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A SIMPLE PROOF OF A SEMI-FREDHOLM PRINCIPLE FOR PERIODICALLY FORCED SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. We prove that a generalized version of a semi-Fredholm principle for the existence of periodic solutions for forced systems with homogeneous nonlinearities recently obtained by Lazer and McKenna can be proved by a simple homotopy argument, which answers a question raised by those authors.

Key words. Periodic solution, periodically forced system, semi-Fredholm principle.

MS Classification. 34 C 25.

1. INTRODUCTION

In a recent paper, Lazer and McKenna [1] have proved the existence of T-periodic solutions for systems of the form

$$u''(t) + V'(u(t)) = p(t), \tag{1}$$

when $V \in C^1(\mathbf{R}^n, \mathbf{R})$ is positively homogeneous of degree two, positive semidefinite and $p \in C^1(\mathbf{R}, \mathbf{R}^n)$ is T-periodic. They use Leray–Schauder degree theory together with two perturbations arguments through systems of the form

$$u''(t) + \varepsilon u'(t) + V'(u(t)) = p(t), \tag{2}$$

with $\varepsilon > 0$ and V positive definite and

$$u''(t) + \delta u(t) + V'(u(t)) = p(t), \tag{3}$$

with $\delta > 0$ and V positive semidefinite. They remark that it does not seem possible to prove the theorem more directly by connecting (1) rather (2) to a linear equation by a homotopy.

We show in this paper that it is indeed possible and, without further complication, we can deal with a more general system which may also depend nonlinearly of u'.

II. A SEMI-FREDHOLM PRINCIPLE FOR PERIODIC SOLUTIONS OF FORCED SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

Recall that a function $W : \mathbf{R}^n \to \mathbf{R}$ is said to be positive (resp. negative) semidefinite if $W(x) \geqq 0$ (resp. $W(x) \leqq 0$) for all $x \in \mathbf{R}^n$, and is said to be positively homogeneous of degree $k \geqq 0$ if $W(tx) = t^k W(x)$ for all $t \geqq 0$ and $x \in \mathbf{R}^n$. We shall call W semidefinite if it is either positive or negative semidefinite. Recall also that if $W \in C^1(\mathbf{R}^n, \mathbf{R})$ and positive homogeneous of degree $k \geqq 1$, then Euler's identity implies that

$$(x, W'(x)) = kW(x)$$

for all $x \in \mathbf{R}^n$. Of course, W' denotes the gradient of W and (x, y) the inner product of x and y in $\mathbf{R}^n$.

We may now state and prove in a direct way a semi-Fredholm principle in the sense of Lazer – McKenna for a larger class of systems.

Theorem 1. *If U and V are in $C^1(\mathbf{R}^n, \mathbf{R})$, positive homogeneous of degree two, semidefinite and such that the system*

$$u''(t) + U'(u'(t)) + V'(u(t)) = 0, \tag{4}$$

has no T-periodic solution other than 0, *then for each $p \in L^1(0, T; \mathbf{R}^n)$ the problem*

$$\begin{aligned} u''(t) + U'(u'(t)) + V'(u(t)) &= p(t), \\ u(0) - u(T) = u'(0) - u'(T) &= 0 \end{aligned} \tag{5}$$

has at least one solution.

Proof. Let $a = \pm 1$ and $b = \pm 1$ be such that aU and bV are positive semidefinite. Observe that the linear system

$$u''(t) + au'(t) + bu(t) = 0, \tag{6}$$

has no T-periodic solution other than 0, because if u is any T-periodic solution of (6), then, taking the inner product of (6) with $u'(t)$, integrating over $[0, T]$ and using the periodicity, we get

$$a \int_0^T |u'(t)|^2 \, dt = 0,$$

so that u is constant, and this constant must be zero as shown by integrating (6) over $[0, T]$. Consequently, it follows from one version of the Leray–Schauder's continuation theorem (see e.g. [2], Theorem IV.5) that (5) will have at least one solution if we can find $r > 0$ such that for each $\lambda \in [0, 1]$ and each possible solution u of the problem

$$\begin{aligned} u''(t) + (1 - \lambda)(au'(t) + bu(t)) + \lambda[U'(u'(t)) + V'(u(t))] &= \lambda p(t), \\ u(0) - u(T) = u'(0) - u'(T) &= 0, \end{aligned} \tag{7}$$

one has $\| u \|_1 < r$, where

$$\| u \|_1 = \max_{t \in [0,T]} | u(t) | + \max_{t \in [0,T]} | u'(t) |.$$

If it is not the case, we can find sequences (λ_k) in $[0, 1]$ and (u_k) in $C^1([0, T], \mathbf{R}^n)$ such that $\| u_k \|_1 > k$ and u_k is a solution of (7) with $\lambda = \lambda_k$ $(k \in \mathrm{N}^*)$. Letting $w_k = u_k/\| u_k \|_1$, so that $\| w_k \|_1 = 1$, for all $k \in \mathrm{N}$, and using the positive homogenity of degree one of U' and V', we get

$$\begin{aligned} w_k''(t) + (1 - \lambda_k)(aw_k'(t) + bw_k(t)) + \lambda_k[U'(w_k'(t)) + V'(u_k(t))] &= \\ = \lambda_k(p(t)/\| u_k \|_1), \\ w_k(0) - w_k(T) = w_k'(0) - w_k'(T) = 0, \end{aligned} \tag{8}$$

for all $k \in \mathrm{N}^*$, which immediately implies that the sequence $(\| w_k'' \|_{L^1})$ is bounded independently of k. Hence, the sequences (w_k) and (w_k') are equibounded and equiuniformly continuous on $[0, T]$, and Ascoli–Arzela's theorem implies the existence of subsequences (λ_{j_k}) of (λ_k), (w_{j_k}) of (w_k) and of $w \in C^1([0, T], \mathbf{R}^n)$ verifying

$$w(0) - w(T) = w'(0) - w'(T) = 0 \tag{9}$$

and such that $w_{j_k} \to w$ and $w_{j_k}' \to w'$ uniformly on $[0, T]$ and $\lambda_{j_k} \to \lambda^*$ for some $\lambda^* \in [0, 1]$. Therefore, if we take the integrated form, from 0 to t, of the differential system in (8) for $k = j_k$ and let $k \to \infty$, we see that

$$w'(t) - w'(0) + \int_0^t \{(1 - \lambda^*)(aw'(s) + bw(s)) + \lambda^*[U'(w'(s)) + V'(w(s))]\}\, ds = 0$$

for all $t \in [0, T]$, and hence w' is absolutely continuous on $[0, T]$ and satisfies the differential equation

$$w''(t) + (1 - \lambda^*)(aw'(t) + bw(t)) + \lambda^*[U'(w'(t)) + V'(w(t))] = 0. \tag{10}$$

If $\lambda^* = 1$, it follows from the assumption on (4) that $w = 0$, a contradiction with $\| w \|_1 = 1$. If $0 \leqq \lambda^* < 1$, then, taking the inner product of (10) with $w'(t)$, integrating over $[0, T]$ and using the conditions (9), we get

$$(1-\lambda^*)\,a\int_0^T |w'(t)|^2\,dt + \lambda^*\int_0^T (U'(w'(t)), w'(t))\,dt = 0,$$

i.e.

$$(1-\lambda^*)\int_0^T |w'(t)|^2\,dt + 2a\lambda^*\int_0^T U(w'(t))\,dt = 0,$$

which implies, by the positive semidefiniteness of aU that

$$\int_0^T |w'(t)|^2\,dt = 0,$$

and hence that w is constant on $[0, T]$, say $w(t) = \bar{w}$ for all $t \in [0, T]$. But then (10) implies, after an inner product with w,

$$(1-\lambda^*)\,b|\bar{w}|^2 + \lambda^*(V'(\bar{w}), \bar{w}) = 0,$$

i.e.

$$(1-\lambda^*)\,|\bar{w}|^2 + 2\lambda^* aV(\bar{w}) = 0,$$

so that $\bar{w} = 0$, as aV is positive semidefinite, and hence $w = 0$, a contradiction with $\|w\|_1 = 1$. Hence, the proof is complete.

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