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Abstract. A criterion for freeness of torsion-free modules over a discrete valuation domain proved by Procházka, which generalizes classical results by Prüfer and Kaplansky, is generalized to torsion-free modules over an almost maximal valuation domain.

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Theorem A (Prüfer). *Let R be a complete discrete valuation domain. Any countably generated torsion-free reduced R-module is free.*

This theorem was explicitely stated by Kaplansky in [3] where the following generalization to modules over maximal valuation domains was also given :

Theorem B (Kaplansky). *Let R be a maximal valuation domain. Any torsion-free R-module of countable rank is completely decomposable.*

Theorem A can be derived from theorem B since, for torsion-free modules over discrete valuation domains, "$\aleph_0$-generated" and "of countable rank" are equivalent, and the only isomorphism classes of submodules of Q, the field of quotients of R, are Q itself and R.

Theorem A can also be viewed as a consequence of the Pontryagin criterion for freeness of modules over PID's in [6] (see also [1]).

A generalization of the Prüfer-Kaplansky theorem was given by Procházka in [5], where both the completeness condition on R and the countability condition on the module are dropped.

Theorem C (Procházka). *Let R be a discrete valution domain. A torsion-free R-module A is free if and only if $\widetilde{R} \otimes_R A$ is a reduced $\widetilde{R}$-module and A belongs to the Baer class $\beta(R)$.*

To explain this result, we recall that $\widetilde{R}$ denotes the completion of R in the ideal topology, and that the Baer class $\mathcal{B}(R)$ is the class of torsion-free R-modules given by : $\mathcal{B}(R) = \bigcup_\alpha \Gamma_\alpha(R)$, where α ranges over the ordinal numbers and $\Gamma_\alpha(R)$ is defined by transfinite induction as follows:

$\Gamma_0(R)$ is the class of countable rank torsion-free R-modules .

$\Gamma_\alpha(R)$ is the class of torsion-free R-modules A with a pure submodule of finite rank B such that $A/B = \oplus_{i\in I} A_i$, where $A_i \in \Gamma_{\alpha_i}(R)$ and $\alpha_i < \alpha$ for all $i \in I$, $(\alpha \geq 1)$.

The proof of Procházka's theorem in [**5**] is elaborate and it is based on the notions of p-basis and p^{∞}-basis (p is a uniformizing element of R).

The goal of this note is to give a simple proof of the Procházka's theorem in the more general setting of modules over almost maximal valuation domains, generalizing also theorem B by Kaplansky. Our generalization of Prüfer-Kaplansky-Procházka theorem states :

Theorem. *Let R be an almost maximal valuation domain and let A be a torsion-free R-module. Then A is free if and only if $\widetilde{R}\otimes_R A$ is an $\widetilde{R}$-homogeneous $\widetilde{R}$-module and $A \in \mathcal{B}(R)$.*

Here too $\widetilde{R}$ denotes the completion of R in the ideal topology; moreover, $\widetilde{R}$ - homogeneous means that every rank-one pure submodule is isomorphic to $\widetilde{R}$. For general facts on modules over valuation domains we refer to [**2**].

Remark 1. The necessity in the theorem is clear, because $A \cong \oplus R$ implies $\widetilde{R} \otimes_R A \cong \oplus \widetilde{R}$, which is trivially $\widetilde{R}$-homogeneous; it is also evident that a free module belong to $\Gamma_1(R) \subseteq \mathcal{B}(R)$. Therefore in the proof of the theorem only sufficiency is needed.

Remark 2. If R is a discrete valuation domain, then the condition that $\widetilde{R}\otimes_R A$ is $\widetilde{R}$-homogeneous ie equivalent to the condition that $\widetilde{R}\otimes_R A$ is a reduced $\widetilde{R}$-module, since a rank-one torsion-free $\widetilde{R}$-module is isomorphic either to $\widetilde{R}$ or to $\widetilde{Q}$.

The proof of the theorem is based on the following two results; the first one is the analogue of the Pontryagin criterion, whose proof can be repeated "mutatis mutandis" (see [**4**]).

Lemma 1. *Let R be a valuation domain and A a torsion-free R-module of countable rank. Then A is free if and only if every pure submodule of finite rank is free.*

Lemma 2. *Let R be an almost maximal valuation domain and B a torsion-free R-module of finite rank. Then B is free if and only if $\widetilde{R}\otimes_R B$ is a $\widetilde{R}$-homogeneous $\widetilde{R}$-module.*

PROOF : The necessity is trivial. Assume That $\widetilde{R}\otimes_R B$ is $\widetilde{R}$-homogeneous. By [**2**, XIV.1.4.]. B has a basic submodule B_0 and B/B_0 is divisible. Moreover the pure-exact sequence

$$0 \longrightarrow \widetilde{R}\otimes_R B_0 \longrightarrow \widetilde{R}\otimes_R B \longrightarrow \widetilde{R}\otimes_R (B/B_0) \longrightarrow 0$$

splits, being $\widetilde{R}\otimes_R B_0$ pure-injective; by hypothesis $\widetilde{R}\otimes_R (B/B_0) = 0$, thus $B = B_0$. But B_0 is free, because $\widetilde{R}\otimes_R B_0$ is also $\widetilde{R}$-homogeneous and so B_0 is R-homogeneous; therefore B is free. ∎

PROOF of the Theorem: By induction on α, if $A \in \Gamma_\alpha(R)$. If $\alpha = 0$, then A has countable rank so, by lemma 1, it is enough to show that any pure submodule of finite rank B is free. But $\widetilde{R}\otimes_R B$, as a pure submodule of $\widetilde{R}\otimes_R A$ is $\widetilde{R}$-homogeneous, hence B is free by lemma 2.

Assume now that $\alpha > 0$ and that the claim is proved for modules in $\Gamma_\beta(R)$ for all $\beta < \alpha$. Consider the pure-exact sequence

$$(1) \qquad 0 \longrightarrow B \longrightarrow A \longrightarrow {}_i\otimes_I A_i \longrightarrow 0$$

where B is a pure submodule of A of finite rank and $A_i \in \Gamma_i(R), \alpha_i < \alpha$. From (1) we obtain the exact sequence

$$0 \longrightarrow \widetilde{R} \otimes_R B \longrightarrow \widetilde{R} \otimes_R A \longrightarrow \bigoplus_i (\widetilde{R} \otimes_R A_i \longrightarrow 0$$

which splits; $\widetilde{R} \otimes_R B$ is $\widetilde{R}$-homogeneous, hence, again by lemma 2, B is free. For each $i \in I$, $\widetilde{R} \otimes_R A_i$ is also $\widetilde{R}$-homogeneous, hence, by induction, A_i is free. Thus (1) splits, being $\bigoplus_i A_i$ a free R-module, and A is free. ■

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