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Jan Chvalina

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CHARACTERIZATIONS OF CERTAIN MONOUNARY ALGEBRAS

(Part I)

JAN CHVALINA, Brno
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INTRODUCTION

The aim of this paper is to give some algebraic characterizations of two types of monounary algebras (i.e. pairs (A, f), where A is a non-void set and f a transformation of A – a mapping of A into itself). The first of those, called nested, is a monounary algebra. The system of all its subalgebras forms a chain with respect to the ordering by means of set inclusion. The notion of a nested monounary algebra is playing an important role in the analyse of centralizers of set transformations (i.e. endomorphism monoids of monounary algebras), especially by studying the realization problems of monounary algebras by closure and topological spaces. Cf. [3] and [4]. The second notion studied here is the notion of a reduced connected monounary algebra. Such monounary algebras are precisely those connected monounary algebras endomorphism monoids of which coincide with monoids of continuous closed self maps of quasi-discrete T_0-spaces on their carrier-sets (see Theorem 3.3 in [3]). Characterizations given here use properties of endomorphism monoids from the point of view of the algebraic theory of semigroups and others are based on notions from groupoid theory and use special binary operations defined on monounary algebras.

Terms and notations concerning monounary algebras are taken from papers [2], [9], [10], [11], [12] and [16], notions from the groupoid and semigroup theory from [7] and [5].

By $\mathbf{N}_0$ we shall denote the set of all non-negative integers, $\mathbf{N} = \mathbf{N}_0 - \{0\}$. The full transformation monoid of a set A, denoted by $T(A)$, is the set of all self maps of the set A with the binary operation – composition of mappings, i.e.

$f, g \in T(A)$, $x \in A$, $f \,.\, g(x) = f(g(x))$ – with the unity id_A. For $n \in \mathbf{N}$, f^n means the n-th iteration of f, $f^0 = \mathrm{id}_A$, $f^{-1}(x) = \{a \in A : f(a) = x\}$. For $X \subseteqq A$, f_X denotes the restriction of the mapping f onto X. A transformation of the set A is the same as a self map of A. An element $h \in T(A)$ is a left zero of $T(A)$ if $h \,.\, g = h$ for every $g \in T(A)$. Clearly, h is a left zero of $T(A)$ iff h is a constant transformation of the set A. If $\mathscr{S}$ is a semigroup, $X \subseteqq \mathscr{S}$ a nonvoid subset then the subsemigroup of $\mathscr{S}$ generated by the set X is denoted by $\langle X \rangle$. If $\mathscr{S} = T(A)$, $X = \{f\}$ then $\langle f \rangle$ is the so called monogenuous monoid with the generator f ($\langle f \rangle = \{f^n : n \in \mathbf{N}_0\}$). For A, $B \subseteqq \mathscr{S}$ we put $A \,.\, B = \{a \,.\, b : a \in A, b \in B\}$.

A monounary algebra (A,f) is said to be connected or briefly a c-algebra if to every pair of its elements a, b there exists a pair of integers $m, n \in \mathbf{N}_0$ with $f^m(a) = f^n(b)$. A maximal (with respect to the set inclusion) connected subalgebra of the algebra (A,f) is called a component of (A,f). If $\{(A_i,f_i) : i \in I\}$ is the system of all the components of (A,f), we write $(A,f) = \sum_{i \in I} (A_i,f_i)$. The algebra is said to be idempotent if $f^2 = f$. A mapping $g \in A^A$ is called an endomorphism of the algebra (A,f) if $g \,.\, f(x) = f \,.\, g(x)$ for every $x \in A$. The endomorphism monoid of (A,f) is denoted by $C(f)$. It is in fact the centralizer of the transformation $f \in A^A$ in the full transformation monoid of the set A. For $a \in A$ we put $[a)_f = \{f^n(a) : n \in \mathbf{N}_0\}$, $(a]_f = \{x \in A : f^n(x) = a, n \in \mathbf{N}_0\}$.

Let (A,f) be a connected monounary algebra, $\{(B_i,f_i) : i \in I\}$ the system of all subalgebras of (A,f). We put $\bigcap_{i \in I} B_i = Z(A,f)$ and the set $Z(A,f)$, denoting mostly by $A_f^{\infty_2}$, is said to be a cycle of the algebra (A,f). If $Z(A,f) = \emptyset$, then the algebra (A,f) is called acyclic. Further, we put $R(A,f) = \mathrm{card}\, Z(A,f) = \mathrm{card}\, A_f^{\infty_2}$ and $R(A,f)$ is called the rank of (A,f). If $R(A,f) = 1$, $A_f^{\infty_2} = \{a\}$ the element a is called a cyclic element of the algebra (component) (A,f) and it is denoted mostly by z_f. Further we put $A_f^{\infty_1} = \{x \in A$: there is a sequence $(x_i)_{i \in \mathbf{N}_0}$ such that $x_0 = x$, $f(x_{i+1}) = x_i$ and $x_i \neq x_j$ for $i, j \in \mathbf{N}_0$, $i \neq j\}$, $A_f^0 = \{x \in A : f^{-1}(x) = \emptyset\}$. Let Ord mean the class of all ordinals. Let $\alpha \in \mathrm{Ord}$, $\alpha > 0$ and suppose that the sets $A_f^\varkappa$ have been defined for all $\varkappa < \alpha$. Then we put $A_f^\alpha = \{x \in A - \bigcup_{\varkappa < \alpha} A_f^\varkappa : f^{-1}(x) \subseteqq \bigcup_{\varkappa < \alpha} A_f^\varkappa$. We suppose thati $\infty_1, \infty_2 \notin \mathrm{Ord}$ and if $\alpha \in \mathrm{Ord}$, then $\alpha < \infty_1 < \infty_2$. Define a map $S_f : A \uparrow \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $S_f(x) = \alpha$ for each $x \in A_f^\alpha$. $S_f(x)$ is called a degree of x. More in detail concerning these notions can be found in [10], [11] and [12].

For $(A,f) = \sum_{i \in I} (A_i,f_i)$ with the property $R(A_i,f_i) \leqq 1$ for each $i \in I$ we define the ordering (induced by f) in this way: $a, b \in A$, $a \leqq_f b$, if there exists $n \in \mathbf{N}_0$ with the property $f^n(a) = b$. If $a \leqq_f b$, $a \neq b$ we write $a <_f b$. A monounary algebra (A,f) is said to be a one-way, two-way infinite chain respectively if $(A, \leqq_f)$ s a chain of the type ω_0, $\omega_0^* \oplus \omega_0$ respectively.

1. BINARY OPERATIONS Δ_f, ∇_f

In this paragraph there will be defined certain binary operations on a connected monounary algebra which will be used later for characterizations of nested and reduced monounary algebras in terms of the groupoid theory. These binary operations denoted by Δ_f and ∇_f, are chosen in such a way that for a certain class of c-algebras including also reduced and nested ones with the rank at most 1 the endomorphism monoids are preserved. Notice that problems of this kind modified for ordered sets and semigroups, groupoids and especially partial groupoids are studied e.g. in papers [6], [8], [13].

Let (A, f) be a connected monounary algebra. If $A_f^{\infty 2} \neq \emptyset$, $a \in A$ then in accordance with Definition 3.3 from [2] we put $\deg(a) =$ the smallest integer n such that $f^n(a) \in A_f^{\infty 2}$. We define a function $\delta : A \times A \to Z$ (the set of all integers) called a *level difference* as it follows:

Let $a, b \in A$. If $A_f^{\infty 2} \neq \emptyset$, i.e. $R(A, f) \geqq 1$, we put $\delta(a, b) = \deg(b) - \deg(a)$, if $A_f^{\infty 2} = \emptyset$, then $\delta(a, b) = m - n$, where m, n are the smallest integers such that $f^n(a) \in [b)_f$, $f^m(b) \in [a)_f$. Evidently it holds $\delta(a, b) + \delta(b, a) = 0$ for each pair $a, b \in A$.

Now, we define binary operations Δ_f, ∇_f on a monounary algebra $(A, f) = \sum_{i \in I} (A_i, f_i)$ (disconnected in general) in the following way. Let $a, b \in A$, $a \in A_i$ $b \in A_j$, $i, j \in I$. We put

$$a \Delta_f b = \begin{cases} f(a) & \text{if} \quad i \neq j \quad \text{or} \quad i = j \quad \text{and} \quad \delta(a, b) \geqq 0, \\ f(b) & \text{if} \quad i = j \quad \text{and} \quad \delta(a, b) < 0, \end{cases}$$

where δ is the level difference on the component (A_i, f_i) of (A, f). Further, for $a, b \in A$ we put

$$a \dot{\nabla}_f b = \begin{cases} f(a) & \text{if} \quad a \Delta_f b = f(b), \\ f(b) & \text{if} \quad a \Delta_f b = f(a), \end{cases}$$

in all cases. It is to be noted that groupoids (A, Δ_f), (A, ∇_f) are neither associative nor commutative in general.

1.1. Lemma. *Let (A, f) be a connected acyclic monounary algebra. Then $a, b \in A$, $g \in C(f)$ is followed by $\delta(g(a), g(b)) = \delta(a, b)$.*

Proof. Let (A, f) be a c-algebra with $R(A, f) = 0$, $g \in C(f)$. Let $a, b \in A$ be elements with $\delta(a, b) = 0$. There exists a positive integer n such that $f^n(a) = f^n(b)$ and $f^m(a) \neq f^m(b)$ for $m < n$. (The case $a = b$ is trivial hence it is not considered). The $f^n . g(a) = g . f^n(a) = g . f^n(b) = f^n . g(b)$and $f^m . g(a) \neq f^m . g(b)$ for $m < n$, thus $\delta(g(a), g(b)) = 0$. If, for $a, b \in A$, there is $\delta(a, b) > 0$ then for every pair of positive integers m, n having the property $f^m(a) = f^n(b)$ it holds $m < n$. Then

$f^m . g(a) = f^n . g(b)$ with $m < n$ and since $g . f(x) \neq g(x)$ for each $x \in A$, we have $\delta(a, b) = \delta(g(a), g(b))$.

In what follows the monoid of all endomorphisms of a groupoid (A, ε) will be denoted by $\boldsymbol{E}(A, \varepsilon)$.

1.2. Proposition. *Let (A, f) be a connected monounary algebra such that either $R(A,f) = 0$ or $R(A,f) = 1$ and $f^2 = f$. Then it holds $\boldsymbol{C}(f) = \boldsymbol{E}(A, \Delta_f) = \boldsymbol{E}(A, \nabla_f)$.*

Proof. We shall show that under the above assumption it holds $\boldsymbol{C}(f) \subseteqq \boldsymbol{E}(A, \Delta_f) \cap \boldsymbol{E}(A, \nabla_f)$, $\boldsymbol{E}(A, \Delta_f) \cup \boldsymbol{E}(A, \nabla_f) \subseteqq \boldsymbol{C}(f)$. Let $g \in \boldsymbol{C}(f)$ and suppose $R(A,f) = 0$. Consider arbitrary elements $a, b \in A$. Assume $\delta(a, b) \geqq 0$. By Lemma 1.1 it holds $\delta(g(a), g(b)) \geqq 0$ thus $g(a\Delta_f b) = g . f(a) = f . g(a) = g(a) \Delta_f g(b)$ and $g(a \nabla_f b) = g . f(b) = f . g(b) = g(a) \nabla_f g(b)$. If $\delta(a, b) < 0$ then also with respect to Lemma 1.1 we get $g(a\Delta_f b) = g(a) \Delta_f g(b)$ and $g(a \nabla_f b) = g(a) \nabla_f g(b)$. If $f^2 = f$ and z_f is the cyclic element of (A, f), i.e. $f(z_f) = z_f$, then $a\Delta_f b = a\Delta_f b = z_f$ for every pair of elements a, $b \in A$ and we have $g(a \varepsilon b) = g(z_f) = z_f = g(a) \varepsilon g(b)$, where ε denotes one of the symbols Δ_f, ∇_f. Thus $g \in \boldsymbol{E}(A, \Delta_f) \cap \boldsymbol{E}(A, \nabla_f)$. Now, let g be an endomorphism of (A, ε), $a \in A$. Then $f . g(a) = g(a) \varepsilon g(a) = g(a\varepsilon a) = g . f(a)$, hence $g \in \boldsymbol{C}(f)$. Therefore we get $\boldsymbol{C}(f) \subseteqq \boldsymbol{E}(A, \Delta_f) \cap \boldsymbol{E}(A, \nabla_f) \subseteqq \boldsymbol{E}(A, \Delta_f) \cup \boldsymbol{E}(A, \nabla_f) \subseteqq \boldsymbol{C}(f)$.

The association of the above defined groupoids (A, Δ_f), (A, ∇_f) to a monounary algebra is related to questions introduced and studied in [15] where among others the so called M-groupoids are treated. These objects are associated to address machines (treated as models of computers) and it is shown that the category of address machines and the category of M-groupoids are equivalent, thus the investigations of some properties of address machines can be replaced by the investigations of corresponding properties of M-groupoids.

Let X be a set, ϱ an equivalence relation on X. If $x \in X$ we put $[x]_\varrho = \{y \in X : x\varrho y\}$.

A triad $G = \langle X, ., \varrho\rangle$ is called an *M-groupoid* (see Def. 1.2.4 in [15] if the following conditions are satisfied:

M.1. $\langle X, .\rangle$ is a groupoid with zero (denoted by 0),

M.2. ϱ is an equivalence relation on X satisfying the following conditions:

M.2.a. $x, y \in X$, $x\varrho y$ implies that for each $z \in X$ it holds $x . z = y . z$.

M.2.b. card $[0]_\varrho = 1$.

Let (A,f) be a monounary algebra. If (A,f) is connected of the rank 1 we put $0_A = z_f$, where $A_f^{\infty 2} = \{z_f\}$. In other cases 0_A denotes a symbol not belonging to A. Put $\bar{A} = A \cup \{0_A\}$, $\varrho = \{[a, b] \in A \times A,\ a \neq z_f \neq b : f(a) = f(b)\} \cup \{[0_A, 0_A]\}$, for a, $b \in A$ we put $a\bar{\Delta}_f b = a\Delta_f b$, $a \bar{\nabla}_f b = a \nabla_f b$ and $0_A\bar{\Delta}_f a = a\bar{\Delta}_f 0_A = 0_A \bar{\nabla}_f a = a \bar{\nabla}_f 0_A = 0_A\bar{\Delta}_f 0_A = 0_A \bar{\nabla}_f 0_A = 0_A$ for each $a \in A$. It is not difficult to prove that $(\bar{A}, \bar{\Delta}_f, \varrho)$, $(\bar{A}, \bar{\nabla}_f, \varrho)$ are M-groupoids and if $R(A,f) = 0$ then each homomorphism $h : (A,f) \to (B, g)$ can be naturally extended onto a homomorphism $\bar{h} : (\bar{A}, \bar{\Delta}_f, \varrho) \to (\bar{B}, \bar{\Delta}_g, \sigma)$ of corresponding M-groupoids by $\bar{h}(a) = h(a)$ for $a \in A$, $a \neq 0_A$ and $\bar{h}(0_A) = 0_B$ (cf. Proposition 1.2). Here by a homomorphism $\bar{h}$ of M-groupoids we mean such a homomorphism of groupoids $(\bar{A}, \bar{\Delta}_f)$, $(\bar{B}, \bar{\Delta}_f)$ that $\bar{h}(0_A) = 0_B$ and $a, b \in \bar{A}$, $a\varrho b$ implies $\bar{h}(a)\, \sigma \bar{h}(b)$; cf. Definition 1.2.5 in [15]. In certain special cases we get that sets of homomorphisms between monounary algebras and M-groupoids, corresponding to them, coincide. Thus the above constructed functor is a realization of a suitable subcategory of the category of monounary algebras and their homomorphisms into the category of M-groupoids. We can consider a lot of various binary operations on a monounary algebra. Paragraph 1.2 in [15] contains a construction of a certain faithful functor from the category of machines into the category of M-groupoids. We get from this construction in our case $a \,.\, b = f(b)$ for every pair $a, b \in A$. It is possible to characterize nested and reduced monounary algebras using the groupoid $(A, .)$, but it has a few usual properties, e.g. $(A, .)$ is not commutative in cases when groupoids (A, Δ_f), (A, ∇_f) are commutative.

We recall some notions of groupoid theory (taken from [7] and [1] § 10) necessary in further development. A groupoid (A, ε) is called *distributive* if it satisfies the identities $a\varepsilon(b\varepsilon c) = (a\varepsilon b)\, \varepsilon(a\varepsilon c)$ and $(b\varepsilon c)\, \varepsilon a = (b\varepsilon a)\, \varepsilon(c\varepsilon a)$. If the operation ε is not associative we denote by $[a^n]$ the set of all elements obtained from the expression $a\varepsilon a\varepsilon \dots \varepsilon a$ (n times) by putting parentheses in all possible ways. A non-empty subset $J \subseteq A$ is a *right* (*left*) *ideal* if $a\varepsilon b \in J$ ($b\varepsilon a \in J$), whenever $a \in J$ and $b \in A$. If J is a left and right ideal simultaneously, then J is simply called an ideal. The least one side or both side ideal containing an element $a \in A$ is called *principal* and is denoted by $J(a)$. If J is an ideal of the groupoid (A, ε) we can define a congruence relation ϱ on (A, ε) as follows:

$$[x, y] \in \varrho \qquad \text{iff either} \qquad x = y \qquad \text{or} \qquad x, y \in J.$$

The corresponding factor-groupoid is denoted by $(A/J, \varepsilon_J)$.

In [1] § 10 there are given two natural non-associative generalizations of the radical: By the *strong radical* of an ideal J (one or both-sides), denoted by $\mathrm{rad}_s\, J$, is meant the set of all elements $a \in A$ such that $[a^n] \cap J \neq \emptyset$ for some integer n. The *weak radical* of J, denoted by $\mathrm{rad}_w\, J$, consists of all $a \in A$ such that $[a^n] \subseteq J$ for some integer n. If (A, ε) is a groupoid then $\mathrm{Id}(A, \varepsilon)$ will denote the set of all the idempotents of (A, ε). A groupoid (A, ε) is called a *BD-groupoid* (in accordance with [7]) if it satisfies the following equivalent conditions (cf. Proposition 1.2 in [7]):

(i) (A, ε) is distributive and $\mathrm{Id}(A, \varepsilon)$ contains just one element,

(ii) there is an element $e \in A$ such that $a\varepsilon e = e = e\varepsilon a$ and $a\varepsilon(b\varepsilon c) = e = (a\varepsilon b)\, \varepsilon c$ for all $a, b, c \in A$.

1.3. Lemma. *Let (A, f) be an idempotent c-algebra. Then the groupoid (A, ε) is a BD-groupoid for $\varepsilon \in \{\Delta_f, \nabla_f\}$.*

Proof. Denote by e the only cyclic element of (A,f). Since for every $x \in A$ there is $f(x) = e$, we have that $a, b \in A$ implies $a\varepsilon b = e$ for $\varepsilon \in \{\Delta f, \nabla_f\}$, hence the above condition (ii) is satisfied.

The below stated assertion following from results of [11] and [12] will be several times used in this paper.

1.4. Proposition. *Let (A,f) be a c-algebra, $a, b \in A$ such a pair of elements that $S_f(f^n(a)) \leqq S_f(f^n(b))$ for each $n \in \mathbf{N}_0$. Then there exists a mapping $g \in C(f)$ with $g(a) = b$.*

Proof follows from Definition 9 and Lemma 2.12 [12] with respect to Definition 8 from the same paper [12].

2. NESTED MONOUNARY ALGEBRAS

2.1. Definition. A monounary algebra is said to be *nested* if the system of all its subalgebras ordered by means of set inclusion forms a chain.

An element $f \in T(A)$ is said to be an *r-potent* if r is the least positive integer with the property $f^r = f$. The cyclic subgroup of $T(A)$ generated by a permutation (i.e. a bijective transformation) g of the set A will be denoted by $\langle f \rangle_G$.

2.2. Theorem. *Let (A,f) be a monounary algebra. The following three assertions are equivalent:*

1° *For every pair of elements $a, b \in A$ there exists an integer $n \in \mathbf{N}_0$ such that either $f^n(a) = b$ or $f^n(b) = a$.*

2° *The algebra (A, f) is nested.*

3° *The algebra (A, f) is connected and if it is acyclic or finite, then $C(f) = \langle f \rangle_G$ and if it is infinite with a non-void cycle, then $C(f) = \langle f, g \rangle$, where g is a connected r-potent with $r = R(A, f)$.*

Proof. Let 1° hold. Let (A_1, f_1), (A_2, f_2) be different subalgebras of (A,f). If $A_1 \neq A_1 \cap A_2 \neq A_2$ then there exist elements $a \in A_1 - A_2$, $b \in A_2 - A_1$ with $f^n(a) \neq b$ and $f^n(b) \neq a$ for each $n \in \mathbf{N}$, which contradicts 1°. Hence A_1, A_2 are comparable, thus 2° holds.

Let 2° hold. Since components are subalgebras, the algebra (A,f) is connected. If (A,f) is a cycle or a two-way infinite chain then by Theorem 2.4 [16] it holds $C(f) = \langle f \rangle_G$, where $\langle f \rangle_G$ is a finite or an infinite cyclic group with the generator f. If (A, f) has one generator, i.e. it is a one-way infinite chain or a cycle with a finite chain, then by Theorem 2.5 [16] we have $C(f) = \langle f \rangle$. Now, let (A, f) be a cycle with an infinite chain. Denote by a an element of $A_f^{\infty_2}$ with $f^{-1}(a) \nsubseteq A_f^{\infty_2}$. Let g be a mapping of the set A onto $A_f^{\infty_2}$ such that $f^k . g(x) = a$, where k is the least integer with the property $f^k(x) = a$. The mapping g defined in this way is connected.

We show that $\{f, g\}$ is a set of generators of $C(f)$. Let $h \in C(f)$. Then either $h(A) = A$ or $h(A) = A_f^{\infty_2}$. Indeed, if for some $a \in A$ the element $h(a)$ belongs to $A - A_f^{\infty_2} = A_f^{\infty_1}$, then for arbitrary $b \in A_f^{\infty_1}$ there exists $n \in \mathbf{N}_0$ with $f^n(b) = h(a)$ or $f^k \,.\, h(a) = b$ for a suitable $k \in \mathbf{N}_0$. There exists $c \in A_f^{\infty_1}$ such that either $f^n(c) = b$ or $f^k(a) = c$. Then $h(c) = b$ for $h \in C(f)$. Now, suppose the first case occurs. Since $x \in A_f^{\infty_2}$ is followed by $h(x) \in A_f^{\infty_2}$ we have that for each $x \in A$ there exists a non-negative integer n with $h(x) = f^n(x)$ (cf. the construction described in Definition 9 [12]). Let $x_1, x_2 \in A$, m be the least integer with $f^m(x_1) = x_2$. Denote by n_1, n_2 the least integers satisfying the conditions $f^{n_1}(x_1) = h(x_2)$, $f^{n_2}(x_2) = h(x_2)$. Then $f^{n_2} \,.\, f^m(x_1) = h \,.\, f^m(x_1) = f^m \,.\, h(x_1) = f^m \,.\, f^{n_1}(x_1)$, i.e. $f^{m+n_2}(x_1) = f^{m+n_1}(x_1)$. There exists an integer $k \in \mathbf{N}_0$ with $k \,.\, R(A, f) = |\, m + n_2 - (m + n_1) \,| = |\, n_2 - n_1 \,|$. From the minimality of m, n_1, n_2 it follows $k = 0$, thus $n_1 = n_2$ and we have $h = f^n$ for a suitable non-negative integer n. Suppose $h(A) = A_f^{\infty_2}$. Since $A_f \neq \emptyset$, it is easy to see that for each non-negative integer n there exists a pair of elements $a, b \in A - A_f^{\infty_2}$ such that $f^n(a) = b$, hence $h \neq f^n$ for each $n \in \mathbf{N}_0$. Let $a \in A_f^{\infty_2}$ be an element with the property $f^{-1}(a) \nsubseteq A_f^{\infty_2}$. Consider the least integer k with $h(a) = f^k(a)$. Let $x \in A_f^{\infty_2}$ and n be the least integer with $x = f^n(a)$. Then $h(x) = h \,.\, f^n(a) = f^n \,.\, h(a) = f^n \,.\, f^k(a) = f^k \,.\, f^n(a) = f^k(x) = f^k \,.\, g(x)$. If $x \in A - A_f^{\infty_2}$ and m is the least integer having the property $f^m(x) = a$, then $f^m \,.\, h(x) = h \,.\, f^m(x) = h(a) = f^k(a)$. According to the definition of the mapping g there is $g(x) = y$, whenever $f^m(y) = a$. Thus $f^m \,.\, h(x) = f^k(a) = f^k \,.\, f^m(y) = f^m \,.\, f^k \,.\, g(x)$. Further $h(x) \in A_f^{\infty_2}$, $g(x) \in A_f^{\infty_2}$, thus also $f^k \,.\, g(x) \in A_f^{\infty_2}$. Consequently $h(x) = f^k \,.\, g(x)$. This equality holds for each $x \in A$, thus $h = f^k \,.\, g$. Consequently the condition 3° is satisfied.

Suppose 3° holds. If $C(f) \in \{\langle f \rangle, \langle f \rangle_G\}$, then by Theorems 2.4 and 2.5 from [16] we get that (A, f) is one of these forms: a cycle, a two-way infinite chain, a cycle with a finite chain, a one-way infinite chain. Then evidently condition 1° is satisfied in this case. Suppose $C(f) = \langle f, g \rangle \neq \langle f \rangle$, is a suitable r-potent, $r \geqq 2$. Admit there exists a pair of elements $a, b \in A$ such that $f^n(a) \neq b$ and at the same time $f^n(b) \neq a$ for each $n \in \mathbf{N}_0$. We shall analyze the three following cases:

(i) $a, b \in A_f^{\infty_1}$, (ii) $a \in A_f^{\infty_1}$, $b \in A - (A_f^{\infty_1} \cup A_f^{\infty_2})$, (iii) $a, b \in A - (A_f^{\infty_1} \cup A_f^{\infty_2})$. In case (i) we have $S_f(a) = S_f(b) = \infty_1$, $S_f(f^n(a)) = S_f(f^n(b))$ for each $n \in \mathbf{N}$ thus by Proposition 1.4 there exist different endomorphisms h_1, h_2 of (A, f) such that $h_1(a) = b$, $h_2(b) = a$. Then evidently $C(f) \neq \langle f, g \rangle$ for every r-potent $g \in T(A)$, which is a contradiction. Consider case (ii). We have $S_f(a) = \infty_1$ again, $S_f(b)$ is an ordinal, thus $S_f(f^n(b)) \leqq S_f(f^n(a))$ for each $n \in \mathbf{N}_0$. By Proposition 1.4 there exists an endomorphism of (A, f), say h_1, with $h_1(b) = a$. If $A_f^{\infty_2} \neq \emptyset$ then $C(f)$ contains a mapping h_2 of A onto $A_f^{\infty_2}$ such that $h_2 \notin \langle f, h_1 \rangle$ and $h_1 \notin \langle f, h_2 \rangle$ and we get a contradiction. If $A_f^{\infty_2} = \emptyset$, we can suppose $\delta(a, b) = 0$. Then h_1 is disconnected and if g is an endomorphism with $\delta(g(b), b) < 0$, then g is not r-potent for any $r \in \mathbf{N}$.

Let the case (iii) occur. If $A_f^{\infty 1} \neq \emptyset$ we can use the same consideration as in the case (ii). Let $A_f^{\infty 1} = \emptyset$. There exists a pair of different elements $a_0, b_0 \in A_f^0$ and a pair of positive integers m, n with $f^m(a_0) = f^n(b_0)$. Suppose that m, n are the least integers with the above property. If $S_f(f^k(a_0)) = k$ for $k = 0, 1, 2, \ldots, m$, then by Proposition 1.4 there exists $h \in C(f)$ with $h(a_0) = f^{n-1}(b_0)$, thus $h \notin \langle f \rangle$ and $h \notin \langle f, g \rangle$ if $A_f^{\infty 2} \neq \emptyset$ and $g \in C(f)$ maps A onto $A_f^{\infty 2}$. This is a contradiction. Let $S_f(f^k(a_0)) > k$ for some $k \in \{1, 2, \ldots, m\}$. Suppose that k is the least integer with this property. There is an element $c_0 \in A_f^0$, $c_0 \neq a_0$ such that $f^k(a_0) = f^p(c_0)$ implies $k > p$. Denote by c_1 the only element of the set $[c_0)_f \cap f^{-1}(f^k(a_0))$. Then evidently $S_f(f^n(a_0)) \leqq S_f(f^n(c_1))$ for each $n \in \mathbf{N}_0$ thus again with regard to Proposition 1.4 there exists an endomorphism of (A, f) which maps the element a_0 onto the element c_1. In the same way as above we get a contradiction. Therefore to the pair $a, b \in A$ there exists an integer $n \in N_0$ such that either $f^n(a) = b$ or $f^n(b) = a$. Hence condition 1° is satisfied, q.e.d.

Recall that the set of all idempotent of a semigroup $\mathscr{S}$ is denoted by $\mathrm{Id}\mathscr{S}$.

Corollary. *Let (A, f) be a nested monounary algebra. Then $\langle \mathrm{Id}\boldsymbol{C}(f) \rangle$ is a submonoid of a monoid consisting the identity and one left zero of $\boldsymbol{T}(A)$.*

Proof. Let (A, f) be a nested monounary algebra. If $R(A, f) = 0$ then by 3° of Theorem 2.2 it holds $\boldsymbol{C}(f) \in \{\langle f \rangle, \langle f \rangle_G\}$. Then $\mathrm{Id}\boldsymbol{C}(f)$ contains the only element id_A for $f^2 \neq f$ and thus $\langle \mathrm{Id}\boldsymbol{C}(f) \rangle$ is trivial. Suppose $R(A, f) > 0$. If $R(A, f) > 1$ then for every $g \in \boldsymbol{C}(f)$, $g \neq \mathrm{id}_A$ having the property $g(A) = A_f^{\infty 2}$ it holds $g^2 = g$. Then $\langle \mathrm{Id}\boldsymbol{C}(f) \rangle$ is trivial again. If $R(A, f) = 1$ and z_f is the cyclic element of (A, f) then the constant mapping $g \in A^A$ with the value z_f belongs to $\mathrm{Id}\boldsymbol{C}(f)$, thus $\mathrm{Id}\boldsymbol{C}(f) = \{\mathrm{id}_A, g\} = \langle \mathrm{Id}\boldsymbol{C}(f) \rangle$.

2.3. Proposition. *Let A be a non-empty set, $f \in \boldsymbol{T}(A)$. The monounary algebra (A, f) is nested iff for every pair of left zeros $g_1, g_2 \in T(A)$ and a suitable $g \in \{g_1, g_2\}$ it holds $\langle f, g_1, g_2 \rangle = \langle f \rangle \cdot g$.*

Proof. Assume (A, f) is a nested monounary algebra, $g_1, g_2 \in \boldsymbol{T}(A)$ are different left zeros, i.e. constant transformation of the set A. Denote by a the value of g_1, by b the value of g_2. By Theorem 2.2 there exists an integer $n \geqq 1$ such that either $f^n(a) = b$ or $f^n(b) = a$. Consider the first possibility. Let $h \in \langle f \rangle \cdot g_2$. Then there exists a non-negative integer k with the property $h = f^k \cdot g_2$ and for each element $x \in A$ we have $h(x) = f^k \cdot g_2(x) = f^k(b) = f^k \cdot f^n(a) = f^{k+n} g_1(x)$, thus $h \in \langle f \rangle \cdot g_1$, hence $\langle f \rangle \cdot g_2 \subseteqq \langle f \rangle \cdot g_1$. Now, let $h \in \langle f, g_1, g_2 \rangle$. Since $g_i \cdot g_j = g_i$ for $i, j \in \{1, 2\}$, $g_i \cdot f^n = g_i$ for $i = 1, 2$ and every $n \in \mathbf{N}_0$, there exists a non-negative integer m such that either $h = f^m \cdot g_1$ or $h = f^m \cdot g_2$. Hence $\langle f, g_1, g_2 \rangle = \langle f \rangle \cdot g_1 \cup \cup \langle f \rangle \cdot g_2 = \langle f \rangle \cdot g_1$. In the same way we get that the assumption $f^n(b) = a$ is followed by $\langle f, g_1, g_2 \rangle = \langle f \rangle \cdot g_2$.

Let $a, b \in A$ be arbitrary elements, g_1, g_2 left zeros of $\boldsymbol{T}(A)$ such that $g_1(x) = a$, $g_2(x) = b$ for each $x \in A$. Then for g_1, g_2 and suitable $g \in \{g_1, g_2\}$ it holds

$\langle f, g_1, g_2 \rangle = \langle f \rangle . g$ according to the supposition. Assume $g = g_1$. Then $\langle f \rangle . g_1 \cup \cup \langle f \rangle . g_2 = \langle f, g_1, g_2 \rangle = \langle f \rangle . g_1$, thus $\langle f \rangle . g_2 \subseteqq \langle f \rangle . g_1$. Further, there exists an integer $n \in \mathbf{N}_0$ with $g_2 = f^n . g_1$. Let $x \in A$ be an arbitrary element. Then $b = g_2(x) = f^n . g_1(x) = f^n(a)$. In the same way we get also that the equality $g = g_2$ is followed by $f^n(b) = a$. Therefore condition 2° from Theorem 2.2 is satisfied.

Another characterizations of a nested monounary algebra use binary operations Δ_f, ∇_f, especially solution sets of equations of the type $a \Delta_f x = b$, $a, b \in A$. If ε denotes a binary operation on the set A, $a, b \in A$. If ε denotes a binary operation on the set A, $a, b \in A$ we put $S(\varepsilon, a, b) = \{x \in A : a \varepsilon x = b\}$, $S_1(\varepsilon, a, b) = = \{x \in A : a \varepsilon x = b, x \neq a\}$, $S_2(\varepsilon, a, b) = \{x \in A : a \varepsilon x = b, x \neq b\}$. By $\sim$ there is denoted the following congruence on the algebra (A, f): $a, b \in A$, $a \sim b$ if either $a = b$ or $a, b \in A_f^{\infty 2}$.

Convention. If (A_0, f_0) is a monounary algebra, we shall write Δ_0, ∇_0 instead of Δ_{f_0}, ∇_{f_0} respectively.

2.4. Theorem. *Let (A, f) be a connected monounary algebra. Denote by ε one of binary operations Δ_0, ∇_0 on a monounary algebra (A_0, f_0) which is a factor-algebra of (A, f) by the congruence $\sim$. The following assertions are equivalent*:

1° *The algebra (A, f) is nested.*

2° (A_0, ε) *is a commutative groupoid such that for every pair of elements $a, b \in A_0$ the equality $a \varepsilon a = b \varepsilon b$ is followed either by $a = b$ or by $a \neq b$, $\{a, b\} \cap \mathrm{Id}(A_0, \varepsilon) \neq \neq \emptyset$.*

3° *If $a, b \in A_0$ is a pair of different elements then either* card $S_2(\Delta_0, a, b) \leqq 1$ *or* card $S_2(\Delta_0, a, b) > 1$ *and $c \in S_1(\Delta_0, a, b)$ implies $S(\Delta_0, c, a) \cap S(\Delta_0, a, b) \neq \emptyset$.*

Proof. Let 1° hold. The factor algebra (A_0, f_0) is nested with at most one-element cycle. If $a, b \in A_0$, $a \neq b$ then either $\delta(a, b) > 0$ or $\delta(a, b) < 0$, thus $a \Delta_0 b = = f_0(a) = b \Delta_0 a$, $a \nabla_0 b = f_0(b) = b \nabla_0 a$ in the first case and $a \Delta_0 b = f_0(b) = b \Delta_0 a$, $a \nabla_0 b = f_0(a) = b \nabla_0 a$ in the second case. Further, let $a, b \in A_0$ be elements satisfying the equality $a \varepsilon a = b \varepsilon b$. Let $a \neq b$. By Theorem 2.2 (2°) there exists $n \in \mathbf{N}$ such that either $f_0^n(a) = b$ or $f_0^n(b) = a$. Consider the first possibility. If $n = 1$ then $f_0^2(a) = f_0(b) = f_0(a) = b$ is a cyclic element of the algebra (A_0, f_0), if $n > 1$ then $b = f_0^n(a) = f_0^{n-1} . f_0(a) = f_0^{n-1} . f_0(b) = f_0^n(b)$, thus $\{b\} = Z(A_0, f_0)$ again. In the case $f_0^n(b) = a$ we get in the same way as above $\{a\} = Z(A_0, f_0)$. Thus $\{a, b\} \cap \mathrm{Id}(A_0, \varepsilon) \neq \emptyset$. Consequently 2° is valid.

Suppose the condition 2° is satisfied. Let $a, b \in A_0$ be elements with the property card $S_2(\Delta_0, a, b) > 1$. Since with respect to the definition of the operation Δ_0 from $\delta(a, b) \geqq 0$ it follows $S_2(\Delta_0, a, b) = \emptyset$ we have $\delta(a, b) < 0$. The groupoid (A_0, Δ_0) is commutative, i.e. $x, y \in A_0$, $\delta(x, y) = 0$ is followed by $f_0(x) = f_0(y)$ and since $S_2(\Delta_0, a, b) \neq \emptyset$, we have $a <_f b$. Admit $f_0(a) \neq b$. Since $S_1(\Delta_0, a, b) = = \emptyset$ (with $a \neq b$) is followed by card $S_2(\Delta_0, a, b) \leqq 1$ for $S_1(\Delta_0, a, b) \cup S_2(\Delta_0, a, b) =$

$= S(\Delta_0, a, b) = \{a\} \cup S_1(\Delta_0, a, b)$, the solution set $S_1(\Delta_0, a, b)$ is non-empty. Then $[f_0^{-1}(b) - \{b\}] \cup S_1(\Delta_0, a, b) = S_2(\Delta_0, a, b)$ and $[f_0^{-1}(b) - \{b\}] \cap \cap S_1(\Delta_0, a, b) \neq \emptyset$. Since card $[(f_0^{-1}(b) - \{b\}) \cap S_1(\Delta_0, a, b)] = 1$ implies card $S_2(\Delta_0, a, b) = 1$, we have that there are different elements $c_1, c_2 \in f_0^{-1}(b) \cap \cap S_1(\Delta_0 a, b)$, i.e. $a\Delta_0 c_1 = b = a\Delta_0 c_2$, i.e. $c_1\Delta_0 c_1 = f_0(c_1) = b = f_0(c_2) = c_2\Delta_0 c_2$. By 2° it holds $c_1 = c_2$, which is a contradiction. Hence $f_0(a) = b$. Now, let $c \in S_1(\Delta_0, a, b)$. Since $\delta(a, c) = 0$ implies $f_0(c) = b$, i.e. $c\Delta_0 c = b = a\Delta_0 a$ thus by 2° we get a contradiction ($a = c$), we have $c <_{f_0} a$. The set $S(\Delta_0, c, a)$ is non-empty. Let $c_1 \in S(\Delta_0, c, a)$. Then $\delta(a, c_1) > 0$, thus $a\Delta_0 c_1 = f_0(a) = b$, i.e. $c_1 \in S(\Delta_0, a, b)$. We get $S(\Delta_0, c, a) \cap S(\Delta_0, a, b) \neq \emptyset$, consequently condition 3° is satisfied.

Suppose 3° is valid. Let $a, b, \in A_0$ be arbitrary elements. Assume $f_0^k(b) \neq a$ for each $k \in \mathbf{N}_0$. Since the algebra (A, f) is connected there exists a pair of integers $m, n \in \mathbf{N}$ such that $f_0^m(a) = f_0^n(b)$. Let m, n be the least integers with the given property. Admit $n \geqq 1$. Then $m \geqq 1$ and we get $\{f_0^{m-1}(a), f_0^{n-1}(b)\} \subset \subset S_2(\Delta_0, f_0^{m-1}(a), f_0^m(a))$. Since $f_0^{m-1}(a) \neq f_0^{n-1}(u)$ we have by 3° that for every element $c \in S_1(\Delta_0, f_0^{m-1}(a), f_0^m(a))$ it holds $S(\Delta_0, c, f_0^{m-1}(a)) \cap S(\Delta_0, f_0^{m-1}(a), f_0^m(a))$ for $c \neq f_0^{n-1}(a)$ and $f_0^{m-1}(a)\, \Delta_0\Delta_0 c = f_0 \cdot f_0^{m-1}(a) = f_0^m(a)$. But for $x \in A_0$ having the property $\delta(x, c) > 0$ it holds $c\Delta_0 x = f_0^{n-1}(a)\, \Delta_0 x = f_0(x)$ and for $x \in A_0$ such that $\delta(x, c) \leqq 0$ it holds $c\Delta_0 x = f_0(c) = f_0(a)$, thus $c\Delta_0 x \neq f_0^{m-1}(a)$ for each $x \in A_0$. From here it follows $S(\Delta_0, c, f_0^{m-1}(a)) = \emptyset$, which is a contradiction. Hence $n = 0, f_0^m(a) = b$ and we have that the algebra (A_0, f_0) is nested and thus (A, f) is nested, too. The proof is complete.

Using notions of a square root and a radical in groupoids, the other simple characterization of a nested monounary algebra can be given. Let (A, ε) be a groupoid, $a \in A$. We put $\sqrt{a} = \{x \in A : x\varepsilon x = a\}$. Every element $b \in \sqrt{a}$ is called a *square root* of the element a in the groupoid (A, ε). If $\sqrt{a} = \emptyset$ then we say that the element a possesses no root, if card $\sqrt{a} = 1$ we say that the element a possesses a unique square root in (A, ε) and we denote this only square root of the element a by $\sqrt{a}$. Notice that for a c-algebra (A, f) with the one-element cycle $\{e\}$ it holds $\sqrt{e} = \mathrm{rad}_w \{e\}$ in the groupoid (A, ∇_f). In what follows there is again denoted by $\sim$ the above defined congruence on (A, f) i.e. (for an arbitrary monounary algebra) equivalence classes of $\sim$ are cycles and singletons disjoint with cycles.

2.5. Theorem. *A monounary algebra is nested iff the following two conditions are satisfied*:

1° *The groupoid* (A_0, Δ_0) $((A_0, \nabla_0))$ *is either ideal-simple or for each its right (left) ideal J it holds* $\mathrm{rad}_s J = A_0$.

2° *The groupoid* (A_0, ε), $\varepsilon \in \{\Delta_0, \nabla_0\}$ *contains at most one idempotent e, at most one element a_0 possessing no root and every element $x \in A_0$, $x \neq e$ possessing at most one square root in the groupoid* (A_0, ε).

Proof. Let (A,f) be a nested monounary algebra. Then (A_0,f_0) is of the same property. Let card $A_0 > 1$. Each subgroupoid of the commutative groupoid (A_0, Δ_0) is its ideal. If J is an ideal then $J = \{f_0^k(a)\colon k = 0, 1, 2, \ldots\}$, $a \in A_0$ and for every $x \in A_0$ there exists an integer n such that $f_0^n(x) \in J$. Then $((\ldots((x\Delta_0 x) \times \Delta_0 x)\ldots)\Delta_0 x) = f_0^n(x) \in J$, ($x$ n-times), thus $x \in \mathrm{rad}_s J$, consequently $\mathrm{rad}_s J = A_0$. Consider the groupoid (A_0, ∇_0). If $A_{f_0}^0 = \emptyset$ then the groupoid (A_0, ∇_0) is ideal-simple. Indeed, denote by J an ideal of (A_0, ∇_0) and admit that there exists an element $a \in A_0$ with $J = \{f_0^k(a)\colon k = 0, 1, 2, \ldots\}$. Denote by b, c elements of A_0 satisfying the conditions $f_0(b) = a$, $f_0(c) = b$. Then $c\nabla_0 a = a\nabla_0 c = b \notin J$, thus $J = A_0$. If $A_{f_0}^0 \neq \emptyset$, i.e. $A_{f_0}^0 = \{a_0\}$ then the underlying set of the only proper ideal J is $\{f_0^k(a_0)\colon k = 1, 2, \ldots\}$ and evidently $\mathrm{rad}_s J = \mathrm{rad}_w J = A_0$. Condition 1° is satisfied.

Let $\varepsilon \in \{\Delta_0, \nabla_0\}$. There is $\mathrm{Id}(A_0, \varepsilon) = Z(A_0, f_0)$ – a one-element cycle. If $a_0 \in A_{f_0}^0$, then $\sqrt{a_0} = \emptyset$. (it is card $A_{f_0}^0 \leqq 1$). For $x \in A_0$ such that $f_0^{-1}(x) \neq \emptyset$ and $f_0(x) \neq x$ it hold card $\sqrt{x} = 1$, where the square root is considered in (A_0, ε). Hence condition 2° is also satisfied.

Suppose (A,f) is a monounary algebra such that the groupoid (A_0, ε), where $\varepsilon \in \{\Delta_0, \nabla_0\}$, on the factor-algebra (A_0, f_0) of (A,f) satisfies conditions 1° and 2°. If (A_{0i}, f_{0i}) is a component of the algebra (A_0, f_0) we have by the definition of operations Δ_0, ∇_0 that A_{0i} is the carrier set of a right, left ideal of the groupoid (A_0, Δ_0), (A_0, ∇_0) respectively. If the algebra (A_0,f_0) contains at least two different components, say (A_{01},f_{01}), (A_{02},f_{02}), then for arbitrary $a \in A_{02}$ it holds $f^n(a) \notin A_{01}$ for every $n \in \mathbf{N}_0$, thus $[a^n] \cap A_{01} = \emptyset$ and we have $\mathrm{rad}_s A_{02} \neq A_0$, which contradicts the assumption. Hence the algebra (A_0, f_0) is connected. Consider a pair of elements a, $b \in A_0$ with $f_0(a) = f_0(b)$. Put $c = f_0(a) = f_0(b)$. Then $a\varepsilon a = b\varepsilon b = c$, where $\varepsilon \in \{\Delta_0, \nabla_0\}$, thus $a, b \in \sqrt{c}$. By 2° there is either $a = b$ or $c\varepsilon c = c$. But in the second case we have $f_0(c) = c$, hence $(A_0, \leqq_{f_0})$ is a chain, therefore the algebra (A_0, f_0) is nested and consequently (A, f) is of the same property.

Recall that a congruence Θ of the monounary algebra (A, f) is an equivalence relation on A satisfying the substitution property: $(a, b) \in \Theta$ implies $(f(a), f(b)) \in \Theta$. A congruence of the monounary algebra (A,f) is said to be fully invariant if, for any $g \in C(f)$, $(a, b) \in \Theta$ implies $(g(a), g(b)) \in \Theta$. Using Theorem 5.1 [16] we get the following characterization:

2.6. Theorem. *Let (A, f) be a c-algebra. The following conditions are equivalent:*

1° *All congruences of (A, f) are fully invariant.*

2° *(A,f) is nested and for each $a \in A$ and every sequence $\{a_n\}_{0 \leqq n < \omega_0}$ of elements from A the sequence of integers $\{\delta(a, a_n)\}_{0 \leqq n < \omega_0}$ has at most one improper cluster point.*

Proof. By Theorem 5.1 [16] we have that 1° is equivalent to the condition that (A,f) is nested but not a two-way infinite chain. Thus it is sufficient to prove that (A,f) is a two-way infinite chain iff there is a point $a \in A$ and a sequence $\{a_n\}_{0 \leqq n < \omega_0} \subseteqq A$ such that the sequence of integers $\{\delta(a, a_n)\}_{0 \leqq n < \omega_0}$ has two improper cluster points ($+\infty$ and $-\infty$). Let (A,f) be two-way infinite, $a \in A$ be an arbitrary element. Putting $a_{2k} = f^k(a)$, $k = 0, 1, 2, \ldots$ and $a_{2k-1} = x$, where $f^k(x) = a$, $k = 1, 2, \ldots$, we obtain the sequence with the above mentioned property. On the contrary, let (A, f) be a c-algebra such that the sequence $\{\delta(a, a_n)\}_{0 \leqq n < \omega_0}$ for some $a \in A$ and $\{a_n\}_{0 \leqq n < \omega_0} \subseteqq A$ has two improper cluster points $-\infty$ and $+\infty$. Then there exist subsequences $\{p_{n_k}\}$, $\{q_{n_k}\}$ of $\{\delta(a, a_n)\}$ with limits $\lim p_{n_k} = -\infty$, $\lim q_{n_k} = +\infty$. Then the set of members of the sequence $\{a_{n_k}\}$ such that $\delta(a, a_{n_k}) = p_{n_k}$ is unbounded from above in $(A_0, \leqq_{f_0})$ (A_0 is a factor-set of A in the above defined congruence $\sim$ on the algebra (A,f)) and similarly A_0 contains a decreesing chain without any lower bound. Hence $R(A,f) = 0$, $A_f^{\infty 1} \neq \emptyset$. Since (A,f) is nested, it is a two-way infinite chain.

A complete survey of obtained results concerning congruences on monounary algebras and other related problems is contained in the paper of L. A. Skornjakov [14].

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J. Chvalina
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia