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# ON CORRECTNESS OF THE GENERALIZED BOUNDARY VALUE PROBLEM FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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*Dedicated to Academician Otakar Borůvka on the occasion of his ninetieth birthday*

**Abstract.** By means of surjectivity theorems in $R^n$ the correctness of the generalized boundary value problem for ordinary differential systems is investigated. A comparison theorem is proved which gives a necessary and sufficient condition for the correctness of the boundary value problem when its uniqueness is assured.

**Key words.** Generalized boundary value problem, surjective mapping, $\tau$-correctness, a subordinate functional, the orientation of a functional.

**MS Classification.** 34 B 15, 47 H 99.

In the sequel the following theorem on surjectivity in $R^n$ from [2], [3] will be used. Here it will be given as

**Lemma 1.** *Let $g$: $R^n \to R^n$ be a continuous map. Then the following statements are true:*

(a) *If $g$ is injective, then $g$ is a homeomorphism of $R^n$ onto itself if and only if it satisfies the condition*

$$\lim_{|x| \to \infty} |g(x)| = \infty. \tag{1}$$

($| \cdot |$ denotes any norm in $R^n$.)

(b) *If $g$ satisfies* (1) *and one of the conditions:*
*Either*
*there is an $x_0 \in R^n$ such that for each $x \in R^n$, $x \neq x_0$,*

$$g(x) - x_0 = k(x - x_0) \qquad \textit{implies } k \geqq 0, \tag{2}$$

*or*
*there is an $x_0 \in R^n$ such that for each $x \in R^n$, $x \neq x_0$,*

(2′) $$g(x) - x_0 = k(x - x_0) \quad \textit{implies } k \leqq 0,$$

*then g is surjective.*

Similarly as in [1], [2] under the generalized boundary value problem for the diferential system

(3) $$x' = f(t, x), \qquad t \in i, x \in R^n,$$

and for the given mapping $F$ of the space $C(i, R^n)$ of all continuous vector functions $x$: $i \to R^n$ we understand the problem to find a solution $x(t)$ of the system (3) in the interval $i$ for which $F(x)$ is a given vector $r \in R^n$, that is

(4) $$F(x) = r.$$

Here and in what follows we suppose that the function $f$ satisfies local Carathéodory conditions in $i \times R^n$ and if $S$ is the set of all noncontinuable solutions of the system (3), then

(5) $$S \cap C(i, R^n) \neq \emptyset.$$

Let in the space $C(i, R^n)$ be a topology $\tau$ given and let the functional $F$: $C(i, R^n) \to \to R^n$ be continuous with respect to this topology.

Further we shall use the following definitions.

**Definition 1.** We shall say that the functional F is injective with respect to the system (3) if it is injective on the set $S \cap C(i, R^n)$.

The functional $F$ is surjective with respect to the system (3) if $F(S \cap C(i, R^n)) = R^n$.

**Definition 2.** The generalized boundary value problem (3), (4) is said to be $\tau$-correct if $F$ is injective and surjective with respect to the system (3) and the inverse mapping $(F|_{S \cap C(i, R^n)})^{-1}$ of the mapping $F|_{S \cap C(i, R^n)}$ is continuous as a mapping from $R^n$ to $C(i, R^n)$.

Denote $x(t, r)$ the solution of the problem (3), (4) (if it exists). Hence $F(x(t, r)) = r$ and the $\tau$-correctness of the problem (3), (4) means that $x(t, r)$ continuously depends on $r$ with respect to the topology $\tau$.

Let the functional $G$: $C(i, R^n) \to R^n$ be continuous with respect to the topology $\tau$.

**Definition 3.** The functional $F$ is said to be subordinate to the functional $G$ with respect to the differential system (3) if the following statement holds:

If the sequence $\{G(x_k)\}$ is bounded in $R^n$, then the sequence $\{F(x_k)\}$ is bounded, too, for each sequence $\{x_k\} \subset S \cap C(i, R^n)$.

**Definition 4.** The functional $G$ is said to have the same (the opposite) orientation as the functional $F$ with respect to the system (3) if the following implication holds: $G(x(t)) = kF(x(t))$ implies $k \geqq 0$ ($k \leqq 0$) for each solution $x(t) \in S \cap C(i, R^n)$ such that $F(x(t)) \neq 0$.

The relation to have the same orientation is reflexive and symmetric.

By means of the notions given above we can state

**Theorem 1.** *Let the boundary value problem* (3), (4) *be $\tau$-correct and let the functional $G : C(i, R^n) \to R^n$ be continuous with respect to the topology $\tau$.*

*Then the following statements are true:*

1. *If the functional $G$ is injective with respect to the system* (3), *then the boundary value problem* (3),

$$G(x) = r \tag{6}$$

*is $\tau$-correct if and only if the functional $F$ is subordinate to the functional $G$ with respect to the system* (3).

2. *If the functional $F$ is subordinate to the functional $G$ with respect to the system* (3) *and the functional $G$ has the same* (the opposite) *orientation as the functional $F$, then the functional $G$ is surjective with respect to the system* (3).

Proof. Define the mapping $H$: $R^n \to C(i, R^n)$ by the relation

$$H(r) = x(t, r) \qquad \text{for each } r \in R^n. \tag{7}$$

Since the boundary value problem (3), (4) is $\tau$-correct, the mapping $H : R^n \to \to C(i, R^n)$ is a homeomorphism of $R^n$ onto $S \cap C(i, R^n)$. Hence the mapping

$$g = GH \tag{8}$$

from $R^n$ into $R^n$ is continuous and if the functional $G$ is injective with respect to the system (3), then $g$ is injective, too. We apply Lemma 1. The condition (1) means that the inverse image of each bounded subset in $R^n$ under the mapping $g$ is bounded in $R^n$.

1. Suppose that the functional $G$ is injective with respect to the system (3) and that the functional $F$ is subordinated to the functional $G$ with respect to the system (3). Let $\{r_k\}$ be an arbitrary sequence of points in $R^n$ and $x_k = x(t, r_k)$ the corresponding sequence of solutions of the system (3) in the interval $i$, i.e. $F(x_k) = r_k$, $k = 1, 2, \ldots$ If the sequence $g(r_k) = G(x_k)$ is bounded, then the sequence $\{F(x_k)\} = \{r_k\}$ is bounded, too. But this means that the condition (1) is fulfilled and thus, by Lemma 1, $g$ is a homeomorphism of the space $R^n$ onto itself. Then $G = gH^{-1}$ is a homeomorphism of the space $S \cap C(i, R^n)$ onto $R^n$ and hence the problem (3), (6) is $\tau$-correct.

If, on the other hand, the problem (3), (6) is $\tau$ correct, then $G$ is a homeomorphic mapping of the space $S \cap C(i, R^n)$ onto $R^n$ and $g$, determined by (8), is a homeomorphism of $R^n$ onto itself. By Lemma 1 the condition (1) is satisfied. Let $\{G(x_k)\}$ be a bounded sequence. In view of the relation $G(x_k) = g(r_k)$ and (1) we get that the sequence $\{r_k\} = \{F(x_k)\}$ is also bounded. Hence the functional $F$ is subordinate to the functional $G$ with respect to (3).

2. If the functional $F$ is subordinate to the functional $G$ with respect to the system (3) and the sequence $\{g(r_k)\} = \{G(x_k)\}$ is bounded, then $\{F(x_k)\} = \{r_k\}$ is bounded, too which means that the condition (1) is fulfilled. The mapping $g$ satisfies the condition (2) with the point $x_0 = 0$ if the equality $G(x(t, r)) = kr = kF(x(t, r))$ implies $k \geqq 0$ for each $r \neq 0$, $r \in R^n$. But this means that the functionals $G$ and $F$ have the same orientation. Similarly the condition (2′) with $x_0 = 0$ is fulfilled if $G$ and $F$ have the opposite orientation.

In applications of Theorem 1 the initial value problem is often compared with the given boundary value problem. As the existence and the uniqueness of the solution to the initial value problem implies the $\tau_0$-correctness of this problem where the topology $\tau_0$ is the topology of uniform convergence (of locally uniform convergence) on $i$ when $i$ is a compact (a noncompact) interval we get the following

**Corollary 1.** (Compare with [2], p. 169). *Let there exist a point $t_0 \in i$ such that for each vector $x_0 \in R^n$ there exists a unique solution $x(t)$ on $i$ to the initial value problem* (3),

(9) $$x(t_0) = x_0$$

*and let the functional $G : C(i, R^n) \to R^n$ be continuous with respect to the topology $\tau_0$. Then the following statements hold:*

1. *If the boundary value problem* (3), (6) *has at most one solution for each vector $r \in R^n$, then this problem is $\tau_0$-correct if and only if the following implication holds:*

(10) *If $\{x_k\}$ is a sequence of solutions of* (3) *on the interval $i$ such that $\{G(x_k)\}$ is bounded, then $\{x_k(t_0)\}$ is bounded.*

2. *If the implication* (10) *as well as the implication:*

(11) *If $G(x) = kx(t_0)$, then $k \geqq 0$ $(k \leqq 0)$ for each solution $x(t)$* of (3) *on $i$ such that $x(t_0) \neq 0$,*

*hold, then the boundary value problem* (3), (6) *has a solution for each $r \in R^n$.*

In the paper [3] two boundary value problems have been compared.

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