Herbert Fleischner; Emanuel Wenger

D_0-favouring Eulerian trails in digraphs

Archivum Mathematicum, Vol. 25 (1989), No. 1-2, 55--60

Persistent URL: http://dml.cz/dmlcz/107339

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 1–2 (1989), 55–60

D_0-FAVOURING EULERIAN TRAILS IN DIGRAPHS

HERBERT FLEISCHNER, EMANUEL WENGER

(Received April 28, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. A characterization for a special class of Eulerian trails in digraphs which traverse a set of arcs of a subdigraph D_0 before any arc of $D_1 = D - D_0$ is traversed, is proved. The most general structure of a subdigraph D_1 to allow such a restricted Eulerian trail is given.

Key words. Directed graph, Eulerian trail, restricted Eulerian trail, spanning in-tree.

MS Classification. 05 C 139

PRELIMINARIES

For notation and terminology, see [2, 4]. Let D be a digraph with vertex set $V(D)$ and $A(D)$. In particular, $V(D)$ and $A(D)$ are always assumed to be finite, $A_v^+ \subset A(D)$ denotes the set of arcs, incident from v, for $v \in V(D)$. For a digraph D and a subdigraph D_1 let $D - D_1 \subseteq D - A(D_1)$ denote the uniquely determined digraph without isolated vertices. The following lemma is folklore.

Lemma 1. *Let D be a digraph and $\text{od}_D(v) \geqq 1$ for all $v \in V(D)$. Then there exists at least one non-trivial strongly connected component C with no arc of D incident from C (that is, $(a, b) \in A(D)$ implies either $a \notin V(C)$, or $b \notin V(D) - V(C)$).*

Lemma 2. *Let D be a digraph satisfying $\text{od}_D(v) \geqq 1$ for all $v \in V(D)$. Suppose D has precisely one (nontrivial) strongly connected component C with no arc of D incident from C. Then there exists a spanning in-tree with root v_0, where v_0 is an arbitrary vertex of C.*

Proof. Let v_0 be an arbitrary vertex of C, and let B_0 be an in-tree with root v_0 containing a maximum number of vertices. If $V(B_0) \neq V(D)$ then we consider $D_0 = \langle V(B_0) \rangle$, the digraph induced by $V(B_0)$. Because of the maximality of B_0 there does not exist an arc (x, y) with $x \in V(D) - V(D_0)$ and $y \in V(D_0)$; furthermore, one easily concludes that $C \subseteq D_0$. $D_1 = D - V(D_0)$ fulfills the assumptions of Lemma 1. Because of Lemma 1 there exists a strongly connected component $C' \subset D_1$ such that no arc of D_1 is incident from C'. By construction it follows that $C' \cap C = \emptyset$ which contradicts the uniqueness of C.

Definition. Let D be a weakly connected eulerian digraph, and let D_0 be a subdigraph of D. An eulerian trail T of D is called *D_0-favouring* if and only if for every $v \in V(D)$, T traverses every arc of D_0 incident from v before it traverses any arc of $D_1 = D - D_0$ incident from v.

Of course, every eulerian trail of D is a D_0-favouring eulerian trail for some D_0 (just take $D_0 = D$). For which subdigraph D_0 of exists a D_0-favouring eulerian trail? There are two known results on the existence of D_0-favouring eulerian trails depending on the structure of $D_1 = D - D_0$.

Theorem 1. *Let D be a weakly connected eulerian digraph, and for given $v \in V(D)$ let $D_0 \subset D$ be chosen such that $D_1 = D - D_0$ is a spanning in-tree of D with root v. Then there exists a D_0-favouring eulerian trail starting (and ending) at v. Conversely, if T is an eulerian trail of D starting (and ending) at v, and if we mark at every $w \in V(D)$, $w \neq v$, the last arc of T incident from w, then D_1, the subgraph of D induced by the marked arcs, is a spanning in-tree with root v (and hence T is a $(D - D_1)$-favouring eulerian trail of D).*

Theorem 1 plays an essential role in establishing the BEST-Theorem which gives a formula for the number of eulerian trails in an eulerian digraph. A proof of Theorem 1 can be found in [1].

Theorem 2. *Let D be an eulerian digraph. Let $D_1 \subseteq D$ be chosen such that $\mathrm{od}_{D_1}(v) \geqq 1$ for every $v \in V(D_1) \subset V(D)$, and let $D_0 = D - D_1$. D has a D_0-favouring eulerian trail if and only if D_1 has precisely one (nontrivial) strongly connected component C_1 with the property that no arc of D_1 is incident from C_1. Moreover, every D_0-favouring eulerian trail of D must start at some vertex of C_1, and for any vertex of C_1 there is a D_0-favouring eulerian trail of D starting at that vertex.*

Theorem 2 was proved by Berkowitz [3].

A GENERAL THEOREM

In view of Theorems 1 and 2, we ask the following question: What is the most general structure a subdigraph D_1 of an eulerian digraph D can have in order to imply the existence of a $(D - D_1)$-favouring eulerian trail T?

Theorem 2 implies that D_1 must not contain more than one nontrivial strongly connected component C_1 with the property that no arc of D_1 is incident from C_1. But this condition is not sufficient even if D_1 is weakly connected; this can be seen from the digraph D^* of Figure 1.

What if we go the other way round? That is, given an eulerian digraph D and $D_1 \subseteq D$, can we find $D_1^+ \subseteq D$ with $D_1 \subseteq D_1^+$ such that D has a $(D - D_1^+)$-favouring eulerian trail T^+ which induces a $(D - D_1)$-favouring eulerian trail T?

This approach and Theorem 1 and Theorem 2 lead to the following theorem which answers our original question.

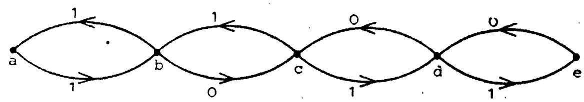

Figure 1

Figure 1. An eulerian digraph D^* having no D_0-favouring eulerian trail (the arcs of D_i are marked with i, $i = 0, 1$).

Theorem 3. *Let D be an eulerian digraph, and let D_1 be a subdigraph of D. Any two of the following statements are equivalent:*

1. *D has a $(D - D_1)$-favouring eulerian trail.*

2. *There exists a digraph D_1^+ with $D_1 \subseteq D_1^+ \subseteq D$ such that for every $v \in V(D)$*
a) $\mathrm{od}_{D_1^+}(v) = \mathrm{od}_{D_1}(v)$ *if and only if* $\mathrm{od}_{D_1}(v) \neq 0$;
b) $\mathrm{od}_{D_1^+}(v) = 1$ *otherwise.*
c) *D_1^+ has precisely one non-trivial strongly connected component C_1 with no arc of D_1^+ incident from C_1.*

3. *There exists a digraph D_1^+ with $D_1 \subseteq D_1^+ \subseteq D$ such that*
a) *D has a $(D - D_1^+)$-favouring eulerian trail;*
b) *for every D_1' with $D_1 \subseteq D_1' \subseteq D_1^+$, if $(x, y) \in A(D_1' - D_1)$, then* $\mathrm{od}_{D_1}(x) = 0$.

4. *D_1 contains a spanning in-forest D_1^- such that*
a) *for some v_0 and for every $x \in V(D_1) - v_0$,* $\mathrm{od}_{D_1^-}(x) = 0$ *if and only if* $\mathrm{od}_{D_1}(x) = 0$, *and* $\mathrm{od}_{D_1^-}(v_0) = 0$;
b) *D has an in-tree B with root v_0 and $D_1^- \subseteq B$.*

Proof. *1. implies 2.* Let T be a $(D - D_1)$-favouring eulerian trail starting at v_0. Define D_1^+ by $D_1^+ = D_1$ if $\mathrm{od}_{D_1}(v) \geqq 1$ for every $v \in V(D)$; otherwise, for every v with $\mathrm{od}_{D_1}(v) = 0$, mark the last arc of T which is incident from v, and let D_1^+ consist of D_1 plus the marked arcs. In any case, $D_1 \subseteq D_1^+$ and D_1^+ satisfies **2. a)**, **2. b)**. Moreover, T is a $(D - D_1^+)$-favouring eulerian trail because of the choice of the elements of $A(D_1^+) - A(D_1)$. It remains to show that D_1^+ has precisely one non-trivial strongly connected component C_1 with no arc of D_1^+ incident from C_1. Because of $\mathrm{od}_{D_1^+}(v) \geqq 1$ for every $v \in V(D_1^+)$ and the finiteness of D_1^+, D_1^+ has at least one non-trivial strongly connected component and, in particular, by Lemma 1 at least one non-trivial strongly connected component C_1^+ with no arc of D_1^+ incident from C_1^+.

T must start and end in a vertex of C_1^+. Otherwise, there exist one or more arcs (v, w) of D such that $v \in V(C_1^+)$ and $w \notin V(C_1^+)$; among these arcs let (v_1, w_1) be the last arc in T, such that $v_1, (v_1, w_1), w_1$ is a section of T. By definition of C_1^+, $(v_1, w_1) \notin A(D_1^+)$, and because of $\mathrm{od}_{D_1^+}(v_1) \geqq 1$ we get a contradiction to the fact that T is a $(D - D_1^+)$-favouring eulerian trail. It's clear now that there can be only one component C_1^+ with the desired property. The implication now follows.

2. implies 3. Take D_1^+ and C_1 as defined by **2 a). b)**, and **c)**. At first it will be proved that D has a $(D - D_1^+)$-favouring eulerian trail.

Properties **2. a), b)**, imply that D_1^+ is a spanning subdigraph of D. Therefore and because of Lemma 1, and property **2. c)** there exists in D a spanning in-tree $B_1^+ \subset D_1$ with root $v_0 \in V(C_1)$ (sce Lemma 2).

Mark all the arcs of B^+. Construct T by starting at vertex v_0 with any arc (v_0, x), choose any unmarked arc incident from x, if such arc exists; otherwise, choose among the marked arcs one which does not belong to B_1^+ if such arc exists; otherwise, choose the arc of B_1^+. Continue this way until this procedure terminates at some $y \in V(D)$. Then $y = v_0$; otherwise, T contains more arcs incident to y than it contains arcs incident from y contradicting D being eulerian. Suppose T does not contain all arcs of D. Then let z be a vertex incident with arcs not contained in T. Since D is eulerian and T is a closed trail, $\mathrm{id}_{D-T}(z) = \mathrm{od}_{D-T}(z) \neq 0$. Moreover, $z \neq v_0$ by the very construction of T. By definition of B_1^+, there is a path $P(z, v_0) \subset \subset B_1^+$ joining z to v_0. Write

$$P(z, v_0) = z, (z, u_1), u_1, \ldots, u_k, (u_k, v_0), v_0;$$

possibly $z = u_k$ and $u_1 = v_0$ (i.e. $P(z, v_0)$ may contain just one arc). By the construction of T it follows that (z, u_1) is not contained in T; therefore, also (u_1, u_2) is not contained in T (note that (u_1, u_2) can be contained in T only if all arcs incident to u_1 are contained in T); a.s.o. In particular, (u_k, v_0) is not contained in T, contradicting the fact that $\mathrm{id}_T(v_0) = \mathrm{od}_T(v_0) = \mathrm{id}_D(v_0) = \mathrm{od}_D(v_0)$. Thus, T contains all arcs of D. This and the construction of T imply that T is a $(D - D_1^+)$-favouring eulerian trail of D.

Now consider any D_1' with $D_1 \subseteq D_1' \subseteq D_1^+$ and suppose $A(D_1' - D_1) \neq \emptyset$; let $(x, y) \in A(D_1' - D_1)$. By definition of D_1^+ in **2. a), b)**, an arc of $D_1^+ - D_1$ is necessarily incident from a vertex z with $\mathrm{od}_{D_1}(z) = 0$. Hence $(x, y) \in A(D_1' - D_1)$ implies $\mathrm{od}_{D_1}(x) = 0$; thus **3. b)** holds as well.

3. implies 4. Start with D_1^+ as described in 3., and consider a $(D - D_1^+)$-favouring eulerian trail T^+ of D. If there is $w \in V(D)$ different from the initial vertex v_0 of T^+ such that the last arc of T^+ incident from w is not in D_1^+, then mark this arc. Note that in this case none of the arcs incident from w lies in D_1^+.

We define

$$D_1^{++} = D_1^+ \qquad \text{if no such } w \text{ exists;}$$

otherwise,

$$D_1^{++} = \langle A(D_1^+) \cup \{a \in A_w^+ / \mathrm{od}_{D_1^+}(w) = 0 \text{ and } a \text{ has been marked}\}\rangle.$$

In any case, by definition of D_1^{++}, T^+ is even a $(D - D_1^{++})$-favouring eulerian trail of D, and D_1^{++} satisfies **3. b)** as well. Moreover, $V(D_1^{++}) = V(D)$.

Marking for every $v \neq v_0$ the last arc of T^+ incident from v yields a spanning subdigraph B and $B \subset D_1^{++}$ follows from the very definition of D_1^{++}. Furthermore, $\mathrm{od}_B(v) = 1$ for all $v \neq v_0$ and $\mathrm{od}_B(v_0) = 0$. Suppose B is not connected; then there exists a weakly connected component B_1 of B which does not contain v_0 and $\mathrm{od}_{B_1}(w) = \mathrm{od}_B(w) = 1$ for all $w \in V(B_1)$. By Lemma 1 there exists at least one nontrivial strongly connected component $C_1 \subseteq B_1$ with no arc of B_1 incident from C_1. Now, if r is the last vertex of T in C_1, such that r, (r, s), s is a section of T, then it follows from the construction of B that $(r, s) \in A(B)$; furthermore $s \in V(C_1)$ because of the definition of C_1. By the choice of r, T terminates in C_1 contradicting the fact, that T is an eulerian trail starting in $v_0 \notin V(B_1) \supset V(C_1)$. Thus B is connected, and $\mathrm{od}_B(v) = 1$ for all $v \neq v_0$, $\mathrm{od}(v_0) = 0$. This implies that B is a spanning in-tree of $D_1^{++} \subseteq D$ rooted at v_0.

Define D_1^- by $V(D_1^-) = V(D_1)$ and $A(D_1^-) = A(B) \cap A(D_1)$; thus D_1^- is a spanning in-forest of D_1 which satisfies *4. b)*. Let (x, y) be any arc of B not in D_1^-; then $x \neq v_0$. If $(x, y) \notin A(D_1^+)$, then it follows from the definition of D_1^{++} and $D_1^{++} \supset D_1$ that $\mathrm{od}_{D_1^+}(x) = 0 = \mathrm{od}_{D_1}(x)$. If $(x, y) \in A(D_1^+)$, then $(x, y) \notin A(D_1)$ by definition of D_1^-; and by **3. b)** with $D_1' = D_1^+$, $\mathrm{od}_{D_1}(x) = 0$ follows.

We summarize: D_1^- is a spanning in-forest of D_1, and if $x \neq v_0$ for some $v_0 \in V(D_1^-)$ (which is the root of B indeed) satisfies $\mathrm{od}_{D_1^-}(x) = 0$ then $\mathrm{od}_{D_1}(x) = 0$ (for, x not being the root of B implies $(x, y) \in A(B - D_1^-)$ for some y). Since $\mathrm{od}_{D_1}(x) = 0$ implies $\mathrm{od}_{D_1^-}(x) = 0$ anyway and $\mathrm{od}_{D_1^-}(v_0) = \mathrm{od}_B(v_0) = 0$, and because $D_1^- \subseteq B$ with $V(B) = V(D)$, the proof of the implication is finished.

4. implies 1. Let $D_1^- \subseteq D_1$ be chosen as described in **4. a)** and let B be a spanning in-forest of D with root v_0 and $D_1^- \subseteq B$. Marking all arcs of B we construct a trail T by starting at vertex v_0 with any arc (v_0, x). Choose any unmarked arc incident from x, if such arc exists; choose the marked arc incident from x, otherwise.

Continuing this way until this procedure terminates we get a $(D - B)$-favouring eulerian trail (for arguments see **2.** implies **3.**).

Because of the freedom to choose the order in which the arcs of $A_v^+ - A(B)$ appear in T for every $v \in V(D)$ we are even able to construct T in such a way that the arcs of $A_v^+ \cap (D - D_1)$ appear in T before any of the arcs of $A_v^+ \cap D_1$ are used. This is true even in the case where an arc $(x, y) \in B$ does not belong to D_1; for, in this case $\mathrm{od}_{D_1^-}(x) = \mathrm{od}_{D_1}(x) = 0$ by **4. a)**, i.e. $A_x^+ \cap A(D_1) = \emptyset$, i.e., $A_x^+ \subseteq D - D_1$. In the case of v_0, if $A_{v_0}^+ \cap A(D_1) \neq \emptyset$, than we proceed in the construction of T by starting along an arc of $A_{v_0}^+ \cap A(D - D_1)$, and each time we arrive in v_0 we continue along an arc of $A_{v_0}^+ \cap A(D - D_1)$ not traversed before, as long as there is such an arc. Consequently, T is a $(D - D_1)$-favouring eulerian trail of D. This finishes the proof of the implication. Theorem 3 now follows.

It is easy to see that Theorem 3 is a generalization of Theorem 1 and Theorem 2. Both Theorems can be derived by using the equivalent statements of Theorem 3 and some details of their proof. We also note that in proving Theorem 3 we used ideas developed originally for the proofs of Theorem 1 and Theorem 2.

REFERENCES

[1] T. van Aardenne – Ehrenfest, N. G. de Bruijn, *Circuits and Trees in Oriented Linear Graphs*, Simon Stevin 28 (1951) 203–217.

[2] L. W. Beineke, R. J. Wilson, *Selected Topics in Graph Theory 2,* Academic Press, London, 1983.

[3] H. W. Berkowitz, *Restricted Eulerian Circuits in Directed Graphs,* Coll. Math., Vol. 39, Fasc. 1 (1978), 185–188.

[4] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications,* American Elsevier, New Jork, and Mac Millan, London, 1976.

Prof. Herbert Fleischner, Emanuel Wenger
Austrian Academy of Sciences
Institute for Information Processing
Sonnenfelsgasse 19/2
1010-Vienna, Austria