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Basic sets of polynomials in Clifford analysis

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Abstract. This paper is concerned with the extension of the theory of basic sets of polynomials in one complex variable, as introduced by J.M.Whittaker and B.Cannon, to the setting of Clifford analysis. This is the natural generalization of complex analysis to Euclidean space of dimension larger than two, where the regular functions have values in a Clifford algebra and are null–solutions of a linear differential operator which linearizes the laplacian. An important subclass of the Clifford regular functions is considered, for which a Cannon theorem on the effectiveness in closed balls is proved. This result is consequently refined in terms of the order and type of entire functions in this subclass.

Keywords: Basic sets of polynomials, cannon sum of basic sets, effectiveness, rate of increase of basic sets, monogenic functions, Clifford algebra

Classification: 41A10, 30G35

1. Introduction.

If m is a positive integer, the *Clifford algebra* $\mathcal{A}_m$ can be defined as the ring extension of the field $\mathbf{R}$ with symbols $e_1, \dots, e_m$ satisfying the fundamental multiplication rule

$$e_j e_k + e_k e_j = -2\delta_{jk}.$$

Clearly, $\mathcal{A}_m$ is then a noncommutative algebra. There are two special cases: $\mathcal{A}_1$ is isomorphic to the complex field $\mathbf{C}$ and $\mathcal{A}_2$ is isomorphic to the quaternion skewfield $\mathbf{H}$. If $m > 2$, $\mathcal{A}_m$ contains zero divisors (for instance, $(1 + e_1e_2e_3)(1 - e_1e_2e_3) = 0$). A general element of $\mathcal{A}_m$ can be written as

$$\alpha = \sum_{A \subseteq M} \alpha_A e_A,$$

where M stands for $\{1, \dots, m\}$, $\alpha_A \in \mathbf{R}$ and if $A = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$, e_A stands for $e_{i_1} e_{i_2} \dots e_{i_k}$. This expansion of α in terms of the e_A is unique, so $\mathcal{A}_m$ is a 2^m–dimensional real vector space. An important *involution* $\bar{\cdot}$ is defined on $\mathcal{A}_m$ by the rules $\overline{\lambda} = \lambda$ if $\lambda \in \mathbf{R}$, $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$, $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$ and $\overline{e}_i = -e_i$ if $i \in M$. A *norm* $|\cdot|$ can be defined on $\mathcal{A}_m$ by

$$|\alpha| = \sqrt{\sum_{A \subseteq M} \alpha_A^2}.$$

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Some care must be taken when using this norm to estimate products: we will always use the formulas $|\alpha\beta| \leq 2^{m/2}|\alpha||\beta|$ in general and $|\alpha\beta| = |\alpha||\beta|$ if $\alpha\overline{\alpha} \in \mathbf{R}$ or $\beta\overline{\beta} \in \mathbf{R}$.

The space $\mathbf{R}^{m+1}$ is identified with a subset od $\mathcal{A}_m$ by associating to $(x_0, x_1, \ldots, x_m)$ the element $x_0 + x_1e_1 + \cdots + x_me_m$ of $\mathcal{A}_m$. The elements of this subset will be referred to as *vectors*. One easily sees that $\overline{x} = x_0 - x_1e_1 - \cdots - x_me_m$ and that $x\overline{x} = \overline{x}x = |x|^2$. As a consequence, one can divide through nonzero vectors, since $x^{-1} = \overline{x}/|x|^2$.

Clifford analysis is a generalization of complex analysis to functions defined on open sets in $\mathbf{R}^{m+1}$. The generalized Cauchy–Riemann operator D is defined by

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + \cdots + e_m\frac{\partial}{\partial x_m};$$

it can act both from the left and the right on functions in $C^1(\mathbf{R}^{m+1}, \mathcal{A}_m)$. The generalization of holomorphic functions in $\mathbf{C}$ are called *monogenic* functions.

Definition. Let f be a smooth $\mathcal{A}_m$–valued function defined in an open set $\Omega \subseteq \mathbf{R}^{m+1}$, then f is *monogenic* in Ω if and only if $Df = 0$ holds in the whole of Ω.

For more details on Clifford algebra and analysis, the reader is referred to [1].

2. Special monogenic functions.

The fundamental reference for special monogenic functions is [7]. For the purposes of this paper, we will give a different, less elaborate exposition of their properties.

We start with polynomials : by definition, a polynomial $P(x)$ is *special monogenic* if and only if $DP(x) = 0$ (so $P(x)$ is monogenic) and there exist constants $a_{ij} \in \mathcal{A}_m$ for which

$$P(x) = \sum_{i,j}{}' \overline{x}^i x^j a_{ij},$$

where the primed sigma stands for a finite sum.

Lemma 1. *If $P_n(x)$ is homogeneous special monogenic polynomial of degree n in x, then*

$$P_n(x) = p_n(x)\alpha,$$

where the polynomials p_n are defined as

$$p_n(x) = \sum_{i+j=n} \frac{((m-1)/2)_i}{i!}\frac{((m+1)/2)_j}{j!}\overline{x}^i x^j,$$

α is some constant in $\mathcal{A}_m$ and for $b \in \mathbf{R}$, $(b)_l$ stands for $b(b+1)\ldots(b+l-1)$.

PROOF : Suppose

$$P(x) \sum_{x+j=n} \overline{x}^i x^j a_{ij}$$

for certain constants $a_{ij} \in \mathcal{A}_m$. The restriction of $P(x)$ to the hyperplane $x_0 = 0$ is

$$x^n \sum_{i+j=n} (-1)^i a_{ij}.$$

The Cauchy–Kowalewski extension (cf. [1]) of $x^n \mid_{x_0=0}$ can be found by noticing that

$$\begin{aligned}\frac{1-\overline{x}}{|1-x|^{m+1}} &= (1-\overline{x})^{-(m-1)/2}(1-x)^{-(m+1)/2}\\ &= \sum_{i,j=0}^{\infty} \frac{((m-1)/2)_i}{i!}\frac{((m+1)/2)_j}{j!}\overline{x}^i x^j,\end{aligned}$$

where the series converges for $|x| < 1$, is a monogenic function and that its restriction to $x_0 = 0$ is

$$\begin{aligned}\frac{1+x}{|1-x|^{m+1}} &= (1+x)^{-(m-1)/2}(1-x)^{-(m+1)/2}\\ &= \sum_{n=0}^{\infty} x^n \sum_{i+j=n} \frac{((m-1)/2)_i}{i!}\frac{((m+1)/2)_j}{j!}(-1)^i.\end{aligned}$$

Splitting these sums in their homogeneous parts, the Cauchy–Kowalewski extension of $x^n \mid_{x_0=0}$ is seen to be proportional to

$$p_n(x) = \sum_{i+j=n} \frac{((m-1)/2)_i}{i!}\frac{((m+1)/2)_j}{j!}\overline{x}^i x^j$$

with a nonzero real proportionality factor. By the uniqueness of the Cauchy–Kowalewski extension the lemma follows. ∎

Lemma 2.

$$\sup_{|x|=r} |p_n(x)| = \binom{m+n-1}{n} r^n.$$

PROOF : Obviously, $\sup_{|x|=r} |p_n(x)| = p_n(r)$. This case be computed more explicitly:

$$\frac{1-r}{|1-r|^{m+1}} = (1-r)^{-m} = \sum_{n=0}^{\infty} \frac{(m)_n}{n!} r^n,$$

so $p_n(r) = \frac{(m)_n}{n!} r^n = \binom{m+n-1}{n} r^n$.

Clearly, $\binom{m+n-1}{n}$ being a polynomial in n, $\limsup_{n\to\infty} \binom{m+n-1}{n}^{1/n} = 1$. ∎

Definition. Let Ω be a connected open subset of $\mathbf{R}^{m+1}$ containing 0 and let f be monogenic in Ω, then f is called *special monogenic* in Ω if and only if its Taylor series near zero (which is known to exist) has the form

$$f(x) = \sum_{n=0}^{\infty} p_n(x) c_n \tag{2.1}$$

for certain constants $c_n \in \mathcal{A}_m$.

3. The radius of regularity of a special monogenic function.

There is a close link between the set on which a special monogenic function is defined and the asymptotic properties of its coefficients c_n.

Lemma 3. *If f is special monogenic in a neighorhood of the closed ball $\overline{B}(r)$, the Taylor series* (2.1) *converges normally in $\overline{B}(r)$.*

PROOF : This is a well-known result of Clifford analysis, a proof can be found in [1], p.81. ∎

Lemma 4. *The series*

$$\sum_{n=0}^{\infty} p_n(x) c_n$$

converges in a neighborhood of $\overline{B}(r)$ if and only if

$$\limsup_{n\to\infty} |c_n|^{1/n} < \frac{1}{r}.$$

PROOF : If $\limsup_{n\to\infty} |c_n|^{1/n} = 1/R < 1/r$ and if $1/R < 1/r_0 < 1/r$ for some r_0,

$$\limsup_{n\to\infty} \sup_{|x|\le r_0} |p_n(x)c_n|^{1/n} \le \limsup_{n\to\infty} \left(\binom{m+n-1}{n} r_0^n |c_n| \right)^{1/n} = r_0/R < 1$$

and the series converges normally on the neighborhood $\overline{B}(r_0)$ of $\overline{B}(r)$.

On the other hand, if the series converges in a neighborhood of $\overline{B}(r)$, it still converges in $\overline{B}(r_0)$ for some $r_0 > r$ and its terms must be bounded there:

$$|p_n(x)c_n| \le M_x$$

for every $x \in \overline{B}(r_0)$ and for every n. In particular, for $x = r_0$, we see that

$$|c_n| \le \frac{M_{r_0}}{\binom{m+n-1}{n} r_0^n}$$

and deduce

$$\limsup_{n\to\infty} |c_n|^{1/n} \le \left(\lim_{n\to\infty} \binom{m+n-1}{n}^{-1/n} \right) /r_0 = 1/r_0 < 1/r.$$

∎

These lemmas lead to the following

Definition. The *radius of regularity* R_f of special monogenic function f is defined by $R_f = 1/(\limsup_{n\to\infty} |c_n|^{1/n})$.

4. Basic sets of polynomials and effectiveness.

In complex analysis, a basic set of polynomials is a basis for the vector space of complex polynomials in one indeterminate; we refer to [6], [8] for this theory. This definition can be adapted to the setting of Clifford algebras by noticing that the set of special monogenic functions is a free module over the $p_n(x)$. A set $\{P_k(x) \mid k \in \mathbf{N}\}$ of special monogenic polynomials is then called *basic* if and only if it is a basis for the set of special monogenic polynomials, i.e. if every p_n can be expressed as a right $\mathcal{A}_m$–linear combination

$$p_n(x) = \sum_{k=0}^{\infty} P_k(x)\pi_{nk}, \qquad \pi_{nk} \in \mathcal{A}_m \tag{4.1}$$

where only a finite number of terms differ from zero, and for any sequence $a_0, a_1, \ldots, a_p \in \mathcal{A}_m$,

$$\sum_{k=0}^{p} P_k(x)a_k = 0 \Rightarrow a_0 = a_1 = \cdots = a_p = 0.$$

If $\deg P_k = k$ for every $k \in \mathbf{N}$, the basic set is called *simple*. By N_n we denote the number of nonzero π_{nk}. If $\limsup_{n\to\infty} N_n^{1/n} = 1$, the basic set is called a *Cannon* basic set.

Given a special monogenic function f such that

$$f(x) = \sum_{n=0}^{\infty} p_n(x)c_n$$

near 0, we can formally express it in terms of the P_k as

$$f(x) = \sum_{k=0}^{\infty} P_k(x)\left(\sum_{n=0}^{\infty} \pi_{nk}c_n\right). \tag{4.2}$$

Definition. If for all special monogenic functions f defined in a neighborhood of $\overline{B}(r)$ this series in terms of the P_k converges normally to f in $\overline{B}(r)$, the basic set $\{P_k(x)\}$ is called *effective* in $\overline{B}(r)$.

5. A Cannon Theorem for Cannon basic sets of special monogenic polynomials.

The *Cannon function* $\lambda(r)$ of a Cannon basic set $\{P_k(x)\}$ is defined as

$$\lambda(r) = \limsup_{n\to\infty} \lambda_n(r)^{1/n},$$

where

$$\lambda_n(r) = \sum_{k=0}^{\infty} \sup_{|x|=r} |P_k(x)\pi_{nk}|.$$

Notice that λ_n and λ are non–decreasing.

Lemma 5. *For all $r > 0$ we have*

$$\lambda(r) \geq r.$$

PROOF : Indeed,

$$\lambda(r) \geq \limsup_{n\to\infty} \sup_{|x|=r} |\sum_{k=0}^{\infty} P_k(x)\pi_{nk}|^{1/n} = \limsup_{n\to\infty} \sup_{|x|=r} |p_n(x)|^{1/n} = r.$$

■

Lemma 6. *If the special monogenic function $f(x) = \sum_{n=0}^{\infty} p_n(x)c_n$ is such that*

$$\sum_{n=0}^{\infty} |c_n|\lambda_n(r) < +\infty, \tag{5.1}$$

the basic series (4.2) represents f in $\overline{B}(r)$.

PROOF : The convergence in (5.1) provides the justification for the rearrangement of terms needed to obtain (4.2) from (2.1) using (4.1) and proves its normal convergence on $\overline{B}(r)$. ■

Lemma 7. *If the special monogenic function $f(x) = \sum_{n=0}^{\infty} p_n(x)c_n$ is such that*

$$\limsup_{n\to\infty} (|c_n|\lambda_n(r))^{1/n} > 1, \tag{5.2}$$

there exists a special monogenic function $g(x) = \sum_{n=0}^{\infty} p_n(x)c'_n$ such that $c'_n = 0$ if $c_n = 0$ and

$$\limsup_{n\to\infty, c_n\neq 0} \left(\frac{|c'_n|}{|c_n|}\right)^{1/n} \leq 1, \tag{5.3}$$

that cannot be represented in $\overline{B}(r)$ by the basic set $\{P_k(x)\}$.

Remark. As a consequence of (5.3), $R_g \geq R_f$.

PROOF : First remark that if $\alpha \in \mathcal{A}_m$, the number

$$\eta(\alpha) = \begin{cases} \overline{\alpha}/|\alpha| & \text{if } \alpha \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

has the property that the real part of $\alpha\eta(\alpha)$ is $|\alpha|$.

Now if for some $k \in \mathbf{N}$ the series

$$\sum_{n=0}^{\infty} N_n|c_n|\,|\pi_{nk}| \tag{5.4}$$

diverges, we take

$$g(x) = \sum_{n=0}^{\infty} p_n(x) N_n |c_n| \eta(\pi_{nk}).$$

Then (5.3) is easily verified and in (4.2) applied to g the coefficient of P_k diverges.

If all series (5.4) converge, we define

$$d_n = N_n |c_n| \max_{k=0}^{\infty} \sup_{|x|=r} |P_k(x)\pi_{nk}|$$

and $x^{(n)}, k_n$ as the values for which d_n is attained in this equation. We construct a sequence of integers $n_0 < n_1 < \ldots$ by taking $n_0 = 0$ and, having determined $n_0, n_1, \ldots, n_{p-1}$, requiring n_p to satisfy

$$\sup_{|x|=r} |P_{k_{n_q}}(x)| \sum_{n=n_p}^{\infty} N_n |c_n| \, |\pi_{nk_{n_q}}| \leq \frac{1}{3 \cdot 2^m}$$

for $q = 0, 1, \ldots, p-1$ and

$$d_{n_p} \geq \max(p, 3 \cdot 2^{m/2} \sum_{l=0}^{p-1} d_{n_l}).$$

The first requirement can be met because all series (5.4) converge, the second one because by (5.2) there exists an $\varepsilon > 0$ such that $d_n \geq (1+\varepsilon)^n$ if $n \geq n_0(\varepsilon)$. Then the special monogenic function

$$g(x) = \sum_{l=0}^{\infty} p_{n_l}(x) N_{n_l} |c_{n_l}| \eta(P_{k_{n_l}}(x^{(n_l)}) \pi_{n_l k_{n_l}})$$

satisfies (5.3) but for every $l > 0$ the supremum over $\overline{B}(r)$ of the modulus of the term involving $P_{k_{n_l}}$ in (4.2) applied to g is larger than $l - l/3 - 1/3 \geq l/3$, contradicting its normal convergence. ∎

Theorem 1. *The Cannon basic set $\{P_k(x)\}$ is effective in $\overline{B}(r)$ if and only if $\lambda(r) = r$.*

PROOF : If $\lambda(r) = r$ and f is special monogenic in a neighborhood of $\overline{B}(r)$,

$$\limsup_{n\to\infty} (|c_n| \lambda_n(r))^{1/n} \leq r/R_f < 1,$$

implying the convergence of $\sum_{n=0}^{\infty} |c_n| \lambda_n(r)$ and the result follows from Lemma 6. If $\lambda(r) = R \neq r$, then, by Lemma 5, $R > r$. Take $r < R_1 < R$ and consider $f(x) = \sum_{n=0}^{\infty} p_n(x)/R_1^n$, then clearly

$$\limsup_{n\to\infty} (\lambda_n(r)/R_1^n)^{1/n} = \lambda(r)/R_1 > 1,$$

so that by Lemma 7 there exists a special monogenic function g such that $R_g \geq R_f = R_1 > r$ that cannot be represented in $\overline{B}(r)$ by the basic set $\{P_k(x)\}$. ∎

6. Rate of increase.

Definition. A monogenic function f defined on the whole of $\mathbf{R}^{m+1}$ is called *entire*.

The rate of increase of an entire special monogenic function can be defined in terms of its Taylor series near zero in analogy with complex analysis (cf. [5]):

Definition. Let $f = \sum_{k=0}^{\infty} p_n(x)c_n$ be the power series expansion of an entire special monogenic function near zero. The *order* ρ of f is then defined by

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|c_n|)},$$

and, if $0 < \rho < \infty$, the *type* σ of f by

$$\sigma = \frac{1}{e\rho} \limsup_{n\to\infty} n|c_n|^{\rho/n}.$$

We will try to refine our result on the effectiveness of Cannon basic sets by taking into account the rate of increase of the functions that are to be represented. This requires a notion of rate of increase for basic sets, defined as follows.

Definition. Let $\{P_k(x)\}$ be a basic set of special monogenic polynomials. Its *order* ω is defined by

$$\omega = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\log \lambda_n(r)}{n \log n},$$

where $\lambda_n(r) = \sum_{k=0}^{\infty} \sup_{|x|=r} |P_k(x)\pi_{nk}|$. If $0 < \omega < \infty$ we define the *type* γ by

$$\gamma = \lim_{r\to\infty} \frac{e}{\omega} \limsup_{n\to\infty} \frac{\lambda_n(r)^{1/n\omega}}{n}.$$

Often we will need to express that the rate of increase of a function or of a basic set is 'less' than a given rate of increase. This means that either the order is smaller than the given order or that the orders are equal but that the type is smaller than the given type.

Our aim is to link the order and type of a basic set to the order and type of the entire functions it represents.

Theorem 2. *A necessary and sufficient condition for a Cannon basic set $\{P_k(x)\}$ to represent all entire special monogenic functions of increase less than order ρ type σ, where $0 < \rho$, $\sigma < \infty$, is that*

$$\limsup_{n\to\infty} \left(\frac{e\rho\sigma}{n}\right)^{1/\rho} \lambda_n(r)^{1/n} \leq 1 \tag{6.1}$$

for all $r \geq 0$.

PROOF : If (6.1) does not hold for $r = r_1$, there exists a $\sigma_1 < \sigma$ such that

$$\limsup_{n\to\infty} \left(\frac{e\rho\sigma_1}{n}\right)^{1/\rho} \lambda_n(r_1)^{1/n} > 1.$$

Then the function $f(x) = \sum_{n=0}^{\infty} p_n(x) c_n$ where $c_0 = 0$ and $c_n(e\rho\sigma_1/n)^{n/\rho}$ for $n > 0$, is entire of order ρ type σ_1 and

$$\limsup_{n\to\infty}(|c_n|\lambda_n(r))^{1/n} > 1$$

for all $r \geq r_1$. By Lemma 7 there exists a special monogenic entire function g of increase less than or equal to order ρ type σ_1 (because of 5.3) that cannot be represented by $\{P_k(x)\}$ in $\overline{B}(r_1)$.

On the other hand, if $r \leq 0$ and $f = \sum_{n=0}^{\infty} p_n(x) c_n$ is of increase less than order ρ type σ_1 where $0 < \sigma_1 < \sigma$, $|c_n| \leq (e\rho\sigma_1/n)^{n/\rho}$ for all $n \geq n_0(\sigma_1)$, so

$$\limsup_{n\to\infty}(|c_n|\lambda_n(r))^{1/n} \leq \limsup_{n\to\infty}(\frac{e\rho\sigma_1}{n})^{1/\rho}\lambda_n(r)^{1/n} < \limsup_{n\to\infty}(\frac{e\rho\sigma}{n})^{1/\rho}\lambda_n(r)^{1/n} \leq 1$$

and, by Lemma 6, f can be represented in $\overline{B}(r)$ by $\{P_k(x)\}$. ■

Corollary. *A necessary and sufficient condition for a Cannon set $\{P_k(x)\}$ of special monogenic polynomials to represent every entire special monogenic function of increase less than order ρ type σ in any ball $\overline{B}(r)$ is that the set is of increase not exceeding order $1/\rho$ type $1/\sigma$.*

PROOF : This is easily verified using the definition of order and type for Cannon sets. ■

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