Eugen Viszus

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On the regularity up to the boundary for higher order quasilinear elliptic systems

EUGEN VISZUS

Abstract. The partial regularity up to the boundary of weak solutions to the Dirichlet problem for higher order quasilinear elliptic systems is proved. The proof of partial regularity is direct.

Keywords: Quasilinear elliptic system, weak solution, partial regularity up to the boundary
Classification: 35J60, 35B65

1. Introduction. Using the direct method from[3], [4], we shall prove partial regularity of weak solutions up to the boundary.

We shall consider the following problem:

$$(1.1)\quad \sum_{j=1}^{N} \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} (-1)^{|\alpha|} D^\alpha (A_{ij}^{\alpha\beta}(x,\delta(u)) D^\beta u^j) = 0, \qquad x \in \Omega, \quad i = 1,\dots,N,$$

$$(1.2)\qquad D^\alpha u^i \mid_\Gamma = 0, \qquad i=1,\dots,N, \quad |\alpha| \le m_i - 1,$$

where $n \ge 2$, $\Omega = Q(0,b) \cap \{x \in \mathbf{R}^n : x_n > 0\}$,
$Q(y,a) = \{x \in \mathbf{R}^n : |x_i - y_i| < a, \quad i = 1,\dots,N\}$, $a > 0$,
$\Gamma = Q(0,b) \cap \{x \in \mathbf{R}^n : x_n = 0\}$, $m_i \ge 1$, m_i is integer for $i = 1,\dots,N$,
$\delta(u) = \{D^\alpha u^i : |\alpha| \le m_i - 1, \quad i = 1,\dots,N\}$.

Let us denote $\kappa = \sum_{i=1}^N \binom{n+m_i-1}{m_i-1}$. We suppose that

$$(1.3)\qquad \begin{cases} A_{ij}^{\alpha\beta} \text{ are uniformly continuous on } \overline{\Omega} \times \mathbf{R}^\kappa \\ |A_{ij}^{\alpha\beta}| \le L \text{ on } \overline{\Omega} \times \mathbf{R}^\kappa, \quad L > 0. \end{cases}$$

$$(1.4)\qquad \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} A_{ij}^{\alpha\beta}(x,\xi)\xi_i^\alpha \xi_j^\beta \ge \nu \|\xi\|^2, \qquad \nu > 0$$

for all $(x,\xi) \in \overline{\Omega} \times \mathbf{R}^\kappa$ and $\xi \in \mathbf{R}^\vartheta$, $\vartheta = \sum_{i=1}^N \binom{n+m_i-1}{m_i}$. By a weak solution of the problem (1.1), (1.2) we mean a function $u \in H^{\underline{m}}(\Omega)$ ($H^{\underline{m}}(\Omega) = H^{m_1} \times \dots \times H^{m_N}(\Omega)$, $H^{m_i}(\Omega)$ – Sobolev space for $i = 1,\dots,N$, $u = (u^1,\dots,u^N)$ – see [8]) such that

$$(1.5)\qquad \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_\Omega A_{ij}^{\alpha\beta}(x,\delta(u)) D^\beta u^j D^\alpha \varphi^i \, dx = 0$$

for all $\varphi \in H_0^{\underline{m}}(\Omega)$ and u satisfies (1.2) in the sense of traces.

The main result of this paper is

Theorem 1.1. *Let* (1.3), (1.4) *be satisfied and let* $u \in H^{\underline{m}}(\Omega)$ *be the weak solution to the problem* (1.1), (1.2). *Then there exists* $\Omega_0 \subset (\Omega \cup \Gamma)$ *(open in* $\Omega \cup \Gamma$*) such that* $u \in C^{\underline{m-1},\mu}_{\text{loc}}(\Omega_0)$, $\mu \in (0,1)$ *and* $\dim_H((\Omega \cup \Gamma) \setminus \Omega_0) \leq n - p$, $p > 2$. *(*$\dim_H$ *- Hausdorff dimension,* $n \geq 3$*).*

This theorem generalizes the result of [1] and [4]. In [1] partial regularity up to the boundary is proved by another indirect approach for systems of second order. In [4], interior regularity is proved by direct approach for systems of higher order. Our proof is interesting from the methodical point of view too.

2. The interior regularity. In this part we shall formulate some assertions, which we do not prove. The proofs of the assertions are analogous to those in [4] or [3].

Theorem 2.1 (L_p-estimate in the interior). *Let* (1.3), *(1.4) be satisfied and let* $u \in H^{\underline{m}}(\Omega)$ *be the weak solution to the* (1.1). *Then there exists* $p > 2$ *such that* $u \in H^{\underline{m},p}_{\text{loc}}(\Omega)$. *Moreover, there exists a constant* $c_1 = c_1(n, N, \underline{m}, L, \nu)$ *such that for all* $x_0 \in \Omega$ *and* $0 < R < \frac{1}{2}\min\{\text{dist}(x_0, \partial\Omega), 2\}$ *the following inequality holds:*

$$(2.1) \qquad \left(\fint_{Q(x^0,R)} |D^{\underline{m}} u|^p \, dx \right)^{2/p} \leq c_1 \fint_{Q(x^0,2R)} |D^{\underline{m}}.u|^2 \, dx.$$

By $\fint_\Omega f\, dx$ *we mean the integral mean value of* f *in* Ω *and* $D^{\underline{m}}u = \{D^\alpha u^i : |\alpha| = m_i, \quad i = 1, \dots, N\}$.

By using the Sobolev's lemma we get

Corollary 2.1. *Let the assumptions of Theorem 2.1 be satisfied and let* $n = 2$. *Then* $u \in C^{\underline{m-1},\mu}_{\text{loc}}(\Omega)$, $\mu = 1 - \frac{n}{p}$.

Let for $u^i \in H^{m_i}(Q(x_0, R))$, $i = 1, \dots, N$, the polynomials $P^i(x) = P^i(x^0, R, u^i, x)$, $x \in Q(x^0, R)$, be such that $\deg(P^i) \leq m_i - 1$ and $\int_{Q(x^0,R)} D^\alpha(u^i - P^i)\, dx = 0$ for all multiindices $\alpha : |\alpha| \leq m_i - 1$.

Let us denote $[P_{x^0,R}] = 1 + \sum_{i=1}^N \sum_{|\alpha|<m_i} |c^i_\alpha|$, where c^i_α are coefficients of polynomial P^i, $i = 1, \dots, N$.

The crucial point in the proof of regularity (for $n \geq 3$) is

Lemma 2.1. *Let the assumptions of Theorem 2.1 be satisfied. Then there exists a constant* $c_2 = c_2(n, N, \underline{m}, L, \nu)$ *such that for all* $x_0 \in \Omega$ *and* $0 < \rho < R < \min\{\text{dist}(x_0, \partial\Omega), 1\}$ *the inequality*

$$(2.2) \qquad \int_{Q(x^0,\rho)} |D^{\underline{m}} u|^2 \, dx \leq c_2 \int_{Q(x^0,R)} |D^{\underline{m}} u|^2 \, dx \cdot \left\{ \left(\frac{\rho}{R}\right)^n + \chi(x^0, R) \right\}$$

holds. Here $\chi(x^0, R) = \left\{ \omega(c_3[R^2 + R^{2-n} \int_{Q(x^0,R)} |D^{\underline{m}} u|^2 \, dx]\right\}^{1-2/p}$, *where* ω *is the modulus of continuity of the functions* $A^{\alpha\beta}_{ij}$, $c_3 = c_3([P_{x^0,R}])$.

Using Lemma 2.1 and the method of induction we could prove

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Then there exists an open set* $\Omega_0 \subset \Omega$ *such that* $u \in C^{\underline{m-1},\mu}_{\mathrm{loc}}(\Omega_0)$, $0 < \mu < 1$ *and* $\dim_H(\Omega\setminus\Omega_0) \le n-p$, $p > 2$.

3. Regularity up to the boundary. In this part we shall prove the regularity of weak solutions to (1.1), (1.2) "near" the boundary Γ. For points $x^0 \in \Gamma$ we shall prove the assertions analogous to those in Part 2. The assertionsa in Parts 2 and 3 will imply Theorem 1.1.

In our proofs we shall use

Lemma 3.1(proved in [9]). *Let* $M = \{u \in H^{1,p}(Q(x^0,R)) : u = 0$ *on* S *with* $\operatorname{meas}(S) \ge c_1[\operatorname{meas}(Q(x^0,R))]$, $c_1 > 0\}$ $1 \le p < \infty$.

Then there exists a constant $c = c(n,p,c_1) > 0$ *such that*

$$\int_{Q(x^0,R)} |u|^p\,dx \le cR^p \int_{Q(x^0,R)} |\nabla u|^p\,dx, \qquad u \in M.$$

Lemma 3.2 (Cacciopoli's inequality). *Let* (1.3), (1.4) *be satisfied and let* $u \in H^{\underline{m}}(\Omega)$ *be the weak solution to* (1.1), (1.2). *Then there exists a constant* $c' = c'(n,N,\underline{m},L,\nu)$ *such that for all* $x^0 \in \Gamma$ *and* $0 < R < \frac{1}{2}\operatorname{dist}(x^0,\partial\Omega\setminus\Gamma)$ *the inequality*

$$\int_{Q'(x^0,R)} |D^{\underline{m}}u|^2\,dx \le \frac{c'}{R^2}\int_{Q'(x^0,2R)} |D^{\underline{m-1}}u|^2\,dx \tag{3.1}$$

holds. ($Q'(x^0,r) = Q(x^0,r)\cap(\Omega\cup\Gamma)$, $x^0 \in \Omega\cup\Gamma$, $r > 0$.)

PROOF : Let $x^0 \in \Gamma$, $0 < R < \frac{1}{2}\operatorname{dist}(x^0,\partial\Omega\setminus\Gamma)$. If $\eta \in C_0^\infty(Q(x^0,2R))$, $0 \le \eta \le 1$, $\eta = 1$ in $Q(x^0,R)$ and $|D^\alpha\eta| \le c_1' R^{-|\alpha|}$, $|\alpha| \le a$, $a = \max\{m_i\}_{i=1}^N$, choosing $\varphi^i = u^i\overline{\eta}^{2a}$ ($i = 1,\dots,N$, $\overline{\eta}$ is the restriction of η to $Q'(x^0,2R)$) in (1.5), we get easily, using the formula of Leibniz:

$$\sum_{i,j=1}^N \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x,\delta(u))D^\beta U^j D^\alpha U^i\,dx = \tag{3.2}$$

$$= \sum_{i,j=1}^N \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x,\delta(u))\Big(\sum_{\gamma<\alpha} L_{\alpha\gamma}(\overline{\eta})D^\gamma u^i\Big)\Big(\sum_{\delta<\beta} M_{\beta\delta}(\overline{\eta})D^\delta u^j\Big)\,dx +$$

$$+ \sum_{i,j=1}^N \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x,\delta(u))D^\alpha U^i\Big(\sum_{\delta<\beta} M_{\beta\delta}(\overline{\eta})D^\delta u^j\Big)\,dx -$$

$$- \sum_{i,j=1}^N \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x,\delta(u))D^\beta U^j\Big(\sum_{\gamma<\alpha} L_{\alpha\gamma}(\overline{\eta})D^\gamma u^i\Big)\,dx,$$

where $U^i = \overline{\eta}^a \cdot u^i$, $i = 1, \ldots, n$ and $L_{\alpha\gamma}(\overline{\eta})$, $M_{\alpha\gamma}(\overline{\eta})$ are polynomials which involve derivatives of order $\leq |\alpha| - |\gamma|$ of $\overline{\eta}$ and being such that $|L_{\alpha\gamma}(\overline{\eta})|, |M_{\alpha\gamma}(\overline{\eta})| \leq c_2' R^{|\gamma|-|\alpha|}$. Using estimates of $L_{\alpha\gamma}$, $M_{\alpha\gamma}$ and (1.3), (1.4), it follows from (3.2) that

(3.3)

$$\sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{Q'(x^0,R)} |D^\alpha u^i|^2 \, dx \leq c_3' \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} R^{2(|\gamma|-|\alpha|)} \int_{Q'(x^0,2R)} |D^\gamma u^i|^2 \, dx.$$

The function $u \in H^{\underline{m}}(\Omega)$ satisfies (1.2) and it may be extended by zero from $Q'(x^0, 2R)$ into $Q(x^0, 2R)$. We denote this extension by $\widetilde{u}$. It is clear that $\widetilde{u} \in H^{\underline{m}}(Q(x^0, 2R))$. Using Lemma 3.1 we obtain the estimate:

$$\sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} R^{2(|\gamma|-|\alpha|)} \int_{Q(x^0,2R)} |D^\gamma \widetilde{u}^i|^2 \, dx \leq c_4' R^{-2} \int_{Q(x^0,2R)} |D^{\underline{m-1}} \widetilde{u}|^2 \, dx.$$

This inequality and (3.3) imply (3.1). ∎

Lemma 3.3. *Let the assumptions of Lemma 3.2 be satisfied. Then there exists a constant $c_5' = c_5'(n, N, \underline{m}, L, \nu)$ such that for all $x^0 \in \Gamma$ and $0 < R < \frac{1}{2}\min\{\operatorname{dist}(x^0, \partial\Omega \setminus \Gamma), 2\}$, the inequality*

$$\text{(3.4)} \qquad \fint_{Q'(x^0,R)} |D^{\underline{m}} u|^2 \, dx \leq c_5' \left\{ \fint_{Q'(x^0,2R)} |D^{\underline{m}} u|^q \, dx \right\}^{2/q}, \qquad q = \frac{2n}{n+2}$$

holds.

PROOF : Dividing both sides of (3.1) by $\operatorname{meas}(Q'(x^0, R))$ we have

$$\text{(3.5)} \qquad \fint_{Q'(x^0,R)} |D^{\underline{m}} u|^2 \, dx \leq c_6' R^{-n-2} \int_{Q'(x^0,2R)} |D^{\underline{m-1}} u|^2 \, dx.$$

Let $\widetilde{u}$ be the extension of u by zero from $Q'(x^0, 2R)$ to $Q(x^0, 2R)$. Using the Sobolev lemma and Lemma 3.1 we obtain

$$\int_{Q'(x^0,2R)} |D^{\underline{m-1}} u|^2 \, dx =$$

$$= \int_{Q(x^0,2R)} |D^{\underline{m-1}} \widetilde{u}|^2 \, dx \leq c_7' R^{-2} \left\{ R^q \int_{Q(x^0,2R)} |D^{\underline{m}} \widetilde{u}|^q \, dx \right\}^{2/q}$$

From (3.5) and this estimate it follows that

$$\fint_{Q'(x^0,R)} |D^{\underline{m}} u|^2 \, dx \leq c_8' R^{-n-2} \left\{ \int_{Q'(x^0,2R)} |D^{\underline{m}} u|^q \, dx \right\}^{2/q}$$

This estimate implies (3.4). ∎

Lemma 3.4. *Let* (1.3), (1.4) *be satisfied and let* $u \in H^{\underline{m}}(\Omega)$ *be a weak solution to* (1.1), (1.2). *Then there exists a constant* $c^* = c^*(n, N, \underline{m}, L, \nu)$ *such that for all* $x^0 \in \Omega \cup \Gamma$ *and* $0 < R < \frac{1}{6}\min\{\text{dist}(x^0, \partial\Omega \setminus \Gamma), 6\}$ *the estimate*

$$(3.6) \qquad \fint_{Q'(x^0,R)} |D^{\underline{m}}u|^2\, dx \le c^* \left\{ \fint_{Q'(x^0,6R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q}, \qquad q = \frac{2n}{n+2}$$

holds.

PROOF : Is is known (from proof of Theorem 2.1) that there exists a constant c_1^* such that for all $x^0 \in \Omega$ and R satisfying the inequality $6R < \min\{\text{dist}(x^0, \partial\Omega), 6\}$ the estimate

$$\fint_{Q(x^0,R)} |D^{\underline{m}}u|^2\, dx \le c_9' \left\{ \fint_{Q(x^0,2R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q} \le c_1^* \left\{ \fint_{Q(x^0,6R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q}$$

holds.

Now, let $x^0 \in \Omega$ and $0 < R < \frac{1}{6}\min\{\text{dist}(x^0, \partial\Omega \setminus \Gamma), 6\}$. There are two possibilities:

a) $2R < \delta$, where $\delta = \text{dist}(x^0, x^1)$, $x^1 \in \Gamma$, x^1 - projection of x^0 on Γ.
b) $2R \ge \delta$.

a) If $2R < \delta$, then $\fint_{Q'(x^0,R)} |D^{\underline{m}}u|^2\, dx \le c_3^* \left\{ \fint_{Q'(x^0,6R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q}$. This estimate follows from the interior estimate.
b) If $2R \ge \delta$ then

$$\fint_{Q'(x^0,R)} |D^{\underline{m}}u|^2\, dx \le c_4^* \frac{(R+\delta)^n}{R^n} c_5' \left\{ \fint_{Q'(x^1,2(R+\delta))} |D^{\underline{m}}u|^q\, dx \right\}^{2/q} \le$$
$$\le c_4^* \frac{(R+\delta)^n}{R^n} c_5' \left(c_5^* \frac{(6R)^n}{(2(R+\delta))^n} \right)^{2/q} \cdot \left\{ \fint_{Q'(x^0,6R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q} \le$$
$$\le c_6^* \left\{ \fint_{Q'(x^0,6R)} |D^{\underline{m}}u|^q\, dx \right\}^{2/q}.$$

Putting $c^* = \max\{c_1^*, c_2^*, c_3^*, c_6^*\}$we have (3.6). ■

Remark 3.1. We know that the weak solution $u \in H^{\underline{m}}(\Omega)$ to (1.1), (1.2) may be extended by zero from Ω into $Q(0, b)$.

The extension $\tilde{u}$ belongs to $H^{\underline{m}}(Q(0, b))$, and it is clear that one has for all $x^0 \in Q(0, b) \setminus (\Omega \cup \Gamma)$ and $0 < R < \frac{1}{6}\,\text{dist}(x_0, \partial Q(0, b))$

$$\fint_{Q(x^0,R)} |D^{\underline{m}}\tilde{u}|^2\, dx \le \fint_{Q(x^1,R)} |D^{\underline{m}}\tilde{u}|^2\, dx \le c_5' \left\{ \fint_{Q(x^1,2R)} |D^{\underline{m}}\tilde{u}|^q\, dx \right\}^{2/q} \le$$
$$\le c_7^* \left\{ \fint_{Q(x^0,6R)} |D^{\underline{m}}\tilde{u}|^q\, dx \right\}^{2/q}$$

(x^1 - projection of x^0, $x^1 \in \Gamma$).

Now we may prove

Theorem 3.1. *Let (1.3), (1.4) be satisfied and let $u \in H^{\underline{m}}(\Omega)$ be the weak solution to the problem (1.1), (1.2). Then there exist $p > 2$ and a constant $c_8^* = c_8^*(n, N, \underline{m}, L, \nu)$ such that $u \in H^{\underline{m},p}(Q'(x^0, R))$ for all $x^0 \in \Omega \cup \Gamma$ and for $0 < R < \frac{1}{6}\min\{\operatorname{dist}(x^0, \partial\Omega \setminus \Gamma), 6\}$.*

Moreover, the estimate

$$\left\{ \fint_{Q'(x^0,R)} |D^{\underline{m}}u|^p \, dx \right\}^{2/p} \leq c_8^* \fint_{Q'(x^0,6R)} |D^{\underline{m}}u|^2 \, dx \tag{3.7}$$

holds.

Proof : We shall use the following

Lemma 3.5 ([4, Proposition 5.1]). *Let $Q \subset \mathbf{R}^n$ be a cube, $g \in L^s(Q)$, $s > 1$, $g(x) \geq 0$ on Q. Let the inequality*

$$\fint_{Q(x^0,R)} g^s \, dx \leq b \left(\fint_{Q(x^0,6R)} g \, dx \right)^s + \theta \fint_{Q(x^0,6R)} g^s \, dx$$

be satisfied for all $x^0 \in Q$ and $R < \min\left\{\frac{1}{6}\operatorname{dist}(x^0, \partial Q), R_0\right\}$ where $b > 1$, $R_0 > 0$, $0 \leq \theta < 1$ are constants. Then $g \in L^p_{\mathrm{loc}}(Q)$ for $p \in [s, s+\varepsilon)$ and

$$\left(\fint_{Q(x^0,R)} g^p \, dx \right)^{1/p} \leq c \left(\fint_{Q(x^0,6R)} g^s \, dx \right)^{1/s},$$

where $Q(x^0, 6R) \subset Q$, $R < R_0$. The constants c, ε depend on b, θ, s, n.

Let $\tilde{u} \in H^{\underline{m}}(Q(0,b))$ be the extension of $u \in H^{\underline{m}}(\Omega)$ by zero from Ω to $Q(0,b)$. Let us put $g = |D^{\underline{m}}\tilde{u}|^q$, $q = \frac{2n}{n+2}$, $s = \frac{2}{q} > 1$.

It is clear that $g \in L^s(Q(0,b))$. Lemma 3.4, Remark 3.1 and Lemma 3.5 imply that there exists $r > \frac{2}{q}$ such that $g \in L^r_{\mathrm{loc}}(Q(0,b))$. Putting $p = q.r > 2$, it is clear that $|D^{\underline{m}}\tilde{u}| \in L^p_{\mathrm{loc}}(Q(0,b))$ and for all $x^0 \in Q(0,b)$ and $0 < R < \frac{1}{6}\min\{\operatorname{dist}(x^0, \partial Q(0,b)), 6\}$ the inequality

$$\left\{ \fint_{Q(x^0,R)} |D^{\underline{m}}\tilde{u}|^p \, dx \right\}^{\frac{1}{p}} \leq c_9^* \left\{ \fint_{Q(x^0,6R)} |D^{\underline{m}}\tilde{u}|^2 \, dx \right\}^{\frac{1}{2}}$$

holds. The assertion of the theorem follows. ∎

Corollary 3.1. *Let the assumptions of Theorem 3.1 be satisfied and let $n = 2$. Then $u \in C^{\underline{m-1},\mu}_{\mathrm{loc}}(\Omega \cup \Gamma)$, $\mu = 1 - \frac{n}{p}$.*

Proof : Theorem 3.1 and Sobolev's lemma imply the result. ∎

For $n \geq 3$ we have

Lemma 3.6. *Let (1.3), (1.4) be satisfied and let $u \in H^{\underline{m}}(\Omega)$ be the weak solution to the problem (1.1), (1.2). Then for all $x^0 \in \Gamma$ and $0 < \rho < R < \min\{\operatorname{dist}(x^0, \partial\Omega \setminus \Gamma), 1\}$ the estimate*

$$\int_{Q'(x^0,\rho)} |D^{\underline{m}}u|^2\,dx \le c'_{10} \int_{Q'(x^0,R)} |D^{\underline{m}}u|^2\,dx \left\{ \left(\frac{\rho}{R}\right)^n + \chi(x^0,R) \right\} \tag{3.8}$$

holds.

$c'_{10} = c'_{10}(n, N, \underline{m}, L, \nu)$,

$\chi(x^0,R) = \left\{ \omega\left(c'_{11}\left[R^2 + R^{2-n} \int_{Q'(x^0,R)} |D^m u|^2\,dx \right] \right) \right\}^{1-2/p}$, $c'_{11} = c'_{11}(n, N, \underline{m})$, $p > 2$, *ω is defined in Lemma 2.1.*

PROOF : Let $x^0 \in \Gamma$, $0 < R < \min\{\operatorname{dist}(x^0, \partial\Omega \setminus \Gamma), 1\}$. Put $A^{\alpha\beta}_{ij0} = A^{\alpha\beta}_{ij}(x^0, \theta)$, θ – the zero-vector in $\mathbf{R}^\kappa$ and let $v \in H^{\underline{m}}(Q'(x^0, \frac{R}{6}))$ be the weak solution to the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} (-1)^{|\alpha|} D^\alpha (A^{\alpha\beta}_{ij0} D^\beta v^j) = 0, \qquad i = 1, \dots, N \quad \text{in } Q'(x^0, \frac{R}{6}), \\ (u - v) \in H^{\underline{m}}_0(Q'(x^0, \frac{R}{6})). \end{cases} \tag{3.9}$$

Then the inequality

$$\int_{Q'(x^0,\rho)} |D^{\underline{m}}v|^2\,dx \le c'_{12} \left(\frac{\rho}{R}\right)^n \int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}}v|^2\,dx, \qquad 0 < \rho < \frac{R}{6}, \tag{3.10}$$

holds. (This fact may be proved by the method in [**10**, Lemma 4.2.11].)

Putting $w = (u - v) \in H^{\underline{m}}_0(Q'(x^0, \frac{R}{6}))$, we have

$$\begin{aligned} \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{Q'(x^0,\frac{R}{6})} A^{\alpha\beta}_{ij0} D^\beta w^j D^\alpha \Phi^i\,dx = \\ = \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{Q'(x^0,\frac{R}{6})} [A^{\alpha\beta}_{ij0} - A^{\alpha\beta}_{ij}(x, \delta(u))] D^\beta u^j D^\alpha \Phi^i\,dx, \end{aligned} \tag{3.11}$$

$$\Phi \in H^{\underline{m}}_0(Q'(x^0, \frac{R}{6})).$$

The inequality (3.10) implies

(3.12)

$$\int_{Q'(x^0,\rho)} |D^{\underline{m}}u|^2\,dx \le c'_{13} \left\{ \left(\frac{\rho}{R}\right)^n \int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}}v|^2\,dx + \int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}}w|^2\,dx \right\}.$$

If we put $\Phi = w$ in (3.11) using (1.4) and the Cauchy-Schwartz inequality, we have

(3.13)
$$\int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}} w|^2\,dx \le c'_{14} \int_{Q'(x^0,\frac{R}{6})} \Big(\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} |A^{\alpha\beta}_{ij0} - A^{\alpha\beta}_{ij}(x,\delta(u))| \Big)^2 \cdot |D^{\underline{m}} u|^2\,dx.$$

From (1.3) it is clear that there exists a function $\omega = \omega(t)$, ω is increasing, continuous, concave, bounded, $\lim_{t\to 0^+} \omega(t) = \omega(0) = 0$, such that

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} |A^{\alpha\beta}_{ij}(x,p) - A^{\alpha\beta}_{ij}(y,q)| \le \omega(|x-y|^2 + |p-q|^2),$$

$$x, y \in \overline{\Omega}, \quad p, q \in \mathbf{R}^{\kappa}$$

Using this fact, we have from (3.13)

$$\int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}} w|^2\,dx \le c'_{15} \int_{Q'(x^0,\frac{R}{6})} \omega^2 |D^{\underline{m}} u|^2\,dx, \tag{3.14}$$

$$\omega = \omega(|x-x^0|^2 + \sum_{i=1}^{N} \sum_{|\alpha| \le m_i - 1} |D^\alpha u^i|^2).$$

Now we shall obtain some estimates. The method of estimating is analogous to that in [**3**, Lemma 2.2] or [**4**, Lemma 3.2].

We estimate the right-hand side of (3.14) using Hölder inequality, Theorem 3.1, and boundedness of ω. For $p > 2$ we obtain

$$\int_{Q'(x^0,\frac{R}{6})} \omega^2 |D^{\underline{m}} u|^2\,dx \le c'_{16} \int_{Q'(x^0,R)} |D^{\underline{m}} u|^2\,dx \left(\fint_{Q'(x^0,R)} \omega\,dx \right)^{1-2/p} \tag{3.15}$$

Jensen inequality and Lemma 3.1 imply

(3.16)
$$\fint_{Q'(x^0,R)} \omega\,dx \le \omega(c'_{17}[R^2 + R^{2-n} \int_{Q'(x^0,R)} |D^{\underline{m}} u|^2\,dx]), \quad c'_{17} = c'_{17}(n, N, \underline{m}).$$

Now (3.14), (3.15), (3.16) imply

$$\int_{Q'(x^0,\frac{R}{6})} |D^{\underline{m}} w|^2\,dx \le \tag{3.17}$$

$$\le c'_{18} \int_{Q'(x^0,R)} |D^{\underline{m}} u|^2\,dx \left\{ \omega\big(c'_{17}[R^2 + R^{2-n} \int_{Q'(x^0,R)} |D^{\underline{m}} u|^2\,dx]\big) \right\}^{1-2/p}$$

From (3.12), (3.17) we have (for $0 < \rho < \frac{R}{6}$)

$$(3.18) \quad \int_{Q'(x^0,\rho)} |D^{\underline{m}}u|^2 \, dx \le c'_{19} \int_{Q'(x^0,R)} |D^{\underline{m}}u|^2 \, dx \left\{ \left(\frac{\rho}{R}\right)^n + \chi(x^0,R) \right\},$$

$$\chi(x^0,R) = \omega^{1-2/p}, \quad p > 2.$$

For $\frac{R}{6} \le \rho < R$ the inequality (3.8) is clear: $c'_{10} = \max\{6^n, c'_{19}\}$. ■

Now we can prove

Theorem 3.2. *Let* (1.3), (1.4) *be satisfied and let* $u \in H^{\underline{m}}(\Omega)$ *be the weak solution to the problem* (1.1), (1.2). *Let us put*

$$\Gamma_1 = \left\{ x \in \Gamma : \lim_{R \to 0+} R^{2-n} \int_{Q'(x,R)} |D^{\underline{m}}u|^2 \, dy = 0 \right\}.$$

Then for all $\overline{x} \in \Gamma_1$ *there exists* $\delta > 0$ *such that*

$$u \in C^{\underline{m-1},\mu}(\overline{Q'(\overline{x},\delta)}), \quad \mu \in (0,1).$$

PROOF : For $\overline{x} \in \Gamma$ and $0 < R < \min\{\operatorname{dist}(\overline{x}, \partial\Omega \setminus \Gamma), 1\}$ we put

$$\Psi(\overline{x},R) = R^{2-n} \int_{Q'(\overline{x},R)} |D^{\underline{m}}u|^2 \, dx.$$

Let $c^*_{10} = \max\{c_2, c'_{10}\}$. It follows from (3.8) that for $0 < \tau < 1$

$$(3.19) \qquad \Psi(\overline{x},\tau R) \le c^*_{10} \Psi(\overline{x},R) \tau^2 \{1 + \chi(\overline{x},R) \cdot \tau^{-n}\}.$$

Now let $\overline{x} \in \Gamma_1$ and $\varepsilon_0 > 0$, $R' < 1$ be chosen by such a way that $\Psi(\overline{x},R) < \varepsilon_0$ for $0 < R < R'$. It follows from the construction of $[P_{\overline{x},R}]$ on $Q'(\overline{x},R)$ that $[P_{\overline{x},R}] \le c^*_{11} \Psi(\overline{x},R) + 2$ for $0 < R < R' < 1$. This fact implies that

$$\sup_{0<R<\operatorname{dist}(\overline{x},\partial\Omega\setminus\Gamma)} [P_{\overline{x},R}] < +\infty \qquad \text{for all } \overline{x} \in \Gamma_1.$$

Let now $0 < \mu < 1$ and choose τ in such a way that

$$(3.20) \qquad 2c^*_{10} \cdot \tau^{2-2\mu} = 1.$$

For $M \ge 8$ denote $c_3(M)$ the constant of Lemma 2.1. Let $\varepsilon > 0$. Then there exists $R_1 > 0$ such that $R^2 + \Psi(\overline{x},R) < \frac{\varepsilon}{c'_{17} \cdot c_3(M)}$ for $0 < R < R_1$. This fact implies: there exists R_2 such that for $0 < R < R_2$

$$(3.21) \qquad \chi(\overline{x},R) < \tau^n.$$

from (3.19), (3.20), (3.21) we have

$$(3.22) \qquad \Psi(\overline{x},\tau R) \le \tau^{2\mu} \Psi(\overline{x},R).$$

By induction we get for every k:

$$\Psi(\overline{x},\tau^k R) \le \tau^{2\mu k}\Psi(\overline{x},R)$$

and hence for every $0<\rho<\overline{R}$, $(\overline{R}<R_2)$:

$$\Psi(\overline{x},\rho) \le \tau^{2-n-2\mu}\left(\frac{\rho}{\overline{R}}\right)^{2\mu}\Psi(\overline{x},\overline{R}). \tag{3.23}$$

It is clear (Ψ is continuous in $\overline{x}$) that there exists $0<\delta<\overline{R}$ such that for every $x^0 \in Q'(\overline{x},\delta)$, $R_{x^0}^2+\Psi(x^0,R_{x^0}) < \frac{\varepsilon}{c'_{17}c_3(M)}$, $R_{x^0}=\operatorname{dist}(x^0,\partial Q(\overline{x},\overline{R})\setminus\Gamma)$. Let $\delta<\frac{\overline{R}}{4}$.

We shall investigate the following cases:

(i) $x^0\in Q'(\overline{x},\delta)\cap\Gamma$,

(ii) $x^0\in Q'(\overline{x},\delta)\cap\Omega$.

(i) For (x^0,R_{x^0}) the inequality

$$\Psi(x^0,\rho)\le\tau^{2-n-2\mu}\left(\frac{\rho}{R_{x^0}}\right)^{2\mu}\Psi(x^0,R_{x^0}),\qquad 0<\rho<R_{x^0} \tag{3.24}$$

holds.

(ii) Let x^1 be the projection of x^0 on Γ and $d_{x^0}=\operatorname{dist}(x^0,\Gamma)=\operatorname{dist}(x^0,x^1)$. If $d_{x^0}\le\rho<\frac{R_{x^0}}{2}$, then $Q'(x^0,\rho)\subset Q'(x^1,2\rho)$ and $\Psi(x^0,\rho)\le 2^{n-2}\Psi(x^1,2\rho)$. Using the case (i) ($x^1\in\Gamma$, $2\rho<R_{x^1}$), we get

$$\Psi(x^0,\rho)\le 2^{n-2}\tau^{2-n-2\mu}\left(\frac{2\rho}{R_{x^1}}\right)^{2\mu}\Psi(x^1,R_{x^1}). \tag{3.25}$$

In the case when $0<\rho<d_{x^0}$, we shall prove

$$d_{x^0}{}^2+\Psi(x^0,d_{x^0})<\frac{\varepsilon}{c'_{17}c_3(M)}, \tag{3.26}$$

$$[P_{x^0,d_{x^0}}]\le\frac{M}{2}. \tag{3.27}$$

Because $d_{x^0}<\delta<\frac{\overline{R}}{4}$, (3.25) implies:

$$\Psi(x^0,d_{x^0})\le 2^{n-2+2\mu}\cdot\tau^{2-n-2\mu}\Psi(x^1,R_{x^1}). \tag{3.28}$$

Let us choose $\overline{R}$ in such a way that

$$\overline{R}^2+\Psi(\overline{x},\overline{R})<\min\left\{2^{2-n-2\mu}\tau^{n-2+2\mu}\frac{\varepsilon}{c'_{17}c_3(M)},2^{-2n}(c^*_{11})^{-1}\tau^{n-2+2\mu}\frac{M}{4}\right\}.$$

Then (3.28) implies (3.26). From $Q(x^0,d_{x^0})\subset Q'(x^1,2d_{x^0})$ it follows that

$$[P_{x^0},d_{x^0}]\le 2^{2n}c^*_{11}\Psi(x^1,2d_{x^0})+2.$$

Using the case (i) ($2d_{x^0} < R_{x^1}$, $x^1 \in \Gamma$) we get:

$$[P_{x^0}, d_{x^0}] \leq 2^{2n} c_{11}^{*} \tau^{2-n-2\mu} \Psi(x^1, R_{x^1}) + 2 \leq \frac{M}{4} + \frac{M}{4} = \frac{M}{2}.$$

(3.26), (3.27) imply (by the arguments from the proof of interior regularity)

$$\Psi(x^0, \rho) \leq \text{const} \left(\frac{\rho}{R_{x^0}} \right)^{2\mu} \tag{3.29}$$

Now (3.24), (3.25), (3.29) imply

$$\Psi(x^0, \rho) \leq \text{const}\rho^{2\mu}, \qquad 0 < \rho \leq \frac{\overline{R} - \delta}{2}, \quad x^0 \in Q'(\overline{x}, \delta). \tag{3.30}$$

Inequality (3.30) and the properties of Campanato spaces (see [8]) imply: $u \in C^{\underline{m-1},\mu}(\overline{Q'(\overline{x}, \delta)})$, $\mu \in (0,1)$. ■

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied. Then there exists $\Gamma_0 \subset \Gamma$ (Γ_0 – open in Γ) such that for every $x \in \Gamma_0$ there exists $\delta > 0$ such that $u \in C^{\underline{m-1},\mu}(\overline{Q'(x, \delta)})$, $\mu \in (0,1)$ and $H_{n-p}(\Gamma \setminus \Gamma_0) = 0$, $p > 2$.*

PROOF : The existence of $\Gamma_0 \subset \Gamma$, Γ_0 – open, follows from Theorem 3.2. It is clear that $\Gamma \setminus \Gamma_0 \subset \sum$, where

$$\sum = \left\{ x \in \Gamma : \limsup_{R \to 0+} R^{2-n} \int_{Q'(x,R)} |D^{\underline{m}} u|^2 \, dy > 0 \right\}.$$

Let now $\tilde{u}$ be the extension of u from $Q'(x, R)$ into $Q(x, R)$ by zero. Then

$$R^{2-n} \int_{Q'(x,R)} |D^{\underline{m}} u|^2 \, dy = R^{2-n} \int_{Q(x,R)} |D^{\underline{m}} \tilde{u}|^2 \, dy.$$

Using Hölder inequality and the fact that $u \in H^{\underline{m},p}_{\text{loc}}(\Omega)$, $p > 2$ we get: $\Gamma \setminus \Gamma_0 \subset \sum \subset \sum_1$ where

$$\sum\nolimits_1 = \left\{ x \in \Gamma : \limsup_{R \to 0+} R^{p-n} \int_{Q(x,R)} |D^{\underline{m}} \tilde{u}|^p \, dy > 0 \right\}.$$

Then Theorem 1 from [6] implies that $H_{n-p}(\Gamma \setminus \Gamma_0) = 0$. ■

Remark 3.2. The proof of Theorem 1.1 follows directly from Theorem 2.2 and Theorem 3.3.

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Matematicko fyzikálna fakulta, Univerzita Komenského, Mlynská dolina, 842 15 Bratislava, Československo

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