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ON ITERATION GROUPS OF CERTAIN FUNCTIONS

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. This paper contains a characterization of iteration groups formed, up to conjugacy, by certain functions of the form

$$\operatorname{Arctan}\frac{a\tan x + b}{c\tan x + d}, \qquad |ad - bc| = 1,$$

under the condition that graphs of different elements of such a group do not intersect each other.

Key words. Iteration groups, Linear differential equations.

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I. INTRODUCTION

For description of global transformations of linear differential equations, it is important to characterize all groups of those transformations that keep a given equation unchanged, see [5] and [6]. This characterization requires the following result concerning iteration groups of certain functions.

II. NOTATION, DEFINITIONS AND SOME BASIC FACTS

In accordance with O. Borůvka [2], the fundamental groups $\mathscr{F}_1$ is defined as the group of all functions $f: \mathbf{R} \to \mathbf{R}$ given by the formula

$$f(t) = \operatorname{Arctan}\frac{a\tan t + b}{c\tan t + d},$$

$a, b, c, d \in \mathbf{R}$, $|ad - bc| = 1$, where Arctan denotes this branch of $\arctan x + k\pi$ that makes function f continuous on $\mathbf{R}$. Then the elements of the fundamental group $\mathscr{F}_1$ are real analytic bijections of $\mathbf{R}$ onto $\mathbf{R}$, they are increasing exactly when $ad - bc = 1$. The group operation "$\circ$" is the composition of functions; for brevity the symbol $\circ$ is sometimes omitted.

Consider the following groups, whose elements are some functions of the fundamental group $\mathscr{F}_1$, restricted to an open interval $I \subset \mathbf{R}$.

$\mathscr{F}_2$: $f\colon (0, \infty) \to (0, \infty)$,

$$f(t) = \operatorname{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad a \in (0, \infty),\ b \in \mathbf{R}.$$

$\mathscr{F}_{3m}$: for each positive integer m
$f\colon (0, m\pi) \to (0, m\pi)$,

$$f(t) = \operatorname{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad a \in (0, \infty),\ b \in \mathbf{R}.$$

$\mathscr{F}_{4m}$: for each positive integer m
$f\colon (0, m\pi - \pi/2) \to (0, m\pi - \pi/2)$,
$f(t) = \operatorname{Arctan}(a \tan t)$, $a \in (0, \infty)$.

Let the topology on $\mathscr{F}_1$ be the relative topology on

$$\{(a, b, c, d) \in \mathbf{R}^4;\ |ad - bc| = 1\},$$

where $\mathbf{R}^4$ is considered with the usual topology.

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two groups whose elements are (some) bijections of intervals I_1 and I_2 onto themselves, respectively. We say that the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are C^k-conjugate (with respect to φ) for some positive integer k if there is a C^k-diffeomorphism φ of interval I_1 onto interval I_2, i.e. $\varphi(I_1) = I_2$, $\varphi \in C^k(I_1)$, $d\varphi(x)/dx \neq 0$ on I_1,
such that

$$\mathscr{G}_2 = \varphi \circ \mathscr{G}_1 \circ \varphi^{-1} := \{\varphi \circ f \circ \varphi^{-1};\ f \in \mathscr{G}_1\}.$$

If $\mathscr{G}_1$ is a topological group the topology on $\mathscr{G}_2$ is induced by the conjugacy.

Let α be an element of a group. For any integer k define the element $\alpha^{[k]}$ as follows:

$\alpha^{[0]}$ is the unit element of the group,
$\alpha^{[k]} = \alpha^{[k-1]} \circ \alpha$ for positive k,
$\alpha^{[k]} = (\alpha^{-1})^{[-k]}$ for negative k,

α^{-1} being the inverse to α. Element $\alpha^{[k]}$ is called the k th iterate of α.

A group is said to be partially (linearly) ordered if the set of its elements is partially (linearly) ordered and, for each its elements α, β and γ, the relation $\alpha \leqq \beta$ implies both $\alpha \circ \gamma \leqq \beta \circ \gamma$ and $\gamma \circ \alpha \leqq \gamma \circ \beta$.

A partially ordered group is called archimedean if the following implication holds:

if $\alpha^{[n]} \leqq \beta$ is satisfied for some elements α and β and for all integers n, then α is the unit element of the group.

The following theorem is due to O. Hölder [3]: *There exists an order preserving isomorphism of any linearly ordered archimedean group into a subgroup of the additive group of real numbers* **R**.

For proof see also for example A. I. Kokorin and V. M. Kopytov [4].

A group is said to be a cyclic group if there exists an element α of it such that all elements are iterates of α. Element α of this property is called a generator of the cyclic group. If, in addition,

$$\alpha^{[m]} \neq \alpha^{[n]}$$

for generator α and different integers m and n, then the group is an infinite cyclic group.

Now, consider an open interval $I \subset \mathbf{R}$. Let $n \geqq 1$ be an integer and $\mathscr{G}$ denote a group of some C^n-diffeomorphisms of I into I. Moreover, suppose that graphs of different elements of $\mathscr{G}$ do not intersect each other (on I).

III. THEOREM

If $\mathscr{G}$ is C^n-conjugate to a closed subgroup of increasing elements of the group $\mathscr{F}_1$, or $\mathscr{F}_2$, or $\mathscr{F}_{3m}$, or $\mathscr{F}_{4m}$,

then either $\mathscr{G}$ is trivial,

or $\mathscr{G}$ is an infinite cyclic group with a generator $h_e \in C^n(I)$, $\mathrm{d}h_e(x)/\mathrm{d}x > 0$ and $h_e(x) \neq x$ on I,

or $\mathscr{G}$ is C^n-conjugate to the group of all translations $\{h_c;\ c \in \mathbf{R}\}$,

$$h_c\colon \mathbf{R} \to \mathbf{R}, \qquad h_c(x) = x + c.$$

Proof

Since different elements of the group $\mathscr{G}$ do not intersect each other on I, $\mathscr{G}$ can be linearly ordered in the following manner:

for $h_1, h_2 \in \mathscr{G}$ we write $h_1 \leqq h_2$,

if either $h_1(x_0) < h_2(x_0)$ for some (then any) number $x_0 \in I$, or $h_1 = h_2$.

Moreover, $\mathscr{G}$ is archimedean, because for $h \neq \mathrm{id}_I$ there holds $h(x) \neq x$ on I an the sequences

$$\{h^{[i]}(x_0)\}_{i=1}^{\infty} \qquad \text{and} \qquad \{h^{[i]}(x_0)\}_{i=-1}^{-\infty}$$

converge to both ends of interval I for any $x_0 \in I$. Due to the Hölder Theorem there exists an order preserving isomorphism of $\mathscr{G}$ onto a subgroup $\tilde{\mathscr{G}}$ of the additive group **R**.

If $\mathcal{G}$ is trivial then $\mathcal{G} = \{\mathrm{id}_I\}$ and $\tilde{\mathcal{G}} = \{0\}$.

Let $\mathcal{G}$ be not trivial and $\tilde{\mathcal{G}} = \{ie;\ i \in \mathbf{Z},\ 0 \neq e \in \mathbf{R}\}$ be an infinite cyclic group generated by a nonzero number e. Denote by h_e this element of group $\mathcal{G}$ that corresponds to the number e. Evidently $h_e \in C^n(I)$, $\mathrm{d}h_e(x)/\mathrm{d}x > 0$ and $h_e(x) \neq x$ on I. Moreover,

$$\mathcal{G} = \{h_e^{[i]};\ i \in \mathbf{Z}\},$$

h_e being a generator of the infinite cyclic group $\mathcal{G}$.

From now, let $\mathcal{G}$ be not trivial, neither it be an infinite cyclic group.

1. Consider first the case when $\mathcal{G}$ is C^n-conjugate to a closed subgroup of the fundamental group $\mathcal{F}_1$ with respect to a C^n-diffeomorphism φ of $\mathbf{R}$ onto I. Let $h \in \mathcal{G}$, $h \neq \mathrm{id}_I$. Then

$$\varphi^{-1}h\varphi(t) = \operatorname{Arctan}\frac{a_{11}\tan x + a_{12}}{a_{21}\tan x + a_{22}} \in \mathcal{F}_1$$

and $a_{11}a_{22} - a_{12}a_{21} = 1$ because $\mathrm{d}h(x)/\mathrm{d}x > 0$ on I.

Case 1a. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix},$$

$b \in \mathbf{R}$, for a non-singular 2 by 2 matrix $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. Without loss of generality, let $\det C = 1$. Denote by ψ one of the continuous functions, element of the group $\mathcal{F}_1$, given by the formula

$$\psi(t) = \operatorname{Arctan}\frac{c_{11}\tan t + c_{12}}{c_{21}\tan t + c_{22}}.$$

It can be verified that

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \operatorname{Arctan}(b^2 \tan t) \in \mathcal{F}_1.$$

Since $h(x) \neq x$ on I, we have

$$\psi^{-1}\varphi^{-1}h\varphi\psi(0) = k\pi$$

for some integer $k \neq 0$.

Case 1b. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix},$$

$\det C = 1$ and $\psi \in F_1$ be defined as in case 1a. Then

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \operatorname{Arctan}(\tan t \pm 1) \in \mathcal{F}_1,$$
$$\psi^{-1}\varphi^{-1}h\varphi\psi(\pi/2) = \pi/2 + k\pi$$

for some $k \in \mathbf{Z} \setminus \{0\}$, otherwize h intersects id_I.
Case 1c. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \cos\omega\pi & \sin\omega\pi \\ -\sin\omega\pi & \cos\omega\pi \end{pmatrix},$$

$\omega \in \mathbf{R} \setminus \mathbf{Z}$, $\det C = 1$ and ψ be defined as above. Then

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = t + \omega\pi \in \mathscr{F}_1.$$

Now, let h and g be two different elements of the group $\mathscr{G}$ that do not belong to the same infinite cyclic group. Denote

$$h_1 := \varphi^{-1}h\varphi \in \mathscr{F}_1 \qquad \text{and} \qquad g_1 := \varphi^{-1}g\varphi \in \mathscr{F}_1.$$

Suppose first that

$$\psi_1^{-1}h_1\psi_1(t) = \operatorname{Arctan}(b_1^2 \tan t), \qquad \text{case 1a for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \operatorname{Arctan}(b_2^2 \tan t), \qquad \text{case 1a for } g,$$

hold for suitable elements ψ_1 and ψ_2 of the fundamental group $\mathscr{F}_1$. Due to the initial values of $\psi_1^{-1}h_1\psi_1$ and $\psi_2^{-1}g_1\psi_2$ at 0, and with respect to the fact that the relation

$$\psi(t + n\pi) = \psi(t) + n\pi,$$

holds for every increasing element ψ of $\mathscr{F}_1$, there exist integers n_1 and n_2 such that either $h_1^{[n_1]}$ and $g_1^{[n_2]}$ coincide and then h and g belong to the some infinite cyclic group, or $h_1^{[n_1]}$ and $g_1^{[n_2]}$ intersect each other, the same being true for $h^{[n_1]}$ and $g^{[n_2]}$. However both cases were excluded from our considerations.

The same argument shows that neither the situation when

$$\psi_1^{-1}h_1\psi_1(t) = \operatorname{Arctan}(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \operatorname{Arctan}(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } g,$$

nor the case when

$$\psi_1^{-1}h_1\psi_1(t) = \operatorname{Arctan}(b^2 \tan t), \qquad \text{case 1a} \qquad \text{for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \operatorname{Arctan}(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } g,$$

can occur.

If one of the functions, say h, is of the form described in case 1c, i.e.

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega\pi, \qquad \omega \in \mathbf{R} \setminus \mathbf{Z},$$

then g cannot be of the form in case 1a

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(b^2 \tan t) \qquad \text{for } k \neq 1,$$

or of the form of the case 1b

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(\tan t + 1),$$

because then there again exist integers n_1 and n_2 such that $h^{[n_1]}$ and $g^{[n_2]}$ intersect each other.

Hence in this case 1 when the group $\mathscr{G}$ is C^n-conjugate to a closed subgroup of the whole fundamental group $\mathscr{F}$, it remains to consider only the situation when

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \qquad \omega_1 \in \mathbf{R} \setminus \mathbf{Z}$$

and either

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(\tan t),$$

or

$$\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \qquad \omega_2 \in \mathbf{R} \setminus \mathbf{Z}.$$

In the first of these cases

$$\psi_2^{-1}g_1\psi_2(t) = t + k_1\pi \qquad \text{for some } k_1 \in \mathbf{Z} \setminus \{0\}$$

due to the initial value of this function at 0. Since

$$\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1}\, \psi_2 g_1 \psi_2^{-1}(\psi_2\psi_1)\,(t)$$

and $\psi_2\psi_1$ is again an increasing element of the fundamental group $\mathscr{F}_1$, i.e. $\psi_2\psi_1(t + k\pi) = \psi_2\psi_1(t) + k\pi$, we have

$$\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1}\,(\psi_2\psi_1(t) + k\pi) = t + k\pi, \qquad k \in \mathbf{Z}.$$

Hence ω_1 is an irrational number, otherwise h_1 and g_1 belong to the same infinite cyclic group and the same is true for the functions h and g, that was already excluded. However, when ω_1 is irrational, then the union of graphs of functions $h_1^{[n_1]}$ and $g_1^{[n_2]}$ for all n_1 and n_2 from $\mathbf{Z}$ is a dense set in $\mathbf{R}^2$. Now we have

$$h = \psi_1\varphi(\text{id} + \omega_1\pi)\,\varphi^{-1}\psi_1^{-1} \qquad \text{and} \qquad g = \psi_1\varphi(\text{id} + k\pi)\,\varphi^{-1}\psi_1^{-1},$$

where $\psi_1\varphi$ is a C^n-diffeomorphism of $\mathbf{R}$ onto I. Since the group $\mathscr{G}$ is closed, we conclude that it is C^n-conjugate to the group of all translations

$$t \mapsto t + c, \qquad \text{for all } c \in \mathbf{R}.$$

Now, let

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \qquad \omega_1 \in \mathbf{R} \setminus \mathbf{Z}, \qquad \text{case 1c for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \qquad \omega_2 \in \mathbf{R} \setminus \mathbf{Z}, \qquad \text{case 1c for } g.$$

Then

$$h_1^{[n_1]}(t) = \psi_1(\psi_1^{-1}(t) + n_1\omega_1\pi),$$
$$g_1^{[n_2]}(t) = \psi_2(\psi_2^{-1}(t) + n_2\omega_2\pi)$$

and the condition $h_1^{[n_1]}(t) \neq g_1^{[n_2]}(t)$ on $\mathbf{R}$ implies

$$\psi_3(t + n_1\omega_1\pi) \neq \psi_3(t) + n_2\omega_2\pi$$

for $\psi_3 := \psi_2^{-1}\psi_1 \in \mathscr{F}$, otherwise $h_1^{[n_1]}$ coincides with $g_1^{[n_2]}$ that shows that h_1 and g_1 belong to the same infinite cyclic group, the case already excluded from our considerations. Since

$$\psi_3(t + \pi) = \psi_3(t) + \pi,$$

we have

$$\psi_3(t) = t + p(t),$$

where p is a π-periodic function: $p(t + \pi) = p(t) \in C^3(\mathbf{R})$. Hence

$$t + n_1\omega_1\pi + p(t + n_1\omega_1\pi) \neq t + p(t) + n_2\omega_2\pi,$$

or

$$p(t + n_1\omega_1\pi) - p(t) \neq (n_2\omega_2 - n_1\omega_1),$$

for all $t \in \mathbf{R}$ and all $n_1, n_2 \in \mathbf{Z}$, $n_1^2 + n_2^2 \neq 0$.

If $n_2\omega_2 - n_1\omega_1 = 0$ for some n_1 and n_2 then either

$$p(t + n_1\omega_1\pi) > p(t) \qquad \text{on } \mathbf{R},$$

or

$$p(t + n_1\omega_1\pi) < p(t) \qquad \text{on } \mathbf{R}.$$

Neither of these cases is possible for any continuous periodic function p.

Hence $n_2\omega_2 - n_1\omega_1 \neq 0$ for all integers n_1 and n_2, $n_1^2 + n_2^2 \neq 0$, that means that ω_1 and ω_2 are rationally independent. Then for each number $t_0 \in \mathbf{R}$ the set

$$\{g_1^{[n_2]} \circ h_1^{[n_1]}(t_0);\ n_1, n_2 \in \mathbf{Z}\}$$

is dense in $\mathbf{R}$, because for different couples (n_1, n_2) and (n_1^*, n_2^*) the values, $g_1^{[n_2]} \circ h_1^{[n_1]}(t_0)$ and $g_1^{[n_2^*]} \circ h_1^{[n_1^*]}(t_0)$ are different, there are infinite number of coumples (n_1, n_2) satisfying $|n_1\omega_1 + n_2\omega_2| < \varepsilon$ for any given $\varepsilon > 0$ and, moreover, ψ_1 and ψ_2 are C^n-diffeomorphisms of $\mathbf{R}$ onto $\mathbf{R}$ for any $n \in \mathrm{N}$ satisfying

$$\psi_1(t) = t + p_1(t), \qquad \psi_2(t) = t + p_2(t),$$

with π-periodic functions p_1 and p_2.

Since φ is a C^n-diffeomorphism of $\mathbf{R}$ onto I, and the group $\mathscr{G}$ is archimedean and closed, the union of graphs of all its elements is the whole square I^2. In such a situation we may apply Theorem 1 of G. Blanton and J. A. Baker [1] which states: "Each group whose elements are C^n-diffeomorphisms of an interval I onto I and such that to each point $(x_0, y_0) \in I \times I$ there exists just one element h of the group satisfying $h(x_0) = y_0$, is formed by functions

$$\chi(\chi^{-1}(x) + c),$$

where χ is a C^n-diffeomorphism of $\mathbf{R}$ onto I and c ranges through the real numbers". In our case we may write

$$G = \chi \circ h_c \circ \chi^{-1},$$

where $h_c\colon \mathbf{R} \to \mathbf{Z}$, $h_c(t) = t + c$, $c \in \mathbf{R}$.

2. Now, suppose that

$$\varphi^{-1}h\varphi(t) = \operatorname{Arctan}\frac{a \tan t}{b \tan t + 1/a}, \qquad t \in R_+,$$

$a \in \mathbf{R}_+$, $b \in \mathbf{R}$, is an element of the two-parametric group $\mathscr{F}_2$ of increasing functions. Since $\lim_{t \to 0_+} \varphi^{-1}h\varphi(t) = 0$, we have

$$\varphi^{-1}h\varphi(\pi) = \pi,$$

hence $\varphi^{-1}h\varphi = \mathrm{id}_{\mathbf{R}_+}$ that is excluded from our considerations.

3*m*. If

$$\varphi^{-1}h\varphi(t) = \operatorname{Arctan}\frac{a \tan t}{b \tan t + 1/a}, \qquad \varphi^{-1}h\varphi;\, (0, m\pi) \to (0, m\pi),$$

$$a \in \mathbf{R}_+,\ b \in \mathbf{R}, \qquad \text{then} \qquad \lim_{t \to 0_+} \varphi^{-1}h\varphi(t) = 0 \qquad \text{and} \qquad \lim_{t \to \pi_-} \varphi^{-1}h\varphi(t) = \pi,$$

because h as well as $\varphi^{-1}h\varphi$ are increasing functions. Hence $m = 1$, otherwise $h = \mathrm{id}_I$ that contradicts to our assumptions. However, if $a \neq 1$ and $b \neq 0$ then the equation

$$\arctan\frac{a \tan t}{b \tan t + 1/a} = t,$$

i.e.

$$a \tan t = (b \tan t + 1/a) \tan t$$

is satisfied for $t_1 \in (0, \pi)$ where

$$\tan t_1 = \frac{a^2 - 1}{ab}.$$

This case is excluded from our considerations. Even the case $b = 0$ impossible since then

$$\varphi^{-1}h\varphi(t) = \arctan(a^2 \tan t)$$

intersects $\mathrm{id}_{(0,\pi)}$ at $\pi/2$.

If $a = 1$ then

$$\varphi^{-1}h\varphi(t) = \arctan\frac{\tan t}{b\tan t + 1}$$
$$= \operatorname{arccot}\frac{1 + b\tan t}{\tan t}$$
$$= \operatorname{arccot}(\cot t + b), \qquad t \in (0, \pi),$$

hence h is conjugate to $x \mapsto x + b$, $x \in \mathbf{R}$ for a fixed $b \in \mathbf{R}$ by means of the function $\varphi \circ \operatorname{arccot} : \mathbf{R} \to I$.

Now, let h and g be two different elements of the stationary group $\mathcal{G}$ that do not belong to the same infinite cyclic group. Then

$$\psi^{-1}h\psi(x) + x + b_1 \qquad \text{and} \qquad \psi^{-1}g\psi(x) = x + b_2$$

on $\mathbf{R}$ where $\psi = \varphi \circ \operatorname{arccot} \in C^n(\mathbf{R})$, and b_1/b_2 is irrational. Since the union of the graphs of functions

$$x \mapsto x + n_1 b_1 + n_2 b_2 \qquad \text{for all } n_1, n_2 \in \mathbf{Z}$$

is dense in $\mathbf{R}^2$, and the group $\mathcal{G}$ is closed, it is C^n-conjugate to the group of all translations:

$$\{x \mapsto x + c,\ c \in \mathbf{R}\}.$$

4*m*. Finally, if

$$\varphi^{-1}h\varphi(t) = \operatorname{Arctan}(a\tan t), \qquad a > 0,$$
$$\varphi^{-1}h\varphi\colon (0, m\pi - \pi/2) \to (0, m\pi - \pi(2),$$

$$\text{then} \qquad \lim_{t\to 0+}\varphi^{-1}h\varphi(t) = 0 \qquad \text{and} \qquad \lim_{t\to \pi/2-}\varphi^{-1}h\varphi(t) = \pi/2,$$

and hence $m = 1$. In this case h is conjugate to the function $x \to x + \ln a$, $x \in \mathbf{R}$ by means of the C^n-diffeomorphism $\varphi \circ \arctan \circ \exp\colon \mathbf{R} \to I$.

Now, analogously to case 3*m*, if h and g are two different elements of $\mathcal{G}$ that do not belong to the same infinite cyclic group, they are C^n-conjugate to $x + b_1$ and $x + b_2$, respectively, with respect to the some C^n-diffeomorphism, the quotient b_1/b_2 being irrational. Hence the group $\mathcal{G}$ is C^n-conjugate to the group

$$\{x \mapsto x + c;\ c \in \mathbf{R}\},$$

that finishes the proof of the theorem.

IV. REMARK

The present paper gives technical details of the proof of Theorem 6.3.5 in the monograph [6], where main steps of the proof were outlined.

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