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ASPECTS OF THE INVERSE PROBLEM TO THE CALCULUS OF VARIATIONS*), **)

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Abstract. The inverse problem to the calculus of variations is that of determining when a given system of differential equations are derivable from a variational principle. Local, global and equivariant aspects of this problem are discussed.

Key words. Euler–Lagrange equations, Helmholtz equations, Cartan forms, Chern–Simons invariants, Noether's Theorem.

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INTRODUCTION

The inverse problem to the calculus of variations is that of determining when a given system of differential equations may be derived from a variational principle. This is an old problem in mathematical physics which dates back to Helmholtz and whose history has been recently recounted by Tonti [37]. At the turn of the century the following facts concerning the inverse problem were known:

(i) The Lagrangian for the given system of equations is not unique In fact, since Euler – Lagrange operator E annihilates divergences of vector fields, the Lagrangian can always be modified by adding a divergence.[1])

(ii) There are certain necessary integrability conditions, henceforth referred to as the Helmholtz conditions, which a system of equations must satisfy in order to be derivable from a variational principle.

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[1]) It should be emphasized that such divergences are nevertheless important in the calculus of variations. They affect the natural boundary conditions, transversality, and the Legendre transformation.

These two observations can be summarized symbolically by the following sequence of spaces and maps:

$$\{\text{vector fields}\} \xrightarrow{\text{div}} \{\text{Lagrangians}\} \xrightarrow[\text{Lagrange}]{\text{Euler}-} \{\text{differential equations}\} \xrightarrow{\text{Helmholtz}} .$$

This sequence is a cochain complex in that the composition of successive maps yields zero.

It was not, however, until just a decade ago that Tulczyjew [39] and Vinogradov [40], [41] realized that this informal sequence was part of a much larger, formal, mathematical structure called the variational bicomplex. This double complex of differential forms on infinite jet bundles plays an extremely important role in the inverse problem to the calculus of variations. Moreover, as Tsujishita [38] observed, this bicomplex is also useful in a wide variety of other problems in differential geometry and mathematical physics. These include characteristic classes and Gilkey's classification of the Pontrjagin and Euler classes, Gelfand – Fuks cohomology, secondary characteristic classes for foliations, and conservation laws for differential equations. Vinogradov [18] has also developed a general theory of characteristics for partial differential equations using the bicomplex. Indeed, the variational bicomplex plays the same role for problems on jet bundles as the de Rham complex plays for problems on manifolds.

The purpose of this paper is to draw attention to the great diversity of current research problems associated with the variational bicomplex. This will be done solely within the context of the inverse problem even though many of the results discussed here are best understood in the larger context of the variational bicomplex. The topics to be presented include: the local theory; variational integrating factors; the global theory; the equivariant inverse problem; and the relationship of the inverse problem to Noether's theorem.

THE LOCAL THEORY

Let $E \xrightarrow{\pi} M$ be a fibered manifold over an n-dimensional manifold M. Local adapted coordinates on E are $(x^i, u^\alpha) \xrightarrow{\pi} (x^i)$ where here, and in the sequel, Latin indices range from 1 to n and Greek indices from 1 to m. Induced coordinates on $J^\infty(E)$, the infinite jet bundle of E are

$$(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots, u^\alpha_{ij\ldots k}, \ldots),$$

where, for $q = j^\infty(s)$ and s a local section of E,

$$u^\alpha_I(q) = u^\alpha_{ij\ldots k}(q) = \frac{\partial^r s^\alpha(x)}{\partial x^i \partial x^j \ldots \partial x^k}, \qquad \text{where } |I| = r.$$

Vector fields on E can be lifted or prolonged to vector fields on $J^\infty(E)$. In particular, if Y is a vertical vector field on E given locally by

$$Y = Y^\alpha \frac{\partial}{\partial u^\alpha}$$

then its prolongation pr Y is the vector field on $J^\infty(E)$ given locally by

$$\mathrm{pr}\, Y = \sum_{|I|=0}^{\infty} D_I Y^\alpha \frac{\partial}{\partial u_I^\alpha}\,.$$

In this equation $D_{ij\ldots k} = D_i D_j \ldots D_k$, where D_i denotes total differentiation with respect to x^i.

Let $\mathscr{L}$ be the vector bundle of horizontal n forms on $J^\infty(E) \xrightarrow{\pi} M$. Locally, a section λ of $\mathscr{L}$ is of the form

$$\lambda = L[u]\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \mathrm{d}x^n,$$

where $L[u]$ indicates that L is a function on $J^\infty(E)$, i.e.

$$L[u] = L(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \ldots, u_{ij\ldots k}^\alpha).$$

We call $\mathscr{L}$ the vector bundle of Lagrangians on E and denote the space of global sections of $\mathscr{L}$ by $\mathscr{L}(E)$. Similarly, let $\mathscr{D}$ be the induced vector bundle of $n + 1$ forms on E over $J^\infty(E) \xrightarrow{\pi} E$ which are horizontal of degree n. A local section Δ of $\mathscr{D}$ is of the form

$$\Delta = F_\alpha[u]\, \mathrm{d}u^\alpha \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \mathrm{d}x^n.$$

We call $\mathscr{D}$ the vector bundle of differential operators on E. A section s of E is a solution to the differential equation $\Delta \in \mathscr{D}(E)$ if

$$[j^\infty(s)]^*(Y \lrcorner \Delta) = 0$$

for all vertical vector fields Y. This simply implies that locally

$$F_\beta(x^i, s^\alpha, s_i^\alpha, s_{ij}^\alpha, \ldots, s_{ij\ldots k}^\alpha) = 0.$$

Let $\lambda \in \mathscr{L}(E)$ be a Lagrangian on E. Then the Euler–Lagrange operator $E(\lambda) \in \mathscr{D}(E)$ is given by

$$E(\lambda) = E_\alpha(L)\, \mathrm{d}u^\alpha \wedge \nu,$$

where $\nu = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \mathrm{d}x^n$ and

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i \frac{\partial L}{\partial u_i^\alpha} + \ldots + (-1)^{|I|} D_I \frac{\partial L}{\partial u_I^\alpha} + \ldots.$$

Standard arguments from the calculus of variations show that the Euler–Lagrange operator is uniquely determined by the Lie derivative condition

$$\mathscr{L}_{\mathrm{pr}\, Y}\lambda = Y \lrcorner E(\lambda) + D\eta, \tag{1}$$

where η is a horizontal $n - 1$ form on $J^\infty(E)$ and $D\eta = D_i(\mathrm{d}x^i \wedge \eta)$. The Euler–Lagrange operator is a natural differential operator in that:

(i) $E(\lambda)(q)$ depends only on the germ of λ at the point $q \in J^\infty(E)$; and

(ii) for any fiber-preserving diffeomorphism $\varphi : E \to E$ and induced map $\Phi : J^\infty(E) \to J^\infty(E)$,

$$E(\Phi^*(\lambda)) = \Phi^* E(\lambda). \tag{2}$$

We pause to remark that these properties alone characterize the Euler–Lagrange operator (Anderson [7]).

Theorem 1. *The only linear, natural differential operator mapping* $\mathscr{L}(E)$ to $\mathscr{D}(E)$ *is, apart from a multiplicative constant, the Euler–Lagrange operator.*

The Helmholtz conditions are very easy to derive from (1) and (2). First note that the Lie derivative $\mathscr{L}_{\mathrm{pr}\,Y}$ commutes with D and therefore, by the uniqueness of the decomposition (1),

$$E(D\eta) = 0.$$

Second, the naturality condition (2) implies that $\mathscr{L}_{\mathrm{pr}\,Y}$ commutes with the Euler–Lagrange operator E, *i.e.*

$$E(\mathscr{L}_{\mathrm{pr}\,Y}\lambda) = \mathscr{L}_{\mathrm{pr}\,Y}E(\lambda).$$

Consequently, on applying the Euler–Lagrange operator to (1), it follows that

$$\mathscr{L}_{\mathrm{pr}\,Y}E(\lambda) = E(Y \lrcorner E(\lambda)).$$

Therefore, if $\Delta = E(\lambda) \in \mathscr{D}(E)$ is an Euler–Lagrange operator it must satisfy the equation

$$\mathscr{L}_{\mathrm{pr}\,Y}\,\Delta = E(Y \lrcorner \Delta) \tag{3}$$

for all vertical vector fields Y. Upon equating the coefficients of $D_I Y^\alpha$ in this formula one obtains the explicit form of the Helmholtz conditions as partial differential equations on the components of Δ. For example, if the fiber of E is one-dimensional and $\Delta = F[u]\,\mathrm{d}u \wedge \nu$ is a second order equation, then (3) is equivalent to

$$\frac{\partial F}{\partial u_i} = D_j \frac{\partial F}{\partial u_{ij}}. \tag{4}$$

In general, one finds that

$$\mathscr{L}_{\mathrm{pr}Y}\Delta - E(Y \lrcorner \Delta) = \sum_{|I|=0}^{k} D_I Y^\beta H^I_{\alpha\beta}(\Delta)\,\mathrm{d}x^\alpha \wedge \nu,$$

where

$$H^I_{\alpha\beta}(\Delta) = \frac{\partial F_\alpha}{\partial u^\beta_I} + (-1)^{|I|}\sum_{|J|=0}^{k-|I|}\binom{|I| + |J|}{|I|}[-D]_J \frac{\partial F_\beta}{\partial u^\alpha_{IJ}}.$$

Theorem 2. *Let E be the trivial bundle $\mathbf{R}^{n+m} \xrightarrow{\pi} \mathbf{R}^n$. If $\Delta \in \mathscr{D}(E)$ satisfies* (3), *then $\Delta = E(\lambda)$, where*

$$\lambda = \int_0^1 u^\alpha F_\alpha[tu]\,\mathrm{d}t. \tag{5}$$

Proof: Let R be the radial vector field $R = u^\alpha \dfrac{\partial}{\partial u^\alpha}$. Then a short set of calculations shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{tF_\alpha[tu]\,\mathrm{d}u^\alpha \wedge v\} = [\mathscr{L}_{pr\,\mathrm{R}}\Delta][tu] = E(R \lrcorner \Delta)[tu] = E(R \lrcorner \Delta[tu]).$$

Integration of this equation with respect to t from 0 to 1 yields (5).

Equation (5) is due to Volterra. Notice that when Δ is homogeneous of degree p, that is

$$F_\alpha[tu] = t^p F_\alpha[u],$$

then (5) reduces to

$$\lambda = \frac{1}{p+1} R \lrcorner \Delta. \tag{6}$$

Theorem 2 does not completely resolve the local inverse problem. Indeed, let $\mathscr{L}^k(E)$ and $\mathscr{D}^k(E)$ denote the spaces of Lagrangians and differential equations of order k and let

$$\mathscr{V}^k(E) = E(\mathscr{L}^k(E)) \subset \mathscr{D}^{2k}(E)$$

be the image of $\mathscr{L}^k(E)$ under the Euler – Lagrange operator. Then Theorem 2 fails to characterize those differential equations $\Delta \in \mathscr{D}^{2k}(E)$ which belong to $\mathscr{V}^k(E)$. This is because the Lagrangian (5) is not of order k but rather of order $2k$. One must therefore establish criteria under which it is possible to add a divergence to the Lagrangian (5) to obtain an equivalent Lagrangian of smallest possible order. This is the minimal order version of the inverse problem whose solution can be stated as follows (Anderson [6]).

Theorem 3. *Let $\Delta \in \mathscr{D}^{2k}(E)$ and suppose that Δ satisfies the Helmholtz conditions. Let*

$$p(t) = \Delta(x^i, u^\alpha, \ldots, u^\alpha_{(k)}, tu^\alpha_{(k+1)}, t^2u^\alpha_{k+2}, \ldots, t^k u^\alpha_{(2k)}),$$

where $u^\alpha_{(l)}$ denotes all derivatives of order l. If $p(t)$ is a polynomial of degree $\leqq k$, then there is a Lagrangian $\lambda \in \mathscr{L}^k(E)$ for which $\Delta = E(\lambda)$.

One can actually refine this result to arrive at a method of undetermined coefficients for finding Lagrangians of lowest possible differential order and polynomial degree.

VARIATIONAL INTEGRATING FACTORS

Given a collection of locally defined, linearly independent one forms ω_α, the Frobenius theorem furnishes necessary and sufficient conditions for the existence of a non-singular matrix $A = [a^\alpha_\beta]$ such that

$$a^\alpha_\beta \omega_\alpha = \mathrm{d} f_\beta$$

for functions f_β. The matrix A is called an integrating factor for the pfaffian system $\omega_\alpha = 0$. Similarly, but with the context of the inverse problem to the calculus of variations, one can seek variational integrating factors (henceforth VIF) for differential equations. A VIF for Δ is a type (1, 1) tensor field on $J^\infty(E)$ of the form

$$A[u] = a^\alpha_\beta[u]\, \mathrm{d}u^\beta \otimes \frac{\partial}{\partial u^\alpha}$$

with is nonsingular and such that

$$A \lrcorner \Delta = E(\lambda)$$

for some Lagrangian λ. In a slightly easier version of the VIF problem attention is restricted to operators $\Delta \in \mathscr{D}^{2k}(E)$ which satisfy the polynomial condition stated in Theorem 3 and VIF are sought which are of order k. In other words, the problem is to determine when the given operator Δ is equivalent to one in $\mathscr{V}^k(E)$. This is still a very difficult problem and not much progress has been made towards a general theory. Nevertheless, two special cases merit review.

First, consider the case of a scalar, quasi-linear second order differential operator (i.e. $m = 1$) $\Delta = F\, \mathrm{d}u \wedge \nu$, where

$$F = A^{ij}(x^h, u, u_h)\, u_{ij} + B(x^h, u, u_h).$$

In this instance the VIF is a single function

$$a = a(x^i, u, u_h).$$

If $a \,.\, F$ is to be an Euler–Lagrange operator, then the Helmholtz conditions

$$\frac{\partial a \,.\, F}{\partial u_i} = D_j \frac{\partial a \,.\, F}{\partial u_{ij}}. \tag{6}$$

must be satisfied. When $n = 1$, *i.e.* when Δ is a second order, ordinary differential equation, equation (6) reduces to a single, first order, linear partial differential equation for a. This admits $2n - 1$ independent solutions. The Lagrangians λ and $\tilde{\lambda}$ which result from different choices of a will be *essentially different* in the sense that their difference $\lambda - \tilde{\lambda}$ is not a divergence.

When $n \geqq 2$ and $\det A^{ij} \neq 0$, it is possible to obtain a closed form solution to the VIF problem (Anderson and Duchamp [9]).

Theorem 4. *Let A_{ij} denote the inverse of the matrix A^{ij}, let*

$$C_k^{ij} = A_{kl}\left(\frac{\partial A^{il}}{\partial u_j} - \frac{\partial A^{jl}}{\partial u_i}\right)$$

and let

$$\Xi = A_{ik}\left(\frac{1}{n-1} C_j^{ij} F + \left(\frac{\partial F}{\partial u_i} - D_j \frac{\partial F}{\partial u_{ij}}\right)\right) \mathrm{d}x^k.$$

Then Ξ is an invariantly defined one form and Δ admits a local variational integrating factor if and only if

$$D\Xi = 0.$$

Moreover, if Ξ is D closed, then the VIF a is uniquely determined up to a multiplicative constant by the equation

$$D[e^a] = \Xi.$$

For example, a straightforward calculation shows that the equation Δ defined by

$$\Delta = [Au_{xx} + Bu_{xy} + Cu_{yy}]\, \mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y$$

where

$$\begin{aligned} A &= au_x^2 + bu_xu_y + cu_y^2 + d \\ B &= -bu_x^2 + 2(a-c)\, u_xu_y + bu_y^2 \\ C &= cu_x^2 - bu_xu_y + au_y^2 + d \end{aligned}$$

and where a, b, c, d are constants is derivable from a variational principle if and only if $b = 0$ or $d = 0$. A complete list of variational integrating factors is given in [9].

For constant coefficient equations

$$\Delta = [A^{ij}u_{ij} + B^i u_j + Cu]\, \mathrm{d}u \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \mathrm{d}x^n$$

VIF exist if and only if

$$\operatorname{rank}\, [A^{ij}, A^i] = \operatorname{rank}\, [A^{ij}].$$

Next, consider the case of a system of second order, ordinary differential equations. Here it is convenient to change notation slightly and write the components of Δ in the form

$$F^\alpha = \ddot{u}^\alpha - f^\alpha(x, u^\alpha, \dot{u}^\alpha)$$

and write the VIF as a type (0, 2) tensor $A = (a_{\alpha\beta})$. The Helmholtz conditions imply immediately that A is symmetric. The case $m = 2$ was solved in the forties by Jesse Douglas [16] using the old Riquier theory for integrating systems of over-determined equations. It would be of considerable interest to rework Douglas' solution within the modern framework of exterior differential systems. With regards to the general problem, a very useful geometric reformulation has been developed by Henneaux [19] and Sarlet *et al.* [33]. See also Thompson [36].

An instructive example – one which I believe reflects both the geometric nature of this problem as well as some of its inherent difficulties, is provided by the spray equations (see Klein [24]) for a linear, symmetric connection Γ on the tangent bundle, *viz.*

$$F^\alpha = \ddot{u}^\alpha - \Gamma^\alpha_{\beta\gamma}\dot{u}^\beta\dot{u}^\gamma.$$

It is easily checked that an *autonomous* VIF $A = [a_{\alpha\beta}(u^\gamma)]$ exists if and only if the Christoffel symbols of $a_{\alpha\beta}$ coincide with the connetcion Γ:

$$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = \Gamma^\alpha_{\beta\gamma}.$$

This implies, in turn, that

$$a_{\alpha\gamma}K^\gamma_{\beta\mu\nu} = a_{\beta\gamma}K^\gamma_{\alpha\mu\nu},$$
$$a_{\alpha\gamma}K^\gamma_{\beta\mu\nu|\varrho} = a_{\beta\gamma}K^\gamma_{\alpha\mu\nu|\varrho} \qquad etc.$$

where K is the curvature tensor of Γ and a vertical slash denotes partial covariant diferentiation. Thus, under certain rank conditions on the curvature tensor, the VIF can be algebraically determined (see Cheng [11]).

The uniqueness of VIF is an interesting problem since different VIF lead to essentially different Lagrangians for the given system of equations (See Crampin [13]). This leads to conservation laws for the equations (see Hojman and Harleston [22] and Sarlet [32]) and ambiguities in the quantization of the mechanical systems (see Henneaux [20] and Dononov [14], [15]). The uniqueness problem can be formulated as follows. Suppose that Δ satisfies the Helmholtz conditions. What additional conditions must be imposed on Δ to imply that the only VIF is a constant multiple of the identity. An important observation, suggested by the work of Henneaux [21], is that if system Δ decouples in some coordinate system (x^i, u^α) into the form

$$\begin{cases} F^1 = \ddot{u}^1 - f^1(x, u^1, \dot{u}^1) \\ F^\mu = \ddot{u}^\mu - f^\mu(x, u^\nu, \dot{u}^\nu) \qquad \mu, \nu = 2, \ldots, m \end{cases}$$

then, in view of our earlier remarks, the system will admit non-trivial VIF. I suspect that if the system does not decouple then the VIF is unique – but then the problem becomes that of determining when a given system decouples. This would seem to be an interesting geometric problem in its own right.

Two other aspects of the VIF problem ought to be mentioned. First, let $\Delta[x, u]$ be a linear, differential operator defined on a domain $D \subset \mathbf{R}^n$. In this case one can require that the VIF $A = [a^\alpha_\beta(x^i)]$ actually gives rise to a self-adjoint (as opposed to a formally self-adjoint) differential operator, that is

$$\langle v, \Delta[x, u]\rangle \overset{\text{def}}{=} \int_D v^\beta a^\alpha_\beta F_\alpha[x, u]\,\mathrm{d}x = \int_D u^\beta a^\alpha_\beta F_\alpha[x, v]\,\mathrm{d}x = \langle u, \Delta[x, v]\rangle.$$

In this context it is appropriate to seek variational integrating factors which are not necessarily smooth functions of x^i. Consider, for example, the operator

$$\Delta(x, u, \dot{u}, \ddot{u}) = [x\ddot{u} + (x - 1)\dot{u}]\, \mathrm{d}u \wedge \mathrm{d}x \qquad \text{on } D = [0, \infty]$$

for which the Helmholtz conditions are

$$x\left(a(x) - \frac{\mathrm{d}}{\mathrm{d}x} a(x)\right) = 0. \tag{7}$$

The smooth solution $a = e^{-x}$ does not make Δ self-adjoint because of boundary contributions at 0. These are eliminated by the distributional solution to (7), *viz.*

$$a(x) = e^{-x} + \delta(x).$$

Other examples of this type can be found in Littlejohn [26] in connection with differential equations for orthogonal polynomial sequences.

Second, consider the case of scalar evolution equations

$$u_t = K(u, u_x, u_{xx}, u_{xxx}, \ldots).$$

Such equations never admit a VIF and are therefore not derivable from a variational principle without the introduction of additional variables. A more interesting question for evolution equations is whether or not the equation is the flow for an infinite dimensional Hamiltonian system. Specifically, is there a Hamiltonian operator $\mathscr{D}$ and a Lagrangian $H = H[u]$ such that

$$K = \mathscr{D} \circ E(H)?$$

This can be viewed as an operator-valued form of the VIF problem. Notice that in this case the uniqueness aspect of the problem impinges upon the theory of bi-Hamiltonian systems. See Olver [29, Chapter 7] for an introduction to this important subject and Olver [30] for a classification of low order Hamiltonian operators.

THE GLOBAL INVERSE PROBLEM

The global inverse problem has a simple cohomological solution.

Theorem 5. a) *Let $\lambda \in \mathscr{L}(E)$ be a globally defined Lagrangian which is variationally trivial, i.e. $E(\lambda) = 0$. Then λ determines a cohomology class $[\lambda] \in H^n(J^\infty(E))$ and λ is globally exact, i.e.*

$$\lambda = D\eta$$

for some global horizontal $n - 1$ form η on $J^\infty(E)$, if and only if this cohomology class vanishes.

b) *Let $\Delta \in \mathscr{D}(E)$ be a globally defined differential operator which satisfies the*

Helmholtz conditions. Then Δ determines a cohomology class $[\Delta] \in H^{n+1}(J^\infty(E))$ and Δ is globally variational, i.e. $\Delta = E(\lambda)$ for a globally defined Lagrangian λ, if and only if this cohomology class vanishes.

Proof. See Vinogradov [40], [41], Takens [35] and Anderson and Duchamp [8].

The global characterization of the image $\mathscr{V}^k(E)$ of the Euler – Lagrange operator on k^{th}-order Lagrangians follows from Theorem 3 and the solution to the global inverse problem on finite order jet bundles given by Anderson and Duchamp [8].

Corollary. *A differential operator $\Delta \in \mathscr{D}^{2k}(E)$ belongs to $\mathscr{V}^k(E)$ if and only if*

(1) *Δ satisfies the Helmholtz conditions,*

(2) *Δ satisfies the polynomial condition*[2]*) of Theorem 3 at each point of $J^{2k}(E)$, and*

(3) *the cohomology class $[\Delta]$ vanishes.*

Representatives of the cohomology classes $[\lambda]$ and $[\Delta]$ can be computed by applying standard spectral sequence techniques to the variational bicomplex. In certain situations these methods give rise to mappings

$$\Theta_0 : \mathscr{L}(E) \to \Omega^n(J^\infty(E))$$

and

$$\Theta_1 : \mathscr{D}(E) \to \Omega^{n+1}(J^\infty(E))$$

with the following properties.

(1) Both Θ_0 and Θ_1 are natural, linear differential operators. Thus, if φ is a local fiber preserving diffeomorphism of E and Φ the induced map on $J^\infty(E)$, then

$$\Theta_0(\Phi^*\lambda) = \Phi^*\Theta_0(\lambda)$$

and

$$\Theta_1(\Phi^*\Delta) = \Phi^*\Theta_1(\Delta).$$

(2) $\Theta_0(\lambda)$ is a d closed n form on $J^\infty(E)$ whenever λ is variationally trivial and a d exact form whenever λ is a global divergence. Hence, when $E(\lambda) = 0$, $\Theta_0(\lambda)$ is a representative of $[\lambda]$ in the Rham cohomology.

(3) $\Theta_1(\Delta)$ is a d closed $(n + 1)$ form on $J^\infty(E)$ whenever Δ satisfies the Helmholtz conditions and a d exact form whenever Δ is globally variational. Hence, when Δ satisfies the Helmholtz conditions, $\Theta_1(\Delta)$ is a representative of $[\Delta]$ in de Rham cohomology.

Let

$$\theta_I^\alpha = \mathrm{d}u_I^\alpha - u_{Ij}^\alpha\,\mathrm{d}x^j$$

denote the contact one forms on $J^\infty(E)$. Then each of the following maps have all of these properties.

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[2]) It is not difficult to see that this condition is an invariant one.

Example 1. Let dim $M = 1$ and for

$$\lambda = L(x, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha, \ldots, u^\alpha_{(k)})\, \mathrm{d}x$$

define

$$\Theta_0(\lambda) = \lambda + \sum_{l=0}^{k-1} P^{(l)}_\alpha \theta^\alpha_{(l)},$$

where

$$P^{(l)}_\alpha = \sum_{j=0}^{k-l} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\frac{\partial L}{\partial u^\alpha_{(j+k+1)}}\right).$$

Example 2. Here the dimensions of M and E are arbitrary but we restrict the map Θ_0 to first order Lagrangians. For $\lambda \in \mathscr{L}^1(E)$, let

$$\Theta_0(\lambda) = \lambda + \sum_{l=0}^{n} \frac{1}{(l!)^2}\left(\frac{\partial^l L}{\partial u^{\alpha_1}_{i_1} \ldots \partial u^{\alpha_k}_{i_k}}\right)\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_k} \wedge \nu_{i_1,\ldots,i_k},$$

where

$$\nu_{i_1,\ldots,i_k} = \frac{\partial}{\partial x^{i_1}} \lrcorner \ldots \frac{\partial}{\partial x^{i_k}} \lrcorner\, \nu.$$

Example 3. Again let dim $M = 1$. We define Θ_1 acting on third order ordinary differential operators

$$\Delta = F_\beta(x, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha, \dddot{u}^\alpha)\, \mathrm{d}u^\beta \wedge \mathrm{d}x$$

by

$$\Theta_1(\Delta) = \Delta + \frac{1}{2}\left(\frac{\partial F_\alpha}{\partial \dot{u}^\beta} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F_\alpha}{\partial \ddot{u}^\beta} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F_\alpha}{\partial \dddot{u}^\beta}\right)\theta^\alpha \wedge \theta^\beta +$$

$$+ \frac{1}{2}\left(\left(\frac{\partial F_\alpha}{\partial \ddot{u}^\beta} + \frac{\partial F_\beta}{\partial \ddot{u}^\alpha}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F_\alpha}{\partial \dddot{u}^\beta} + 2\frac{\partial F_\beta}{\partial \dddot{u}^\alpha}\right)\right)\theta^\alpha \wedge \dot{\theta}^\beta +$$

$$+ \frac{1}{2}\left(\frac{\partial F_\alpha}{\partial \dddot{u}^\beta} + \frac{\partial F_\beta}{\partial \dddot{u}^\alpha}\right)\theta^\alpha \wedge \ddot{\theta}^\beta - \frac{1}{2}\left(\frac{\partial F_\alpha}{\partial \dddot{u}^\beta}\right)\dot{\theta}^\alpha \wedge \dot{\theta}^\beta.$$

The generalization of this formula to ordinary differential equations of arbitrary order and to first order partial differential equations can be found in Anderson [5].

Several remarks are now in order. Firstly, the forms defined in examples 1 and 2 are actually Lepagean equivalents for the given Lagrangian. In fact, they are the *unique* Lepage equivalents such property (3) holds. The form in example 2 was discovered independently by Betounes [10] and Rund [31]. Secondly, the form introduced in the third example is a type of Lepage equivalent for differential operators. Indeed, for second order operators which satisfy the Helmholtz condition, this form coincides with that introduced in this conference by Krupka.

Thirdly, I believe that the examples presented here constitute the only situations where the cohomology classes $[\lambda]$ and $[\Delta]$ can be represented by natural differential operators acting on $\mathscr{L}(E)$ and $\mathscr{D}(E)$. Finally, if the base manifold is endowed with a symmetric, linear connection Γ then it is possible to construct invariant differential operators $\Theta_\Gamma(\lambda)$ and $\Theta_\Gamma(\Delta)$ which represent these classes (Anderson [5]). These are very complicated constructions which generalize the work of Kolář [25], Masqué [28] and Ferraris [17] on the construction of global Lepage equivalents.

THE EQUIVARIANT INVERSE PROBLEM

Let G be a transformation group on E which preserves fibers. The induced transformation group on $J^\infty(E)$ will also be denoted by G. Let $\mathscr{L}_G(E)$ be the space of G-invariant Lagrangians on E and $\mathscr{D}_G(E)$ the space of G-invariant differential operators on E. Then, on account of the naturality of the Euler-Lagrange operator,

$$E : \mathscr{L}_G(E) \to \mathscr{D}_G(E).$$

Consequently, it is possible to formulate the following equivariant version of the inverse problem to the calculus of variations. Let $\Delta \in \mathscr{D}_G(E)$ be a G-invariant differential operator which satisfies the Helmholtz conditions. Does there exist a G-invariant Lagrangian $\lambda \in \mathscr{L}_G(E)$ such that $\Delta = E(\lambda)$?

As a trivial example, consider the differential operator $\Delta = (\ddot{u} + 1)\,\mathrm{d}u \wedge \mathrm{d}x$. This operator is translationally invariant in both the independent and dependent variables but there exists no Lagrangian for Δ with these symmetries.[3] In general the equivariant inverse problem is another difficult aspect to the inverse problem. Not much work has been done in this direction. Two special cases have been solved and these point to a remarkable connection between the inverse problem and the theory of secondary characteristic classes.

First, let E be the bundle of metrics on M so that a section $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ of E is a symmetric, positive-definite type (0, 2) tensor field on M. Let $G = \mathrm{Diff}$ be the pseudo-group of all local diffeomorphisms on M. The space $\mathscr{L}_{\mathrm{Diff}}$ consists of all natural Riemannian Lagrangians on M. A Lagrangian $\lambda[g]$ belongs to $\mathscr{L}_{\mathrm{Diff}}(E)$ if and only if for each local diffeomorphism φ and induced diffeomorphism Φ,

$$\varphi^*\lambda[g] = \lambda[\Phi^*g].$$

Roughly speaking, $\lambda[g]$ is a natural Riemannian Lagrangian if and only if it is obtained from the inverse of the metric g^{ij}, its curvature tensor R_{ijhk}, repeated covariant derivatives of the curvature tensor $R_{ijhk|p|q\ldots|r}$ and the volume form

[3]) To see this, view Δ as an equation on the two torus T^2. The cohomology class $[\Delta]$ as determined by $\Theta_1(\Delta)$ is simply $\mathrm{d}u \wedge \mathrm{d}x$ and this is not exact on T^2.

$dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_n}$ by contraction of indices. Natural differential operators $\Delta[g]$ have a similar structure.

Consider, for example, the *Cotton tensor*

$$C[g] = C^{ij}[g]\, dg_{ij} \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where

$$C^{ij} = \varepsilon^{ihk} R^j_{h|k} + \varepsilon^{jhk} R^i_{h|k}.$$

This tensor is a symmetric, third order tensor defined on three manifolds. The vanishing of the Cotton tensor is necessary and sufficient for the local conformal flatness of g.

In Chern and Simons [12] the following locally defined, second order Lagrangian is given foı the Cotton tensor:

$$\lambda[g] = \left(\frac{1}{2}\Gamma^j_{ih}\Gamma^i_{jk,l} + \Gamma^j_{ih}\Gamma^m_{jk}\Gamma^i_{ml}\right) dx^i \wedge dx^j \wedge dx^k. \tag{8}$$

This Lagrangian is not natural. Horndeski [23] showed, by direct calculation, that the Cotton tensor satisfies the Helmholtz conditions. Aldersley showed that the Cotton tensoı is not derivable from a natuıal Lagrangian. He assumed that a natural Lagrangian exists and proved, because C^{ij} is homogeneous of degıee -1, that there must be a natural Lagrangian which is homogeneous of degree 0. Standard techniques from invariant theory show that such a Lagrangian does not exist on three manifolds. In Anderson [4] it was established that, for $n = 3$, the Cotton tensor is the only obstruction to the solution of the equivariant inverse problem – all other natural, locally variational operatoıs on Riemannian metrics admit natural variational principles.

More generally, let $P(A)$ be an $O(n)$-invariant polynomial which is homogeneous of degree l and let ω and Ω be the matrices of connection one forms and curvatuıe two foıms. Then, on the frame bundle of M, the $2l$ form $P(\Omega)$ is exact and

$$d[TP(\omega)] = P(\Omega),$$

where $TP(\omega)$ is called the *transgression* of $P(\Omega)$. When $P(\Omega) = 0$, $TP(\omega)$ defines a Chern–Simons secondary characteristic class. In particular, if $n = 4m - 1$ and $l = 2m$ then $P(\Omega) = 0$. Under these circumstances, let $f = \left\{\frac{\partial}{\partial x^i}\right\}$ be the local coordinate frame and set

$$\lambda_P[g] = f^*[TP(\omega)].$$

This is a top dimensional form depending on the metric and its first and second derivatives. It is not a natural Lagrangian. For example, with $P(A) = \text{trace}[A^2]$ it is easily verified that $\lambda_P[g]$ coincides with the Lagrangian (7). Furtheımore, it is possible to pıove, by direct calculation, that the Euler–Lagrange operators

$$C_P[g] = E(\lambda_P)\,[g]$$

are natural differential operators. Aldersley and Horndeski [1] called the tensors $C_P[g]$ generalized Cotton tensors. A simple modification of the above argument of Aldersley shows that there does not exist a natural Lagrangian for the generalized Cotton tensors. These tensors are therefore obstructions to the solution of our equivariant inverse problem. They are the only obstructions.

Theorem 6. *Let $\Delta[g]$ be a natural differential operator on the bundle of Riemannian metrics of M and suppose that Δ satisfies the Helmholtz conditions. If $n \neq 3 \bmod 4$, then there is a natural Lagrangian $\lambda[g]$ for which*

$$\Delta = E(\lambda).$$

If $n = 3 \bmod 4$, then there is a natural Lagrangian λ and an invariant polynomial P such that

$$\Delta = E(\lambda) + C_P.$$

This decomposition is unique.

Sketch of Proof: For simplicity, let us assume that $\Delta[g]$ is a polynomial natural differential operator. Define

$$\Delta_t[g] = \Delta(g_{ij},\, tg_{ij,h_1},\, t^2 g_{ij,h_1h_2},\, \ldots,\, t^k g_{ij,h_1h_2\ldots h_k})$$

and expand this natural differential operator in powers of t:

$$\Delta_t[g] = \Delta_0[g] + t\Delta_1[g] + t^2\Delta_2[g] + \ldots + t^p\Delta_p[g]. \tag{9}$$

It is readily established that each coefficient $\Delta_r[g]$ is a natural differential operator which satisfies the Helmholtz conditions and, in addition, is homogeneous of degree $p = (n - r - 2)/2$. Consequently equation (6), *viz.*

$$\lambda_r[g] = \frac{1}{p+1}\, g_{ij} F_r^{ij}[g]$$

can be used to construct natural Lagrangians for each $\Delta_r[g]$, $r \neq n$. Thus, with $t = 1$, (9) yields

$$\Delta[g] = E(\lambda)\,[g] + B[g],$$

where $\lambda[g]$ is a natural Lagrangian and $B[g]$ is a natural, locally variational operator which is homogeneous of degree zero. Classical invariant theory now implies that $B[g]$ is of second order in the derivatives of the metric if n is even and linear in the third derivatives of the metric if n is odd. The final, and most difficult step in the proof, is to use the Helmholtz conditions to show that there is an invariant polynomial P such that

$$B[g] = C_P[g].$$

For our second example of an equivariant inverse problem, let E be the cotangent bundle of Minkowski space M and let G be the group of gauge transformations

$$\psi \to \psi + \mathrm{d}\varphi,$$

where $\psi = \psi_i \, \mathrm{d}x^i$ is a section of E and φ is a function on M. Let $\mathscr{L}_{\text{Lor-G}}(E)$ and $\mathscr{D}_{\text{Lor-G}}(E)$ be the spaces of Lorentz invariant and gauge invariant Lagrangians and differential operators on E. A Lagrangian $\lambda[\psi] \in \mathscr{L}_{\text{Lor-G}}(E)$ assumes the form

$$\lambda[\psi] = L(\psi_i, \psi_{i,j}, \psi_{i,jh}, \ldots, \psi_{i,jh\ldots k})\, \nu$$

and satisfies

$$\lambda[\psi] = \lambda[\Lambda \,.\, \psi]$$

for every Lorentz transformation Λ and

$$\lambda[\psi] = \lambda[\psi + \mathrm{d}\varphi]$$

for every function φ. The Lagrangian for the source-free Maxwell's equations, *viz.*

$$\lambda = F^{ij}F_{ij}\nu, \qquad \text{where} \qquad F_{ij} = \psi_{i,j} - \psi_{j,i}$$

is an example of such a Lagrangian. The solution to the equivariant inverse problem in this case is found in Anderson [5].

Theorem 8. *Let $\Delta[\psi] \in \mathscr{D}_{\text{Lor-G}(E)}$ be a Lorentz invariant, gauge invariant differential operator which satisfies the Helmholtz conditions. If n is even, then there is a $\lambda[\psi] \in \in \mathscr{L}_{\text{Lor-G}}(E)$ such that*

$$\Delta[\psi] = E(\lambda)\,[\psi].$$

If $n = 2m + 1$ is odd, then there is a $\lambda \in {}_{\text{Lor-G}}(E)$ and a constant a such that

$$\Delta[\psi] = E(\lambda)\,[\psi] + a\,\Sigma[\psi],$$

where

$$\Sigma[\psi] = \varepsilon^{ih_1k_1\ldots h_mk_m} F_{h_1k_1} \ldots F_{h_mk_m}\, \mathrm{d}\psi_i \wedge \nu.$$

This decomposition is unique.

Sketch of Proof: As in the Riemannian metric case, $\Delta[\psi] = F^i[\psi]\,\mathrm{d}\psi_i \wedge \nu$ is decomposed into its homogeneous components. However, unlike the Riemannian metric case, the Volterra Lagrangian

$$\lambda[\psi] = \frac{1}{p+1}\psi_i F^i[\psi]\,\nu \tag{10}$$

fails to produce an invariant (in this case gauge invariant) Lagrangian. This is the essential and non-trivial difference between this theorem and the previous one. To overcome this difficulty it is shown that because Δ is gauge invariant and locally variational, it is divergence-free, *i.e.*

$$D_iF^i = 0.$$

This changes the original problem into that of finding all gauge invariant, divergence-free type (1, 0) tensors. It is shown that, apart from the tensor Σ, there exists a Lorentz invariant, gauge invariant, skew-symmetric tensor $S^{ij}[\psi]$ such that

$$F^i[\psi] = D_j S^{ij}[\psi].$$

Consequently, the Lagrangian (10) can be integrated by parts to obtain the equivalent, but gauge invariant Lagrangian

$$\lambda = -\frac{1}{2} F_{ij} S^{ij} \nu.$$

Note that Σ is the Euler–Lagrange operator of the non-gauge invariant Lagrangian

$$\lambda[\psi] = \frac{1}{m+1} \psi \wedge [d\psi]^m,$$

and that this is the transgression of the Chern form $[d\psi]^{m+1}$.

The two cases treated thus far can be coupled by considering gauge invariant, natural differential operators $\Delta[g, \psi]$. In this case the obstructions to the solution of the equivariant inverse problem are generated by the obstructions in the individual cases. The equivariant inverse problem for connections on principle fiber bundles (for example, Yang–Mills equations) is currently under investigation. Finally, there appears to be some relationship between the equivariant inverse problem and the existence of anomalies in quantum field theory (see Witten [42]). An important and timely problem is to make this connection precise.

NOETHER'S THEOREM

This brief survey will conclude by discussing the relationship between Noether's Theorem and the inverse problem to the calculus of variations. Specifically, I would like to address the following problem which was first posed by Takens [34]. Suppose that Δ is a differential operator and $\mathfrak{g}$ is a Lie algebra of infinitesimal symmetries of Δ, each element of which generates a conservation law for Δ (in a manner to be made precise momentarily). Under what circumstances does this imply that Δ is locally variational? I feel that this is an important question whoes answer will enhance our understanding of the role of the calculus of variations in mathematical physics. In any event Takens' question does lead to some very interesting mathematical problems.

To formulate Takens' problem precisely, consider a vector field

$$X = a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha}$$

on E. The associated evolutionary vectoı field is

$$X_{\text{ev}} = [b^\alpha - u_i^\alpha a^i]\frac{\partial}{\partial u^\alpha}$$

and the associated total vector field is

$$X_{\text{tot}} = a^i D_i.$$

The prolongation of X is given in terms of these associated vector fields by

$$\text{pr}\, X = \text{pr}\, X_{\text{ev}} + X_{\text{tot}\, X}.$$

Recall that $H^I_{\alpha\beta}(\Delta)$ denote the components of the Helmholtz conditions and set, for

$$Y = Y^\alpha[u]\frac{\partial}{\partial u^\alpha},$$

$$H_Y(\Delta) = \sum_{|I|=0}^{k} D_I Y^\beta H^I_{\alpha\beta}(\Delta)\, du^\alpha \wedge \nu.$$

It is not difficult to prove the following Lie derivative formula.

Theorem 7. *Let X be a vector field on E and let Δ be a differential operator. Then*

(11) $$\mathscr{L}_{\text{pr}\, X}\Delta = E(X_{\text{ev}} \lrcorner\, \Delta) + H_{X\text{ev}}(\Delta).$$

Corollary. *Suppose that Δ is locally variational. Then a vector field X on M is an exact symmetry of Δ, i.e.*

(12) $$\mathscr{L}_{\text{pr}\, X}\Delta = 0$$

if and only if X_{ev} is a characteristic for a local conservation law for Δ, i.e.

(13) $$E(X_{\text{ev}} \lrcorner\, \Delta) = 0.$$

Proof: If Δ is locally variational, then $H_{X\text{ev}}\Delta$ vanishes foı any vector field X and the result follows immediately from (11). Equation (13) implies, at least locally, that there exists a horizontal $(n - 1)$ form η on $J^\infty(E)$ such that

$$X_{\text{ev}} \lrcorner\, \Delta = D\eta.$$

and consequently η is conserved along solutions of Δ.

This is a slightly different version of Noether's Theorem since this theorem is based directly upon the symmetries of the operator instead of the symmetries of the Lagrangian. This theoıem includes *both* the first and second Noether Theorems.

Example. Let E be the bundle of metrics oveı M. Then a vector field

$$X = a^i\frac{\partial}{\partial x^i}$$

on M lifts to the vector field

$$\tilde{X} = a^i \frac{\partial}{\partial x^i} - 2g_{ih}a^h{}_{,j} \frac{\partial}{\partial g_{ij}}$$

on E. Note that

$$\tilde{X}_{ev} = -a_{i|j} \frac{\partial}{\partial g_{ij}}, \qquad \text{where } a_i = g_{ij}a^j.$$

If

$$\Delta[g] = F^{ij}[g]\, \mathrm{d}g_{ij} \wedge \nu$$

is a natural tensor, then the infinitesimal invariance criteria for natural tensors states that for any vector field X on M

$$\mathscr{L}_{\mathrm{pr}\,\tilde{X}} \Delta[g] = 0.$$

Hence, if $\Delta[g]$ is also locally variational, then

$$E(X_{ev} \lrcorner \Delta) = 0$$

i.e.

$$E[a_{i|j}F^{ij}] = -E[a_i F^{ij}{}_{|j}] = 0$$

for all vector fields X on M. From this last equation it now follows (Anderson [3]) that *every locally variational, natural tensor* $\Delta(g)$ *is divergence-free*:

$$F^{ij}{}_{|j} = 0.$$

In a similar way one can establish that every conformally invariant, locally variational, natural tensor is trace-free.

Takens' problem can now be precisely formulated as follows. Suppose $\mathfrak{g}$ is a Lie algebra of vector fields on E which are exact symmetries of Δ and which are characteristics for conservation laws for Δ. When does this imply that Δ is locally variational?

Theorem 8. *Let E be the trivial bundle $\mathbf{R}^{n+m} \xrightarrow{\pi} \mathbf{R}^n$ and let $\mathfrak{g}$ be a p-dimensional Lie algebra of translations on the base space $\mathbf{R}^n$. If $\Delta[u]$ is a polynomial in the dependent variables and their derivatives of degree $\leqq p$, then $\Delta[u]$ is locally variational.*

The case $p = 1$ was established by Takens. The theorem is false if the polynomial condition on $\Delta[u]$ is omitted. For example, the operator

$$\Delta = [3u_{xx}(u_{xx}u_{yy} - u_{xy}^2) + u_{xxx}(u_x u_{yy} - u_y u_{xy}) + \\ + u_{xxy}(-u_x u_{xy} + u_y u_{xx})]\, \mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y$$

has infinitesimal symmetries

(14) $$X = \frac{\partial}{\partial x} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y}.$$

These are both characteristics for conservation laws for Δ:

$$X_{ev} \lrcorner \Delta = \left(-u_x \frac{\partial}{\partial u}\right) \lrcorner \Delta =$$
$$= D\{[u_{xx}u_x(-u_xu_{xy} + u_yu_{xx})]\,dx - [u_{xx}u_x(u_xu_{yy} - u_yu_{xy}]\,dy\}$$

and

$$Y_{ev} \lrcorner \Delta = \left(-u_y \frac{\partial}{\partial u}\right) \lrcorner \Delta =$$
$$= D\{[u_{xx}u_y(-u_xu_{xy} + u_yu_{xx})]\,dx - [u_{xx}u_y(u_xu_{yy} - u_yu_{xy}]\,dy\}.$$

However, $\Delta[u]$ is a scalar, third order differential operator and therefore cannot be derived from a variational principle.

Sketch of Proof: To illustrate the general argument it suffices to consider the case $p = 2$ and $m = 1$. Pick coordinates on $\mathbf{R}^n$ such that $\mathfrak{g}$ is spanned by the vector fields (14). Since these vector fields are exact symmetries of Δ and are also characteristics for conservation laws, we can deduce from Theorem 7 that

$$H_{X_{ev}}(\Delta) = 0 \qquad \text{and} \qquad H_{Y_{ev}}(\Delta) = 0.$$

Written out in detail, these two equations lead to a pair of equations of the form

$$u_xA + u_{xx}B + u_{xy}C + u_{xxx}D + u_{xxy}E + \dots = 0$$

and

$$u_yA + u_{yx}B + u_{yy}C + u_{yxx}D + u_{yxy}E + \dots = 0,$$

where $A, B, C, \dots$ are the various components of the Helmholtz conditions $H^I(\Delta)$. Because Δ is a polynomial of degree $\leqq 2$, the coefficients $A, B, C, \dots$ are polynomials of degree $\leqq 1$. The theorem will be proved by showing that these coefficients vanish.

Solve for A in terms of $C, D, E, \dots$ by Cramer's rule to obtain

$$\begin{vmatrix} u_x & u_{xx} \\ u_y & u_{yx} \end{vmatrix} A = -\begin{vmatrix} u_{xy} & u_{xx} \\ u_{yy} & u_{yx} \end{vmatrix} C - \begin{vmatrix} u_{xxx} & u_{xx} \\ u_{yxx} & u_{yx} \end{vmatrix} D - \begin{vmatrix} u_{xxy} & u_{xx} \\ u_{yyx} & u_{yx} \end{vmatrix} E - \dots.$$

The left-hand side of this equation is therefore in the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} u_{xx} & u_{xy} & u_{xxx} & u_{xxy} & \cdots \\ u_{yx} & u_{yy} & u_{yxx} & u_{yxy} & \cdots \end{pmatrix}.$$

This is an example of a Hankel matrix and a deep theorem from algebraic geometry due to D. Eisenbud asserts that the polynomial ideal $\mathscr{I}$ generated by the maximal minors of such a matrix is a prime ideal. Consequently, either the determinant $(u_xu_{yx} - u_yu_{xx})$ or the polynomial A belongs to $\mathscr{I}$. The determinant cannot belong to I since every polynomial in $\mathscr{I}$ vanishes when

$$u_{xy} = u_{yy} = u_{yxx} = u_{yyx} = \dots = 0.$$

Therefore $A \in \mathscr{I}$. Since A is a polynomial of degree 1 and the generators of $\mathscr{I}$ are of degree 2, $A = 0$. In a similar fashion the other coefficients $B, C, D, \ldots$ are shown to vanish.

It is unknown if this theorem can be extended to include other symmetries such as rotations in the space of independent variables. As mentioned earlier every locally variational natural differential operator $\Delta[g]$ on the bundle of metrics is necessarily divergence-free. It is unknown if every divergence-free, natural differential operator is locally variational. This is true for second order operators by virtue of a theorem due to Lovelock [27].

This concludes this brief survey of the inverse problem.

In summary, it now appears that the both the local and global versions of the inverse problem to the calculus of variations are well understood. Much work remains to be done on the variational integrating factor problem and on the equivariant inverse problem.

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