Miroslav Bartušek

Monotonicity theorems for second order non-linear differential equations

Archivum Mathematicum, Vol. 16 (1980), No. 3, 127--135

Persistent URL: http://dml.cz/dmlcz/107065

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 127—136, 1980

MONOTONICITY THEOREMS FOR SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

MIROSLAV BARTUŠEK, Brno

(Received November 23, 1978)

1. Consider the differential equation

(1) $$y'' + f(t, y, y') = 0$$

where f is continuous on $D = \{(t, y, v): t \in [a, b), b \leqq \infty, y \in R, v \in R\}, f(t, y, v)\, y > 0$ for $y \neq 0$.

A non-trivial solution y of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_1^\infty$ such that $a \leqq t_k < t_{k+1}, y(t_k) = 0, y(t) \neq 0$ on $(t_k, t_{k+1}), k = 1, 2, \ldots,$ $\lim_{k \to \infty} t_k = b$ holds.

In all the paper we shall omit the trivial solution $y \equiv 0$ from our considerations.

Let y be an oscillatory solution of (1) and $\{t_k\}_1^\infty$ the sequence of all its zeros. Then there exists exactly one sequence of numbers $\{\tau_k\}_1^\infty$ called the sequence of extremants of y, such that $t_k < \tau_k < t_{k+1}, y'(\tau_k) = 0$ holds. This is a consequence of the following lemma (see [1], [2]):

Lemma. *Let y be an arbitrary non-trivial solution of (1) and $t_1 < t_2$ its consecutive zeros ($y(t) \neq 0$ for $t \in (t_1, t_2)$). Then t_1, t_2 are the simple zeros of y, there exists exactly one number τ such that $t_1 < \tau < t_2, y'(\tau) = 0$ holds. Further,*

$$f(t, y(t), y'(t)) > 0, \qquad t \in (t_1, \tau),$$
$$f(t, y(t), y'(t)) < 0, \qquad t \in (\tau, t_2).$$

Denote $D_1 = \{(t, y, v) : (t, y, v) \in D, y > 0\}$, $D_2 = \{(t, y, v) : (t, y, v) \in D, y < 0\}$, $D_3 = \{(t, y, v) : (t, y, v) \in D, v \neq 0\}$, $D_4 = \{(t, y, v) : (t, y, v) \in D_1, v > 0\}$, $D_5 = \{(t, y, v) : (t, y, v) \in D_2, v > 0\}$, $D_6 = \{(t, y, v) : (t, y, v) \in D_1, v < 0\}$, $J_k = [t_k, \tau_k]$, $L_k = [\tau_k, t_{k+1}]$, $k = 1, 2, 3, \ldots$

Consequently, we must state some of the following assumptions on the function $f(t, y, v)$

(2) $\quad f(t, -y, v) = -f(t, y, v)$ in D,

(3) $\quad f(t, y, -v) = f(t, y, v)$ in D,

(4) $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial y}$ exist in D, $\frac{\partial f}{\partial v}$ exists in D_3,

(5) f is decreasing (increasing) with respect to t in $D_1(D_2)$,

(6) f is increasing (decreasing) with respect to t in $D_1(D_2)$,

(7) $\frac{\partial}{\partial y} f(t, y, v) \geqq 0$ in D,

(8) f is non-increasing (non-decreasing) with respect to v in $D_4(D_5)$,

(9) f is non-decreasing (non-increasing) with respect to v in $D_4(D_5)$,

(10) f is non-decreasing (non-increasing) with respect to v in $D_4(D_6)$.

Put $\Delta_k = t_{k+1} - t_k$, $\delta_k = \tau_k - t_k$, $\gamma_k = t_{k+1} - \tau_k$, $k = 1, 2, 3, \ldots$ Thus $\Delta_k = \delta_k + \gamma_k$. Our aim is to find conditions under which the sequences $\{|y(\tau_k)|\}_1^\infty$, $\{|y'(t_k)|\}_1^\infty$ (i.e. the sequences of the absolute values of all local extremes of the solution y and its derivative) and $\{\Delta_k\}_1^\infty$ are monotone. This problem was studied e.g. in [3–7], but for the special cases of the differential equation (1):

$$y'' + f(t, y)\, g(y') = 0 \quad \text{in [3], [4], [7],}$$
$$y'' + f(t, y) = 0 \quad \text{in [6],}$$
$$y'' + \varphi(t) f(y)\, h(y') = 0 \quad \text{in [5].}$$

We use the method of "local inverse functions" used in [3]. As the oscillatory solution $y(t)$ is monotone on J_k or L_k, there exist the inverse functions $T_{1,k}(z)$ and $T_{2,k}(z)$ to $|y(t)|$ on J_k and L_k, respectively, $z \in [0, |y(\tau_k)|]$, $k = 1, 2, \ldots$ Similarly, as $y''(t) = 0 \Leftrightarrow t = t_k$, the function $y'(t)$ is monotone on J_k or L_k; let us denote the inverse function to $|y'(t)|$ on J_k and L_k by $T_{1,k}^*(z)$ $z \in [0, |y'(\tau_k)|]$ and $T_{2,k}^*(z)$, $z \in [0, |y'(\tau_{k+1})|]$, respectively.

The differential equation (1) has been investigated in [3], too. The basic results are given in the following

Theorem 1. *Let y be an oscillatory solution of* (1).

(i) *Let* (5), (8), ((6), (9)) *and* (3) *be valid. Then*

$$|y'(T_{1,k})| \geqq |y'(T_{2,k})|, \qquad \tau_k - T_{1,k} \leqq T_{2,k} - \tau_k,$$
$$(|y'(T_{1,k})| \leqq |y'(T_{2,k})|, \qquad \tau_k - T_{1,k} \geqq T_{2,k} - \tau_k),$$

$z \in [0, |y(\tau_k)|]$, $k = 1, 2, \ldots$ *holds, so that, in particular, the sequence* $\{|y'(t_k)|\}_1^\infty$ *is non-increasing (non-decreasing) and* $\delta_k \leqq \gamma_k$ $(\delta_k \geqq \gamma_k)$.

(ii) *Let* (5), (10), ((6), (8)) *and* (2) *be valid. Then the sequence* $\{|y(\tau_k)|\}_1^\infty$ *is non-decreasing (non-increasing) and*

$$|y'(T_{1,k})| \leqq |y'(T_{2,k})|, \qquad z \in [0, |y(\tau_k)|],$$
$$(|y'(T_{1,k})| \geqq |y'(T_{2,k})|, \qquad z \in [0, |y(\tau_{k+1})|]),$$

holds.

2. **Theorem 2.** *Let y be an oscillatory solution of* (1) *and let* (3), (4), (5), (9) *be valid. Then*

$$|y'(T_{1,k})| \geqq |y'(T_{2,k})|, \qquad \tau_k - T_{1,k} \leqq T_{2,k} - \tau_k,$$

$z \in [0, |y(\tau_k)|]$, $k = 1, 2, \ldots$ *holds, so that, in particular, the sequence* $\{|y'(t_k)|\}_1^\infty$ *is non-increasing and* $\delta_k \leqq \gamma_k$, $k = 1, 2, \ldots$ *holds.*

Proof. Let $y(t) > 0$ on (t_k, t_{k+1}). If $y < 0$, the proof is similar. Thus especially $f(t, y(t), y'(t)) > 0$, $y''(t) < 0$ on this interval, $y'(t) > 0$ for $t \in [t_k, \tau_k)$, $y'(t) < 0$ for $t \in (\tau_k, t_{k+1}]$ (see Lemma). Let k be an arbitrary integer. Put for the simplicity $T_1 = T_{1,k}$, $T_2 = T_{2,k}$, $y'_1 = y'(T_1)$, $y'_2 = y'(T_2)$, $y''_1 = y''(T_1)$, $y''_2 = y''(T_2)$. From this and from the assumptions of the theorem we obtain for the fixed $z \in [0, y(\tau_k))$:

$$\frac{d}{dz}(y'_1 - |y'_2|) = \frac{y''_1}{y'_1} + \frac{y''_2}{y'_2} = \frac{1}{y'_1 |y'_2|}[-|y'_2| \cdot f(T_1, z, y'_1) + y'_1 \cdot f(T_2, z, y'_2)] =$$
$$= \frac{1}{y'_1|y'_2|}[(y'_1 - |y'_2|) f(T_2, z, y'_2) + |y'_2| (f(T_2, z, y'_2) - f(T_1, z, y'_2)) +$$
$$+ |y'_2| (f(T_1, z, y'_2) - f(T_1, z, y'_1))], \tag{11}$$

$$\frac{d}{dz}(y'_1 - |y'_2|) < \frac{1}{y'_1|y'_2|}[(y'_1 - y'_2) f(T_2; z, y'_2) + |y'_2| \times$$
$$\times (f(T_1, z, y'_2) - f(T_1, z, y'_1))], \tag{12}$$

$$\frac{d}{dz}(y''_1 - y''_2) = \frac{d}{dz}(-f(T_1, z, y'_1) + f(T_2, z, y'_2)) =$$
$$= -\frac{1}{y'_1}\frac{\partial}{\partial t} f(T_1, z, y'_1) - \frac{\partial}{\partial y} f(T_1, z, y'_1) - \frac{y''_1}{y'_1}\frac{\partial}{\partial v} f(T_1, z, y'_1) +$$
$$+ \frac{1}{y'_2}\frac{\partial}{\partial t} f(T_2, z, y'_2) + \frac{\partial}{\partial y} \quad (T_2, z, y'_2) + \frac{y''_2}{y'_2}\frac{\partial}{\partial v} f(T_2, z, y'_2). \tag{13}$$

Now we show by the indirect proof that

$$y'_1 - |y'_2| \geqq 0 \qquad \text{for } z \in [0, y(\tau_k)] \tag{14}$$

holds. Let $\xi \in [0, y(\tau_k))$ be a number such that $y'_1(\xi) - |y'_2(\xi)| < 0$. The validity of the following relation follows from (12)

$$y'_1(\eta) - |y'_2(\eta)| = 0 \Rightarrow \frac{d}{dy}(y'_1(\eta) - |y'_2(\eta)|) < 0$$

and thus the following relation must be valid

$$y'_1 - |y'_2| < 0 \qquad \text{for } z \in [\xi, y(\tau_k)). \tag{15}$$

From this and from (13) we have

$$\frac{d}{dz}(y_1'' - y_2'') \geqq$$

$$\geqq \frac{1}{|y_2'|}\left\{-\frac{\partial}{\partial t} f(T_1, z, y_1') - y_1'' \frac{\partial}{\partial v} f(T_1, z, y_1') - y_2'' \frac{\partial}{\partial v} f(T_2, z, y_2')\right\} -$$

$$- \frac{\partial}{\partial y} f(T_1, z, y_1') + \frac{\partial}{\partial y} f(T_2, z, y_2'), \qquad z \in [\xi, y(\tau_k)).$$

As

$$\lim_{z \to y(\tau_k)} \frac{\partial}{\partial t} f(T_1, z, y_1') = \frac{\partial}{\partial t} f(\tau_k, y(\tau_k), 0) < 0,$$

$$\lim_{z \to y(\tau_k)} \left[-y_1'' \frac{\partial}{\partial v} f(T_1, z, y_1') - y_2'' \frac{\partial}{\partial v} f(T_2, z, y_2')\right] = 0,$$

(we must use the assumption $f(t, y, v) = f(t, y, -v)$) we can see that

$$\lim_{z \to y(\tau_k)} \frac{d}{dz}(y_1'' - y_2'') = \infty.$$

Thus there exists a number $\xi_1 \geqq \xi$ such that $\frac{d}{dz}(y_1'' - y_2'') \geqq 0$ for $z \in I = [\xi_1, y(\tau_k)']$ holds and from the fact that $y_1'' - y_2'' = 0$ for $z = y(\tau_k)$ we can conclude that $y_1'' - y_2'' \leqq 0$ on I. According to (11)

$$\frac{d}{dz}(y_1' - |y_2'|) = \frac{1}{y_1'}\left(y_1'' - \frac{y_1'}{|y_2'|} y_2''\right) \leqq \frac{1}{y_1'}(y_1'' - y_2'') \leqq 0$$

on I and (see (15)) $y_1'(z) - |y_2'(z)| \leqq y_1'(\xi_1) - |y_2'(\xi_1)| < 0$, $z \in I$. However, it is a contradiction because $y_1' - |y_2'| = 0$ for $z = y(\tau_k)$. Thus we proved that (14) is valid and the first part of the statement is proved.

Consider two functions $h_1(z) = \tau_k - T_1(z) \geqq 0$, $h_2(z) = T_2(z) - \tau_k \geqq 0$, $z \in [0, y(\tau_k)]$. From the proved part (14) of the theorem it follows that

$$\frac{d}{dz}[h_1(z) - h_2(z)] = -\frac{1}{y_1'} - \frac{1}{y_2'} \geqq 0, \qquad z \in [0, y(\tau_k)).$$

The function $h_1 - h_2$ is non-decreasing and with regard to $h_1(z) = h_2(z) = 0$ for $z = y(\tau_k)$ we can conclude that $h_1 \leqq h_2$, i.e. $\tau_k - T_1(z) \leqq T_2(z) - \tau_k$, $z \in [0, y(\tau_k)]$. The theorem is proved.

The following theorem can be proved in the same way as Theorem 2.

Theorem 3. *Let y be an oscillatory solution of* (1) *and let* (3), (4), (6) *and* (8) *be valid. Then*

$$|y'(T_{1k})| \leqq |y'(T_{2k})|, \qquad \tau_k - T_{1k} \geqq T_{2k} - \tau_k,$$
$$z \in [0, |y(\tau_k)|], \qquad k = 1, 2, 3, \ldots$$

holds, so that particularly the sequence $\{|y'(t_k)|\}_1^\infty$ *is non-decreasing and* $\delta_k \geqq \gamma_k$, $k = 1, 2, 3, \ldots$ *holds.*

Theorem 4. *Let y be an oscillatory solution of* (1) *and let* (3), (4), (5) *and* (7) *be valid. Then*

$$|y(T_{1k}^*)| > |y(T_{2k}^*)|, \qquad z \in (0, |y'(t_{k+1})|],$$

$k = 1, 2, 3, \ldots$ *holds. The sequence* $\{|y'(t_k)|\}_1^\infty$ *itself is decreasing.*

Proof. Let $y(t) > 0$ on (t_k, t_{k+1}). If $y < 0$, the proof is similar. Thus $y''(t) < 0$, $f(t, y(t), y'(t)) > 0$ on this interval, $y'(t) > 0$ for $t \in [t_k, \tau_k)$, $y'(t) < 0$ on $(\tau_k, t_{k+1}]$. Let k be an arbitrary integer number. Put for the simplicity $T_1 = T_{1k}^*$, $T_2 = T_{2k}^*$, $y_1 = y(T_1)$, $y_2 = y(T_2)$, $y_1'' = y''(T_1)$, $y_2'' = y''(T_2)$, $I = (0, c)$, $c = \min(y'(t_k), |y'(t_{k+1})|)$. We have for $z \in I$

$$\text{(16)} \qquad \frac{d}{dz}(y_1 - y_2) = z\left(\frac{1}{y_1''} - \frac{1}{y_2''}\right),$$

$$\frac{d}{dz}(y_1'' - y_2'') = \frac{1}{y_1''}\left[-\frac{\partial}{\partial t} f(T_1, y_1, z) - \frac{\partial}{\partial y} f(T_1, y_1, z) - \frac{\partial}{\partial v} f(T_1, y_1, z)\right] +$$

$$+ \frac{1}{y_2''}\left[-\frac{\partial}{\partial t} f(T_2, y_2, z) - \frac{\partial}{\partial y} f(T_2, y_2, z) + \frac{\partial}{\partial v} f(T_2, y_2, z)\right].$$

According to (17) and $y_1'' - y_2'' = 0$ for $z = 0$ we can see that

$$\lim_{z \to 0} \frac{d}{dz}(y_1'' - y_2'') < 0.$$

There exists an interval $I_1 = (0, \xi)$ such that $y_1'' - y_2'' < 0$ on I_1. Further, it is shown that we can put $I_1 = I$. On the other hand let η be the smallest number $\eta \in I$ such that $y_1''(\eta) - y_2''(\eta) = 0$. Then $y_1''(z) - y_2''(z) < 0$, $z \in (0, \eta)$,

$$\text{(18)} \qquad y_1''(0) = y_2''(0) \neq 0, \qquad y_1(0) = y_2(0) \neq 0$$

and according to (16) $\frac{d}{dz}(y_1 - y_2) > 0$, $z \in (0, \eta)$.

Therefore

$$\text{(19)} \qquad y_1 - y_2 > 0 \qquad \text{for } z \in (0, \eta].$$

Consequently,

$$0 = y_1''(\eta) - y_2''(\eta) =$$
$$= [-f(T_1, y_1, \eta) + f(T_2, y_1, \eta)] + [-f(T_2, y_1, \eta) + f(T_2, y_2, \eta)] <$$
$$< -f(T_2, y_1, \eta) + f(T_2, y_2, \eta).$$

The inequality $y_1 < y_2$ following from the notation $\frac{\partial}{\partial y} f \geqq 0$ is a contradiction to (19). Therefore

$$y_1''(z) - y_2''(z) < 0, \qquad z \in I \tag{20}$$

and $y_1(z) - y_2(z) > 0$, $z \in (0, c]$ (use (20), (16) and (18)). As a consequence, we have $y_2(c) = 0$, $y_1(c) \geqq 0$ wherefrom $c = |y'(t_{k+1})|$, $|y'(t_k)| > |y'(t_{k+1})|$. The statement of the theorem is proved.

The following theorem can be proved in the same way as Theorem 4.

Theorem 5. *Let y be an oscillatory solution of* (1) *and let* (3), (4), (6) *and* (7) *be valid. Then*

$$|y(T_{1k}^*)| < |y(T_{2k}^*)|, \qquad z \in (0, |y'(t_k)|], \qquad k = 1, 2, 3, \ldots$$

In particular, the sequence $\{|y'(t_k)|\}_1^\infty$ is increasing.

Theorem 6. *Let y be an oscillatory solution of* (1) *and let* (2), (4), (5) *and* (7) *be valid. Then*

$$|y(T_{2k}^*)| \leqq |y(T_{1,k+1})|, \qquad z \in [0, |y'(t_{k+1})|]$$

holds, so that, especially, the sequence $\{|y(\tau_k)|\}_1^\infty$ is non-decreasing.

Proof. Let $y'(t) > 0$ on (τ_k, τ_{k+1}). If $y' < 0$ holds, the proof is similar. Thus $y(t) < 0, f(t, y(t), y'(t)) > 0, y''(t) > 0$ on $[\tau_k, t_{k+1})$ and $y(t) > 0, f(t, y(t), y'(t)) < 0'$ $y''(t) < 0$ on $(t_{k+1}, \tau_{k+1}]$ (see Lemma). Let k be an integer number. Put for the simplicity $T_2 = T_{2,k}^*$, $T_1 = T_{1,k+1}^*$, $y_1 = y(T_1)$, $y_2 = y(T_2)$, $y_1'' = y''(T_1)$, $y_2'' = y''(T_2)$ and $I = [0, y'(t_{k+1}))$. Then we get for the fixed $z \in I$:

$$\frac{\mathrm{d}}{\mathrm{d}z}(|y_2| - y_1) = -\frac{z}{y_2''} - \frac{z}{y_1''} = \tag{21}$$

$$= \frac{z}{y_2''|y_1''|}\{[f(T_2, |y_2|, z) - f(T_1, |y_2|, z)] + [f(T_1, |y_2|, z) - f(T_1, y_1, z)]\}.$$

Now, considering the assumptions of the theorem, we have

$$|y_2(\eta)| - y_1(\eta) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}z}(|y_2(\eta)| - y_1(\eta)) > 0. \tag{22}$$

The following relation will be proved indirectly:

$$|y_2(z)| - y_1(z) \leqq 0, \qquad z \in I. \tag{23}$$

Let a number $\xi \in I$ exist such that $|y_2(\xi)| - y_1(\xi) > 0$, then it follows from (22) that

$$|y_2(z)| - y_1(z) > 0 \qquad \text{for } z \in I_1 = [\xi, y'(t_{k+1})). \tag{24}$$

Furthermore, if $y_2'' = |y_1''|$ for some $z \in I_1$, then

$$0 = y_2'' - |y_1''| = -f(T_2, y_2, z) - f(T_1, y_1, z) =$$
$$= [f(T_2, |y_2|, z) - f(T_1, |y_2|, z)] + [f(T_1, |y_2|, z) - f(T_1, y_1, z)] \geqq$$
$$\geqq f(T_1, |y_2|, z) - f(T_1, y_1, z)$$

and because f is non-decreasing with respect to y we obtain the relation $y_2'' - |y_1''| = 0 \Rightarrow |y_2| \leqq y_1$. Taking (24) into consideration, one of the following inequalities is valid

(25) $$y_2'' - |y_1''| > 0 \qquad \text{on } I_1,$$

(26) $$y_2'' - |y_1''| < 0 \qquad \text{on } I_1.$$

But if (26) is valid, then

$$0 > y_2'' - |y_1''| = -f(T_2, y_2, z) - f(T_1, y_1, z) =$$
$$= [f(T_2, |y_2|, z) - f(T_1, |y_2|, z)] +$$
$$+ [f(T_1, |y_2|, z) - f(T_1, y_1, z)] \geqq 0, \qquad z \in I_1$$

and we get the contradiction. Thus (25) is valid and it follows from (21) and (24) that

$$\frac{\mathrm{d}}{\mathrm{d}z}(|y_2| - y_1) = z\left(-\frac{1}{y_2''} - \frac{1}{y_1''}\right) > 0, \qquad z \in I_1,$$

$$|y_2(z)| - y_1(z) > |y_2(\xi)| - y_1(\xi) > 0, \qquad z \in [\xi, y'(t_{k+1})).$$

Especially for $z = y'(t_{k+1})$ $|y_2| - y_1 > 0$, which is a contradiction, as $y_1 = y_2 = 0$ for $z = y'(t_{k+1})$. So we have proved that the inequality (23) is valid. For $z = 0$ in particular, we get $|y(T_2)| \leqq y(T_1)$.

Theorem 7. *Let the assumptions of Theorem 6 be fulfilled. Let* $\frac{1}{f}\frac{\partial f}{\partial y}$ *be non-increasing with respect to t and y in* D_1 *and let* $\frac{1}{f}\frac{\partial f}{\partial v}$ *be non-decreasing with respect to t and y in* D_4 *and non-increasing with respect to t and y in* D_6. *Then*

$$T_{2k}^* - \tau_k \leqq \tau_{k+1} - T_{1,k+1}^*, \quad z \in [0, |y'(t_{k+1})|], \textit{ so that } \gamma_k \leqq \delta_{k+1}, k = 1,2,3,\ldots$$

holds.

Proof. Let $y'(t) > 0$ on (τ_k, τ_{k+1}). If $y' < 0$, the proof is similar. Let $T_1, T_2, y_1, y_2, y_1'', y_2''$ be of the same meaning as in Theorem 6. We prove the inequality

(27) $$y_2''(z) - |y_1''(z)| \geqq 0, \qquad z \in (0, y'(t_{k+1})] = I$$

by the indirect proof. Let $\xi \in I$ be such number that $y_2''(\xi) - |y_1''(\xi)| < 0$. Then there exists $\eta > \xi$ whereby

(28) $$y_2''(z) - |y_1''(z)| < 0, \qquad z \in [\xi, \eta) \subset I,$$

$y_2''(\eta) = |y_1''(\eta)|$ (use the fact that $y_2''(z) = |y_1''(z)|$ for $z = y'(t_{k+1})$) and

$$\frac{d}{dz}(\ln y_2'' - \ln |y_1''|) =$$

$$= \frac{1}{y_2''^2}\left[\frac{\partial}{\partial t} f(T_2, |y_2|, z) - z\frac{\partial}{\partial y} f(T_2, |y_2|, z) + \frac{\partial}{\partial v} f(T_2, |y_2|, z) \,.\, y_2''\right] +$$

$$+ \frac{1}{y_1'' |y_1''|}\left[-\frac{\partial}{\partial t} f(T_1, y_1, z) - z\frac{\partial}{\partial y} f(T_1, y_1, z) - y_1''\frac{\partial}{\partial v} f(T_1, y_1, z)\right] <$$

$$< \frac{z}{y_2''}\left[-\frac{\frac{\partial}{\partial y} f(T_2, |y_2|, z)}{f(T_2, |y_2|, z)} + \frac{\frac{\partial}{\partial y} f(T_1, y_1, z)}{f(T_1, y_1, z)}\right] +$$

$$+ \frac{\frac{\partial}{\partial v} f(T_2, |y_2|, z)}{f(T_2, |y_2|, z)} - \frac{\frac{\partial}{\partial v} f(T_1, y_1, z)}{f(T_1, y_1, z)}.$$

As $|y_2(z)| < y_1(z)$, $z \in [0, y'(t_{k+1}))$, then $\frac{d}{dz}(\ln y_2'' - \ln |y_1''|) < 0$ and thus the function $\frac{y_2''}{|y_1''|}$ is decreasing. As $\frac{y_2''(\eta)}{|y_1''(\eta)|} = 1$, we can conclude that $y_2''(z) \geqq |y_1''(z)|$, $z \in [\xi, \eta]$. This is a contradiction to (28), so that (27) is valid.

Consider two functions $h_2(z) = T_2(z) - \tau_k$, $h_1(z) = \tau_{k+1} - T_1(z)$, $z \in [0, y'(t_{k+1})]$. Then

$$\frac{d}{dz}[h_1(z) - h_2(z)] = -\frac{1}{y_1''} - \frac{1}{y_2''} \geqq 0, \qquad z \in [0, y'(t_{k+1})).$$

The function $h_1 - h_2$ is non-decreasing and with respect to $h_1(0) = h_2(0) = 0$ we can conclude that $h_1 \geqq h_2$, i.e. $T_2(z) - \tau_k \leqq \tau_{k+1} - T_1(z)$. The theorem is proved. The following theorem can be proved similarly to Theorems 6 and 7.

Theorem 8. *Let y be an oscillatory solution of* (1) *and let* (2), (4), (6) *and* (7) *be valid. Then*

$$|y(T_{2k}^*)| \geqq |y(T_{1,k+1}^*)|, \qquad z \in [0, |y'(t_{k+1})|]$$

holds, so that in particular, the sequence $\{|y(\tau_k)|\}_1^\infty$ is non-increasing. If, in addition, $\frac{1}{f}\frac{\partial f}{\partial y}$ is non-decreasing (non-increasing) with respect to $t(y)$ in D_1, $\frac{1}{f}\frac{\partial f}{\partial v}$ is non-increasing (non-decreasing) with respect to $t(y)$ in $D_4(D_6)$, then

$$T_{2k}^* - \tau_k \geqq \tau_{k+1} - T_{1,k+1}^*, \qquad z \in [0, |y'(t_{k+1})|].$$

It should be emphasized that $\gamma_k \geqq \delta_{k+1}$, $k = 1, 2, \ldots$ holds.

Corollary 1. *Let y be an oscillatory solution of* (1) *and let* (2), (3), (4), (5) *and* (7) *be valid. Further, let* $\frac{1}{f}\frac{\partial f}{\partial y}$ *be non-increasing with respect to t and y in* D_4 *and* $\frac{1}{f}\frac{\partial}{\partial v}$ *non-decreasing with respect to t and y in* D_4. *Then the sequence* $\{| y'(t_k) |\}_1^\infty$ *is non-increasing,* $\{| y(\tau_k) |\}_1^\infty$ *and* $\{\Delta_k\}_1^\infty$ *are non-decreasing.*

Corollary 2. *Let y be an oscillatory solution of* (1) *and let* (2), (3), (4), (6) *and* (7) *be valid. Further, let the function* $\frac{1}{f}\frac{\partial f}{\partial t}$ *be non-decreasing with respect to t and non-increasing with respect to y in* D_4 *and* $\frac{1}{f}\frac{\partial f}{\partial v}$ *be non-increasing with respect to t and non-decreasing with respect to y in* D_4. *Then the sequence* $\{| y'(t_k) |\}_1^\infty$ *is non-decreasing,* $\{| y(\tau_k) |\}_1^\infty$ *and* $\{\Delta_k\}_1^\infty$ *are non-increasing.*

REFERENCES

[1] М. Бартушек: О нулях колеблющихся решений уравнения $(p(x)x')' + f(t, x, x') = 0$. Дифф. урав., *XII*, №4, 621-625.

[2] M. Bartušek: *On Zeros of Solutions of the Differential Equation* $(p(t)y')' + f(t, y, y') = 0$. Arch. Math., *XI*, No. 4, 187—192.

[3] M. Bartušek: *Monotonicity Theorems concerning Differential Equations* $y'' + f(t, y, y') = 0$. Arch. Math., *XII*, No. 4, 1976, 169—178.

[4] M. Bartušek: *On Zeros of Solutions of the Differential Equation* $y'' + f(t, y)g(y') = 0$. Arch. Math., XV, 3, 129—132.

[5] I. Bihari: *Oscillation and Monotonity Theorems Concerning Non-linear Differential Equations of the Second Order.* Acta Math. Acad. Sci. Hung., *IX*, No. 1—2, 1958, 83—104.

[6] K. M. Das: *Comparison and Monotonity Theorems for Second Order Non-linear Differential Equations.* Acta Math. Sci. Hung., *XV*, No. 3—4, 1964, 449—456.

[7] А. Г. Катрамов: *Об асимптотическом поведении колеблющихся решений уравления* $\ddot{x} + f(t, x)g(\dot{x}) = 0$. Дифф. ураблення, *VIII*, №6, 1972, 1111-1115.

M. Bartušek
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia