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Abstract. A proof of several characteristics of paracompactness in the general localic setting using a procedure very similar to the classical one is presented. Of these, the ones concerning full normality and σ–local finiteness have been proved by other techniques before; we add characterizations by the existence of locally finite quasirefinements, and of σ–discrete refinements.

Keywords: paracompact, σ–locally finite, σ–discrete, fully normal

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Paracompactness is one of these classical topological notions one can immediately transfer to locale (frame) theory. In classical topology one knows a variety of equivalent properties of spaces. The equivalence of the most important of them has been proved in the general localic setting as well: full normality was treated by J.R.Isbell ([3]) and C.H.Dowker and D.Papert Strauss ([1], where one has also other equivalent properties, most notably the statement on partitions of unity); recently, Sun Shu–Hao ([8]) proved that the paracompactness is equivalent with the existence of σ–locally finite refinements.

Still, there may be some interest in the question as to whether one can prove an equivalence theorem along the classical line (as e.g. in [5]). There is an obvious obstacle: namely the extensive use of general (not necessarily open) covers. In the present notes we shall show that, nevertheless, it can be done, the obstacle being removed by considering quasicovers (that is, system whose joins are dense). While thus imitating the set–topological approach we also obtain the equivalence of paracompactness with the existence of locally finite quasirefinements (which roughly corresponds to the classical characteristics by the existence of locally finite not necessarily open refinements), and with the existence of σ–discrete refinements. Moreover, we think that thus obtained proof of the full normality and the σ–local finiteness results may appear, in some sense, more lucid.

1. Preliminaries.

1.1. As usual (see, e.g., [4]), a **frame** is a complete lattice A satisfying the distributivity law $(\bigvee_j a_i) \wedge b = \bigvee_j(a_i \wedge b)$. Because of the distributivity, there is, for each $a \in A$, the largest element b such that $b \wedge a = 0$, namely $\bigvee\{x | x \wedge a = 0\}$. It is called the **pseudocomplement** of a and denoted by $\neg a$.

The relation $\lhd$ is defined by

$$a \lhd b \text{ if and only if } \neg a \vee b = 1.$$

(Note that, trivially, $a \lhd b$ implies $a \leq b$, and $x \leq a \lhd b \leq y$ implies $x \lhd y$.) A frame A is said to be **regular** if

$$\text{for each } a \in A, \quad a = \bigvee\{x | x \lhd a\}.$$

1.2. Let X be a subset of a frame A and let a be in A. We put (see [7]).

$$Xa = \bigvee\{x \in X | x \wedge a \neq 0\}.$$

Let X, Y be subsets of A. We set

$$XY = \{Xy | y \in Y\}, \quad X \wedge Y = \{x \wedge y | x \in X, y \in Y\},$$

and write

$$X \prec Y$$

if for each $x \in X$ there is a $y \in Y$ such that $x \leq y$.

The following are trivial observations:

(1.2.1) $X \wedge Y \prec X$ and $X \wedge Y \prec Y$.
(1.2.2) If $X \prec X_1$ and $a \leq a_1$ then $Xa \leq X_1 a_1$. Consequently, if $X \prec X_1$ and $Y \prec Y_1$ then $XY \prec X_1 Y_1$.
(1.2.3) $Xa \wedge b \neq 0$ if and only if $a \wedge Xb \neq 0$.
(1.2.4) $\bigvee(X \wedge Y) = \bigvee X \wedge \bigvee Y$.
(1.2.5) $\bigvee XY = X \bigvee Y$.
(1.2.6) $(X_1 \wedge \cdots \wedge X_n)(Y_1 \wedge \cdots \wedge Y_n) \prec X_1 Y_1 \wedge \cdots \wedge X_n Y_n$.

1.3. A subset $X \subseteq A$ is said to be a **cover** if $\bigvee X = 1$. The following are trivial observations:

(1.3.1) By (1.2.4), in particular, if X, Y are covers, then $X \wedge Y$ is a cover.
(1.3.2) If X is a cover of a regular frame A, then

$$\{y | y \lhd x \text{ for some } x \in X\}$$

is a cover of A.
(1.3.3) If X is a cover then, for each $a, Xa \vee \neg a = 1$; that is, $a \lhd Xa$.

An element $x \in A$ is said be **dense** if $a \wedge x \neq 0$ for each $a \neq 0$. A subset $X \subseteq A$ is said to be a **quasicover** if $\bigvee X$ is dense (that is, if for each $a \neq 0$ in A there is an $x \in X$ such that $a \wedge x \neq 0$). Obviously,

(1.3.4) If X is a cover and Y a quasicover, then XY is a cover.

Let $X \prec Y$. If X is a (quasi) cover we say that X is a **(quasi)refinement** of Y.

1.4. A cover X is said to **finitize** (resp. **separate**) a subset $Y \subseteq A$ if for each $x \in X$ there are only finitely many (resp at most one) $y \in Y$ such that $x \wedge y \neq 0$. A set $Y \subseteq A$ is said to be **locally finite** (resp. **discrete**) if there is a cover X finitizing (resp. separating) Y. It is said to be σ–**locally finite** (resp. σ–**discrete**) if we can write $Y = \bigcup_{n=1}^{\infty} Y_n$ with Y_n locally finite (resp. discrete). Obviously, for covers X, X' and X_i, and subsets Y, Y_i,

(1.4.1) If X finitizes (resp. separates) Y and $X' \prec X$ then X' finitizes (resp. separates) Y.
(1.4.2) If X_i, finitize Y_i for $i = 1, 2, \ldots, n$ then $X_1 \wedge \cdots \wedge X_n$ finitizes $Y_1 \wedge \cdots \wedge Y_n$.

In consequence of (1.4.1), in the definition of σ-local finiteness we may require that, moreover, $Y_1 \subseteq Y_2 \subseteq \ldots$

1.5. *A* non-void system $\mathcal{U}$ of covers of A is said to be **a uniformity** on A if

(u1) $U \in \mathcal{U}$ and $U \prec V$ implies $V \in \mathcal{U}$,
(u2) if $U, V \in \mathcal{U}$ then $U \wedge V \in \mathcal{U}$, and
(u3) for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \prec U$.

1.5.1. Remark. Put

$$U^* = \{\bigvee S | S \subseteq U \text{ such that for each } a, b \in S, a \wedge b \neq 0\},$$
$$U^\times = \{\bigvee S | S \subseteq U \text{ such that } \bigwedge S \neq 0\}.$$

Obviously

$$U^\times \subseteq U^* \subseteq UU \subseteq (U^\times)^\times.$$

Consequently, (u3) can be replaced by any of the following two conditions:

(u3*) For each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^* \prec U$.
(u3$^\times$) For each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^\times \prec U$.

The condition (u3*) was used in [7], (u3$^\times$) will be handy in one of the proofs bellow.

2. Various characteristics of paracompactness.

2.1. *A* frame is said to be **paracompact** if each cover has a locally finite refinement.

2.2. Proposition. *Let A be a regular frame. If each cover has a locally finite quasirefinement then the system of all covers of A is a uniformity.*

PROOF : Let U be a cover and let Y be a locally finite quasirefinement of $\{x | x \triangleleft u, u \in U\}$ (recall (1.3.2)). Thus, for each $y \in Y$ there is a $u_y \in U$ such that $y \triangleleft u_y$. Put

$$W = \{w | \forall y \in Y (w \leq u_y \text{ or } w \leq \neg y)\}.$$

This will be shown to be a cover such that $W^\times \prec U$.

Let X be a cover finitizing Y, let $\in X$. We have

$$x \wedge \bigvee W =$$
$$= \bigvee\{x \wedge w | \forall y \in Y (w \leq u_y \text{ or } w \wedge y = 0)\} =$$
$$= \bigvee\{x \wedge w | \forall y \in Y (x \wedge w \leq u_y \text{ or } x \wedge w \wedge y = 0)\} =$$
$$= \bigvee\{x \wedge w | \forall y \in Y, x \wedge y \neq 0 (x \wedge w \leq u_y \text{ or } x \wedge w \wedge y = 0)\} \geq$$
$$\geq \bigvee\{x \wedge w | \forall y \in Y, x \wedge y \neq 0 (w \leq u_y \text{ or } w \wedge y = 0)\} =$$
$$= x \wedge \bigvee\{w | \forall y \in Y, x \wedge y \neq 0 (w \leq u_y \text{ or } w \wedge y = 0)\}$$

Let $y_1, \ldots, y_n$ be those elements of Y for which $x \wedge y_i \neq 0$. Thus,

$$\bigvee\{w | \forall y \in Y, x \wedge y \neq 0 (w \leq u_y \text{ or } w \wedge y = 0)\} =$$
$$= \bigvee\{w | \forall i, w \leq u_{y_i} \text{ or } w \leq \neg y_i\} \geq$$
$$\geq \bigvee\{\bigwedge_{i=1}^{n} a_i | a_i = u_{y_i} \text{ or } a_i = \neg y_i\} = \bigwedge_{i=1}^{n} (u_{y_i} \vee \neg y_i) = 1.$$

Hence, $x \wedge \bigvee W = x$ and since X is a cover, $\bigvee W = 1$.

Now let $w_i \in W, i \in J$, be such that $w = \bigwedge_J w_i \neq 0$. Since Y is a quasicover, there is a $y \in Y$ such that $w \wedge y \neq 0$. Then, for each $i \in J, w_i \nleq \neg y$ and hence $w_i \leq u_y$ so that, finally, $\bigvee w_i \leq u_y$. Thus, $W^\times \prec U$. ∎

Remark. The proof can be made very close to the classical one (see [5]) by defining, first, an element w of $A \otimes A$ by putting

$$w_0 = \bigwedge\{u_y \otimes u_y \vee \neg y \otimes \neg y | y \in Y\}$$

and then considering $W = \{w | w \otimes w \leq w_0\}$. Then, of course, one has to use some properties of products of locales (coproducts of frames), which has been avoided here.

2.3. Proposition. *Let the system of all covers of A be a uniformity and let each cover have a locally finite quasirefinement. Then A is paracompact.*

PROOF : Let U be a cover of A, let X be a cover such that $XX \prec U$, let Y be a locally finite quasirefinement of X and let Z finitize Y. Let Z_1 be a cover such that $Z_1 Z_1 \prec Z$. Put $W = Z_1 \wedge X$. Thus,

$$WW \prec Z \text{ and } W \prec X.$$

Finally put $V = WY$. It is a cover (see (1.3.4)) and we have

$$V = WY \prec XX \prec U.$$

Let $w \in W$ and let $w \wedge W_{y_i} \neq 0$ for some $y_i \in Y$. Thus, by (1.2.3), $Ww \wedge y_i \neq 0$ and since $WW \prec Z$ and Z finitizes Y, y_i are only finitely many. Thus, W finitizes V. ∎

2.4. Proposition. *Let each cover of a regular frame A have a σ-locally finite refinement. Then A is paracompact.*

PROOF : By 2.2 and 2.3 it suffices to prove that each cover has a locally finite quasirefinement. Let U be a cover and let Y be a refinement such that $Y = \bigcup_{n=1}^{\infty} Y_n$ with $Y_1 \subseteq Y_2 \subseteq \dots$ locally finite. Let X_n finitize Y_n.

For $y \in Y$ put $n(y) = \min\{n | y \in Y_n\}$ and choose an antireflexive well-ordering R on Y such that $n(x) < n(y)$ implies xRy.

Put $Z_n = X_n \wedge Y_n$. By (1.2.4), $\bigvee Z_n = \bigvee Y_n$ and hence $Z = \bigcup Z_n$ is a cover.

Put $\widetilde{y} = y \wedge \neg \bigvee\{x | xRy\}, V = \{\widetilde{y} | y \in Y\}$. Let $a \in A$ be non-zero. Let y be first in R such that $y \wedge a \neq 0$. Then $a \wedge \widetilde{y} = a \wedge y \neq 0$. Thus, V is a quasirefinement of U.

Finally we will show that Z finitizes V. Indeed, let z be in Z_n. Then there is a $y \in Y_n$ such that $z \leq y$. Hence, if $n(x) > n, z \leq \bigvee\{u | uRx\}$ so that $z \wedge \widetilde{x} = 0$. Consequently, if $z \wedge \widetilde{y_i} \neq 0, y_i$ are in Y_n, and since $\widetilde{y_i} \leq y_i$ and $z \leq z_1$ for some $z_1 \in X_n, \widetilde{y_i}$ are only finitely many. ∎

2.5. Proposition. *Let the system of all covers of A be a uniformity. Then each cover of A has a σ-discrete refinement.*

PROOF : Take a cover U of A. Putting $U = U_0$ choose inductively covers U_n such that

$$U_n U_n \prec U_{n-1}.$$

Choose an antireflexive well-ordering R on U_1 and write

$$u\overline{R}v \text{ for } (uRv \text{ or } u = v).$$

For $u \in U_1$ define inductively $u^{(n)}$ by putting

$$u^{(1)} = u, \qquad u^{(n+1)} = U_{n+1}u^{(n)}.$$

Put

$$p_u^{(n)} = \bigvee\{v^{(n)} | vRu\}.$$

by (1.2.5) we have

$$p_u^{(n+1)} = U_{n+1}p_u^{(n)}. \tag{1}$$

Now put

$$\widetilde{u}^{(n)} = u^{(n)} \wedge \neg p_u^{(n+1)}, \qquad \widetilde{U} = \{\widetilde{u}^{(n)} | u \in U_1, n = 1, 2, \dots\}.$$

I. $\widetilde{U}$ is a cover:

We will prove that

$$\text{for each } u, \quad \bigvee_{n=1}^{\infty} \bigvee\{\widetilde{v}^{(n)} | v\overline{R}u\} = \bigvee_{n=1}^{\infty} \bigvee\{v^{(n)} | v\overline{R}u\}. \tag{2}$$

The equality obviously holds if u is the first element in R. Let it hold for all wRu. We have

$$\bigvee_{n=1}^{\infty}\bigvee\{\widetilde{v}^{(n)}|v\overline{R}u\}=\bigvee_{n=1}^{\infty}(\bigvee\{\widetilde{v}^{(n)}|vRu\}\vee\widetilde{u}^{(n)})=$$

$$=\bigvee_{n=1}^{\infty}(\bigvee_{wRu}\bigvee\{\widetilde{v}^{(n)}|v\overline{R}w\}\vee\widetilde{u}^{(n)})=\bigvee_{n=1}^{\infty}(\bigvee_{wRu}\bigvee\{v^{(n)}|v\overline{R}w\}\vee\widetilde{u}^{(n)})=$$

$$=\bigvee_{n=1}^{\infty}(\bigvee\{\widetilde{v}^{(n)}|vRu\}\vee\widetilde{u}^{(n)})=\bigvee_{n=1}^{\infty}(p_u^{(n)}\vee(u^{(n)}\wedge\neg p_u^{(n+1)})).$$

Since $p_u^{(k)}\lhd p_u^{(k+1)}$ (by (1) and (1.3.3)) we proceed:

$$\cdots=\bigvee_{n=1}^{\infty}(p_u^{(n+2)}\vee(u^{(n)}\wedge\neg p_u^{(n+1)}))=$$

$$\bigvee_{n=1}^{\infty}(p_u^{(n+2)}\vee(u^{(n)})\wedge(p_u^{(n+2)}\vee\neg p_u^{(n+1)}))=$$

$$=\bigvee_{n=1}^{\infty}(p_u^{(n+2)}\vee(u^{(n)})=\bigvee_{n=1}^{\infty}(p_u^{(n)}\vee u^{(n)})=\bigvee_{n=1}^{\infty}\bigvee\{v^{(n)}|v\overline{R}u\}.$$

Now, by (2),

$$\bigvee\widetilde{U}=\bigvee_{u\in U_1}\bigvee_{n=1}^{\infty}\bigvee\{\widetilde{v}^{(n)}|v\overline{R}u\}\geq\bigvee U_1=1.$$

II. $\widetilde{U}$ refines U:

Take $u\in U_1$. Since $U_1U_1\prec U$, there is a $v\in U$ such that

$$U_1u=U_1u^{(1)}\leq v.$$

Since

$$U_{n+1}u^{(n+1)}=U_{n+1}U_{n+1}u^{(n)}\leq U_n^{(n)}$$

we obtain by induction $U_nu^{(n)}\leq v$, and consequently $\widetilde{u}^{(n)}\leq u^{(n)}\leq v$.

III. $\widetilde{U}$ is σ–discrete:

Put $\widetilde{U}_n=\{\widetilde{u}^{(n)}|u\in U_1\}$. We have $\widetilde{U}=\bigcup_{n=1}^{\infty}\widetilde{U}_n$. We will show that U_{n+1} separates $\widetilde{U}_n$. Indeed let $x\wedge\widetilde{u}^{(n)}\neq 0$ for an $x\in U_{n+1}$ and let uRv. We have $x\wedge u^{(n)}\neq 0$ and hence $x\leq u^{(n+1)}\leq p_v^{(n+1)}$ so that $x\wedge\neg p_v^{(n+1)}=0$ and hence $x\wedge\widetilde{v}^{(n)}=0$. ∎

2.6. Theorem. *Let A be a regular frame. Then the following statements are equivalent:*

(1) *A is paracompact,*
(2) *each cover of A has a locally finite quasirefinement,*
(3) *the system of all covers of A is a uniformity,*
(4) *each cover of a has a σ–discrete refinement,*
(5) *each cover of A has a σ–locally finite refinement.*

PROOF : Trivially (1)$\Rightarrow$ (2), (2)$\Rightarrow$(3) is in 2.2, (3)$\Rightarrow$(4) is in 2.5, (4)$\Rightarrow$ (5) is trivial and (5)$\Rightarrow$(1) follows from 2.2 and 2.3. ■

2.7. Remark. In a regular locale A, the system $\mathcal{U}$ of all covers is admissible in the sense that $A = A_{\mathcal{U}}$ (see [**7**], or $A = [A : \mathcal{U}]$ in the notation of [**6**]). Thus, the system of all covers of a paracompact frame makes it to a uniform frame (see [**3**], [**7**]).

3. Remarks on full normality.

3.1. The characteristics of paracompactness which we have encountered as (3) in Theorem 2.6, and amounting in fact to

(3') for each cover U of A there is a cover V of A such that $VV \prec U$,

corresponds to the property which is in the classical case refered to as **full normality** (and this expression is used in [**1**] and [**3**] in the general context, too). This is justified by the fact that the normality of a topological space is equivalent to the statement that

for each finite open cover there is a finite star refinement.

It is perhaps worth showing explicitly that this holds for general frames as well and hence using the atribute "fully normal" for frames satisfying (3') is indeed justified.

3.2. Recall that a frame is **normal** if for a_1, a_2 such that $a_1 \vee a_2 = 1$ there are b_1, b_2 such that $a_i \vee b_i = 1$ and $b_1 \wedge b_2 = 0$. In fact we have

Lemma. *A is normal if and only if for any finite cover $\{a_1, \ldots, a_n\}$ there are $b_i, i = 1, \ldots, n$ such that $a_i \vee b_i = 1$ and $\bigwedge\limits_{i=1}^{n} b_i = 0$.*

PROOF by induction: Let the statement hold for n and let $\bigvee_{i=1}^{n+1} a_i = 1$. We have $b_1, \ldots, b_{n-1}, x$ such that $a_i \vee b_i = 1$ for $i \leq n-1, a_n \vee a_{n+1} \vee x = 1$ and $x \wedge \bigwedge\limits_{i=1}^{n-1} b_i = 0$. By normality there are b_n, y such that $a_n \vee b_n = 1, a_{n+1} \vee x \vee y = 1$ and $b_n \wedge y = 0$. Again, there are b_{n+1}, z such that $a_{n+1} \vee b_{n+1} = 1, x \vee y \vee z = 1$ and $b_{n+1} \wedge z = 0$. Thus,

$$\bigwedge_{i=1}^{n+1} b_i = (\bigwedge_{i=1}^{n-1} b_i \wedge b_n \wedge b_{n+1}) \wedge (x \vee y \vee z) = 0.$$

■

3.3. Lemma. *Let A be normal and let $a \vee b = 1$. Then there is a finite cover U of A such that $UU \prec \{a, b\}$.*

PROOF : Choose u, v such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$ and, further, u', v' such that $b \vee v' = 1 = v \vee u'$ and $u' \wedge v' = 0$. It is easy to check that $U = \{v', b \wedge v, u' \wedge a, u\}$ has the required properties. ■

3.4. Proposition. *Let A be a frame. Then the following statements are equivalent:*

(1) *A is normal,*

(2) *for each finite cover U there is a finite cover U' such that $U'U' \prec U$,*

(3) *for each finite cover U there is a cover U' such that $U'U' \prec U$.*

PROOF : (1)⇒(2): If $U = \{a_1, \dots, a_n\}$, we have, by 3.2, $U_1 \wedge \cdots \wedge U_n \prec U$ where $U_i = \{a_i, b_i\}$. Choose by 3.3 U_i' such that $U_i'U_i' \prec U_i$. Put $U' = U_1' \wedge \cdots \wedge U_n'$. By (1.2.6), $U'U' \prec U$.

(2)⇒(3) trivially.

(3)⇒(1): Let $a_1 \vee a_2 = 1$, let $UU \prec \{a_1, a_2\}$. Put

$$b_i = \bigvee \{x | x \in U, x \not\leq a_i\}.$$

Obviously $a_i \vee b_i = 1$. Now let $u_i \in U$ be such that $u_i \not\leq a_i$. Then $Uu_1 \leq a_2$; since $u_2 \not\leq a_2$ necessarily $u_1 \wedge u_2 = 0$. Thus, by distributivity, $b_1 \wedge b_2 = 0$. ∎

REFERENCES

[1] Dowker C.H., Strauss D., *Paracompact frames and closed maps*, Sympos. Math. **16** (1975), 93-116.

[2] Dowker C.H., Strauss D., *Sums in the category of frames*, Houston J.Math. **3** (1977), 7-15.

[3] Isbell J.R., *Atomless parts of spaces*, Math.Scand. **31** (1972), 5-32.

[4] Johnstone P.T., "Stone Spaces," Cambridge Univ.Press, Cambridge, 1982.

[5] Kelley J.L., "General Topology," The University Series in Higher Mathematics, Van Nostrand, 1955.

[6] Kříž I., Pultr A., *Systems of covers of frames and resulting subframes*, Proc.14th Winter School, Suppl.ai Rend.del Circ.Mat.diPalermo, Ser.II –No 14 (1987), 353-364.

[7] Pultr A., *Pointless uniformities I.Complete regularity*, Comment.Math.Univ.Carolinae 25 (1984), 91-104.

[8] Sun Shu–Hao, *On paracompact locales and metric locales*, Comment.Math.Univ.Carolinae **30** (1989), 101-107.

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