbf{R}$, k *is a homeomorphism of* I *on* J *having the inverse function* h, $h \in C^1(J, I)$ *for* $\lambda \neq 0$ *and* $u = v \circ h \in C^1(J, \mathbf{R}\setminus\{0\})$. *Moreover, the transformation* (19) *converts every equation* $E(P, I)$ *into the equation*

$$z' = \left[\frac{u'(t)}{u(t)} + \lambda h'(t) \operatorname{tr} P(h(t))\right] z.$$

Proof. Let T satisfy (**B3**) and (Z3). Analogously to Section 4 we can prove that T_1 is independent of the variable Y. We put

$$k(x) = T_1(x, Y),$$
$$F(t, Y) = T_2(h(t), Y),$$

where h is the inverse function to k. In the same way as in the proof of Theorem 1, for all the equations $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ and all their solutions $Y \in C^1(I, \mathbf{GL}_n^+)$ we derive the relation

(21) $$F(t, Y(h(t))\, B) = F(t, Y(h(t)))\, \chi(B),$$

where χ is a continuous homomorphism of $\mathbf{GL}_n^+$ into $\mathbf{GL}_1^+$. Let $t \in J$ be fixed and let $Y \in C^1(I, \mathbf{GL}_n^+)$ be a solution of some equation $E(P, I)$, $P \in C^r(I\ \mathbf{L}_n)$ such that $Y(h(t)) = E$. Substituting this Y into (21) and using Lemma 2 we get

$$F(t, B) = F(t, E)\, (\det B)^\lambda = u(t)\, (\det B)^\lambda.$$

T_2 is not identically zero, hence, there exists a solution $Y \in C^1(I, \mathbf{GL}_n^+)$ of some equation $E(P, I)$ such that

(22) $$z(t) = u(t)\, (\det Y(h(t)))^\lambda$$

is not identically 0. Since z is a solution of a scalar equation of the first order, $z(t) \neq 0$ for every $t \in J$ and that is why $u(t) \neq 0$ on J. If $\lambda = 0$ we have immediately $u \in C^1(J, \mathbf{R}\setminus\{0\})$. If $\lambda \neq 0$, for every $t_0 \in J$ we can find solutions $Y_1, Y_2 \in C^1(I, \mathbf{GL}_n^+)$ of two different equations (1) with the coefficients in $C^r(I, \mathbf{L}_n)$ such that

$$\left(\frac{\det Y_1}{\det Y_2}\right)'(h(t)) \neq 0$$

on a neighbourhood of the point t_0. Writing equalities (22) for Y_1 and Y_2 instead of Y and dividing the former by the latter, we obtain

$$\frac{z_1(t)}{z_2(t)} = \left(\frac{\det Y_1}{\det Y_2}\right)^\lambda (h(t)).$$

The left-hand side of this formula is a function of the class C^1 and the function $(\det Y_1/\det Y_2)^\lambda$ has the inverse function of the class C^1 on some neighbouhood of t_0. Hence, h is of the class C^1 on this neighbourhood. Then from (22) we get $u \in C^1(J, \mathbf{R}\setminus\{0\})$. Putting $v = u \circ k$, this part of the proof is over.

It is easy to find out that every transformation (19) satisfies (**A3**) and (**B3**). By differentiation of (22) one can make sure that the transformation (19) converts every equation $E(P, I)$ into the equation (20).

Remark. There is a possibility to generalize Theorem 3 and to describe all pointwise transformations converting every system (1) with $P \in C^r(I, \mathbf{L}_n)$ or $P \in C(I, \mathbf{M}_n)$ into a system (1) of order m, $1 \leqq m < n$. The proof would be proceeded similarly as those of Theorem 1 and Theorem 3. It is based on the fact, the author have learnt recently, that every continuous homomorphism $G\colon \mathbf{GL}_n^+ \to \mathbf{GL}_m^+$, $1 \leqq m < n$, is of the form

$$G(A) = \exp\left[C \log(\det A)\right],$$

where $C \in M_m$ and $\exp B = \sum_{i=0}^{\infty} B^i/i!$, ([6]).

Consider the assumptions

(**A4**) *$T = (T_1, T_2)$ is a continuous mapping of $I \times \mathbf{GL}_n^+$ into $J \times \mathbf{M}_m$, $T(I \times \mathbf{GL}_n^+) \cap \cap (J \times \mathbf{GL}_m) \neq 0$.*

(**B4**) *For every equation $E(P, I)$ with $P \in C^r(I, \mathbf{L}_n)$ or $P \in C(I, \mathbf{M}_n)$ there is an equation $E(R, J)$ with $R \in C(J, \mathbf{M}_m)$ such that for every solution $Y \in C^1(I, \mathbf{GL}_n^+)$ of the equation $E(P, I)$ the mapping T is injective on the graph of the function Y and the couples (t, Z)*

$$t = T_1 = T_1(x, Y(x)), \qquad Z = T_2(x, Y(x))$$

for $x \in I$ form a graph of a function $Z(t)$ representing a solution of $E(R, J)$.

For $1 \leqq m < n$ the following generalization of Theorem 3 holds:

A mapping $T = (T_1, T_2)$ satisfies (**A4**) *and* (**B4**) *if and only if*

$$T_1(x, Y) = k(x),$$

$$T_2(x, Y) = M(x) \exp\left[C \log(\det Y)\right],$$

where k is a homeomorphism of I on J having the inverse h, $C \in \mathbf{M}_m$, $h \in C^1(J, I)$ for $C \neq 0$, $M \in C(I, \mathbf{GL}_m)$ and $M \circ h \in C^1(J, \mathbf{GL}_m)$.

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