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Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 2, 377--385

Persistent URL: http://dml.cz/dmlcz/106548

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,2(1987)

FIXED POINTS, EQUILIBRIA AND MAXIMAL ELEMENTS IN LINEAR TOPOLOGICAL SPACES

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Abstract: In this paper we have proved some generalizations of the fixed-point theorems of Browder and Tarafdar in linear topological spaces. These results are used to prove some general theorems on the existence of maximal elements and equilibria in linear topological spaces.

Key words: Fixed points, maximal element, equilibrium.

Classification: 47H05, 90A14

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1. Introduction. Browder [1968 Theorem 1] proved that a nonempty convex-valued multifunction with open inverse images defined on a compact, convex subset of a Hausdorff linear topological space, has a fixed-point. This theorem of Browder was generalized by Tarafdar [1977] who replaced the assumption that the multifunction has open inverse images by a weaker condition. Tarafdar used this generalized theorem to prove the existence of a solution to a nonlinear variational inequality. Tarafdar's theorem was also used in Mehta and Tarafdar [1985], Tarafdar and Mehta [1984] to prove generalized versions of the Gale-Nikaido-Debreu theorem in mathematical economics.

The object of this paper is to prove some generalizations of the Browder and Tarafdar theorems and to give some applications. In section 2, a generalization of the Browder and Tarafdar fixed-point theorems is proved in a locally convex linear topological space. In section 3, a different approach is employed and a generalization of the Browder and Tarafdar theorems is proved, in a linear topological space using a recent theorem of

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x) This work was done while visiting the University of Bonn. I should like to thank Professor Hildebrand for his hospitality and encouragement. Financial support from the Sonderforschungsbereich is gratefully acknowledged.

Fan [1984]. Finally, in section 4, these generalizations are used to prove some general theorems on the existence of maximal elements and the existence of equilibria of qualitative games or abstract economies.

We shall use the following notation. If K is a subset of a linear topological space, then co K denotes the convex hull of K, cl K denotes the topological closure of K and int K denotes the topological interior of K.

2. A fixed-point theorem in a locally convex space. The following theorem has been proved by Browder [1968, p. 285].

Theorem 2.1. Let K be a compact, convex subset of a Hausdorff linear topological space. Let T be a multifunction on K into 2^K such that

(a) for each $x \in K$, $T(x)$ is a non-empty convex subset of K;

(b) for each $x \in K$, $T^{-1}(x) = \{y \in K : x \in T(y)\}$ is open in K.

Then there is a point $x_0 \in K$ such that $x_0 \in T(x_0)$.

Remark. Instead of condition (b) in the above theorem one can make the following weaker assumption:

(b´) for each $x \in K$, there is a y in K such that $x \in \text{int}\,[T^{-1}(y)]$.

Tarafdar [1977] has proved the existence of a fixed point of T under the weaker condition (b´).

The question arises if the assumption in the Browder and Tarafdar theorems, that the domain of the multifunction is compact, can be significantly weakened. We show in this section that this can be done in locally convex linear topological spaces. More specifically, we prove that a multifunction defined on a paracompact, convex subset X of a locally convex linear topological space with values in a compact subset D of X, and which satisfies the other conditions, has a fixed-point.

The method of proof we employ in this section is similar to that of Browder and consists of two steps. First, a partition-of-unity argument is used to get a continuous selection for the given multifunction. Secondly, a fixed-point theorem is applied to get a fixed-point for the continuous selection. The fixed-point theorem we use is a generalization due to Himmelberg [1972] of the classical fixed-point theorem of Fan.

Theorem 2.2. Let X be a paracompact, convex, nonempty subset of a Hausdorff locally convex linear topological space. Let D be a compact subset of X and $T:X \longrightarrow 2^D$ a multifunction such that

(i) for all x in X, $T(x)$ is convex and nonempty;

(ii) for each x in X, there exists a y in D such that $x \in \operatorname{int} T^{-1}(y)$.

Then there exists a point $x_0 \in D$ such that $x_0 \in T(x_0)$.

Proof. Let $0_y = \operatorname{int} T^{-1}(y)$ for y in D. By assumption (ii), $0 = \{0_y : y \in D\}$ is an open cover of X. Since X is paracompact, there exists an open locally finite refinement $V = \{V_a : a \in A\}$ of 0 where A is an index set [Kelley, 1955, p. 156]. Since every paracompact space is normal [Kelley, 1955, p. 159], there exists a family of continuous functions $\{f_a : a \in A\}$ on X with non-negative real values, such that for each $x \in X$, $\sum_{\alpha \in A} f_a(x) = 1$ and each f_a vanishes outside V_a [Kelley, 1955, p. 171].

Now for each $a \in A$, there exists a y_a in D such that $V_a \subseteq 0_{y_a} \subseteq T^{-1}(y_a)$ since V refines 0. Define a function f on X as follows: $f(x) = \sum_{\alpha \in A} f_a(x) y_a$. Since V is locally finite, each point x has a neighbourhood which intersects only finitely many sets of the family $\{V_a : a \in A\}$. Consequently, only finitely many functions have a non-zero value for each x. Thus f is a continuous function.

If $f_a(x) = 0$, $x \in V_a \subseteq 0_{y_a} \subseteq T^{-1}(y_a)$ so that $y_a \in T(x)$. Hence, by assumption (i), $f(x) \in T(x)$ for all $x \in X$, since $f(x)$ is a convex combination of the points y_a in $T(x)$. This proves the existence of a continuous selection f for the multifunction T on X.

Now f is a continuous single-valued function on a convex subset X of a separated locally convex topological vector space with values in a compact subset D of X. It follows that f is an upper semi-continuous multifunction on X with closed and convex values. Hence, by a theorem of Himmelberg [1972, Theorem 2] f has a fixed-point x_0. Consequently, $x_0 = f(x_0) \in T(x_0)$ and T has a fixed point.

q.e.d.

Remark 1. The above theorem shows that in a locally convex linear topological space, it is possible to generalize the Browder and Tarafdar theorems, to multifunctions defined on paracompact, convex subsets.

Remark 2. If we assume that for $y \in D$, $T^{-1}(y)$ is open in X, then assumption (ii) is automatically satisfied since $T(x)$ is non-empty for each $x \in X$. Hence, the fixed-point theorem of Yannelis and Prabhakar [1983] is a special case of the above theorem.

3. A generalization of the Browder and Tarafdar theorems. The object of this section is to generalize the Browder and Tarafdar theorems in arbitrary linear topological spaces and to prove them in a different way. The proofs of Browder´s theorem [1968] and the previous fixed-point theorem were based on a partition of unity. The approach used in the next theorem does not rely on this method. Instead, we prove the existence of a fixed-point by using a recent generalization of the classic Knaster-Kuratowski-Mazurkiewicz theorem due to Fan [1984].

Theorem 3.1. Let K be a convex subset of a Hausdorff linear topological space E and $T:K \longrightarrow 2^K$ a multifunction such that

(i) for each $x \in K$, $T(x)$ is nonempty and convex;

(ii) for each $y \in K$, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ contains an open subset O_y of K;

(iii) $\cup\{O_y : y \in K\} = K$;

(iv) there exists a nonempty subset K_0 of K such that $\bigcap_{x \in K_0} \{[O_x]^c\}$ is compact and K_0 is contained in a compact convex subset of E.

Then T has a fixed point.

Proof. Suppose that T has no fixed point. Then $x \notin T^{-1}(x)$ for all x. This implies that $x \notin O_x$ for all x.

For $x \in K$, define $F(x) = [O_x]^c$. Since O_x is open in K for each $x \in K$, it follows that $F(x)$ is a relatively closed subset of K for each $x \in K$.

By assumption (iv), $\bigcap_{x \in K_0} F(x_0)$ is compact for some K_0 contained in a compact, convex subset.

Let $\{x_1, x_2, \dots, x_n\}$ be a finite subset of X. We want to show that the convex hull S of $\{x_1, x_2, \dots, x_n\}$ is contained in $\bigcup_{i=1}^{n} F(x_i)$. We argue by contradiction. Suppose there is $x \in S$ such that $x \notin \bigcup_{i=1}^{n} F(x_i)$. Hence, for each $i=1,\dots,n$, $x \notin [O_{x_i}]^c$ which implies that for each i, $x \in O_{x_i}$. Now, by assumption (ii), $O_{x_i} \subseteq$ $\subseteq T^{-1}(x_i)$, so that $x \in T^{-1}(x_i)$ for all i. This implies that $x_i \in$ $\in T(x)$ for all i. By assumption (i), $T(x)$ is a convex set for all x which implies that $x \in T(x)$ and this contradicts our supposition that T has no fixed points. We conclude, therefore, that the convex hull of any finite subset $\{x_1, x_2, \dots, x_n\}$ is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$.

Thus all the conditions of Theorem 4 of Fan [1984] are satisfied and we conclude that $\bigcap_{x \in K} F(x) \neq \emptyset$. Let x_0 be a point in this intersection. Then $x_0 \in F(x) = [O_x]^c$ for all x in K, and this contradicts assumption (iii). The contradiction proves the existence of a fixed point for T.

q.e.d.

Remark. If K is compact, as in Browder [1968] or Tarafdar [1977] condition (iv) is automatically satisfied so that these theorems are a special case of the above theorem.

4. Applications. Let K be a subset of some linear topological space E. With each binary relation P on K one can associate a multifunction $T: K \longrightarrow 2^K$ in the following way: $y \in T(x)$ if and only if $(x,y) \in P$. Conversely, if $T: K \longrightarrow 2^K$ is a multifunction, then a binary relation P is defined on K by the condition that $(x,y) \in P$ if and only if $y \in T(x)$. Hence, we have the following:

Definition. A point x_0 is said to be a <u>maximal element</u> of the multifunction $T: K \longrightarrow 2^K$ if $T(x_0) = \emptyset$.

We now prove the following theorems on the existence of maximal elements of a multifunction T.

Theorem 4.1. Let X be a non-empty, paracompact, and convex subset of a separated locally convex linear topological space and D a compact, convex subset of X.
Let $T: X \longrightarrow 2^D$ be a multifunction such that

(i) for each $x \in X$, $x \notin \operatorname{co} T(x)$;

(ii) for each $x \in X$, there exists a y in D such that $x \in \operatorname{int} Q^{-1}(y)$, where $Q(x) = \operatorname{co} T(x)$ for x in X.

Then there is a maximal element, i.e. a point x_0 such that $T(x_0)=$ $=\emptyset$.

Proof. If there is no maximal element, $T(x)$, and therefore $Q(x)$, is nonempty for each $x \in X$, so that $Q:X \longrightarrow 2^D$ is nonempty, convex-valued multifunction. Assumption (ii) of Theorem 4.1 implies that assumption (ii) of Theorem 2.2 is satisfied. By Theorem 2.2, we conclude that there exists a point x_0 such that $x_0 \in T(x_0) \subseteq Q(x_0)$. This is a contradiction since assumption (i) implies that $x \notin Q(x)$ for all $x \in X$. Hence, T has a maximal element.

Remark. Assumption (ii) of Theorem 4.1 is weaker than the assumption used by Yannelis and Prabhakar that for all $y \in D$, $T^{-1}(y)$ is open in X. The reason for this is the following. If $T^{-1}(y)$ is open, then $Q^{-1}(y)$ is also open, where $Q(y)=\mathrm{co}\, T(y)$ [1983, p. 239]. Hence, Theorem 4.1 is a generalization of the result of Yannelis and Prabhakar [1983, p. 240].

Theorem 4.2. Let K be a convex subset of a Hausdorff linear topological space E and $T:K \longrightarrow 2^K$ a multifunction such that:

(i) for each $x \in K$, $T(x)$ is convex;

(ii) for each $x \in K$, $x \notin T(x)$;

(iii) for each $y \in K$, $T^{-1}(y)$ contains an open subset O_y of K;

(iv) $\bigcup\{O_y : y \in K\}=K$;

(v) there exists a nonempty subset K_0 of K such that $\bigcap_{x \in K_0}\{[O_x]^c\}$ is compact and K_0 is contained in a compact convex subset of E.

Then T has a maximal element.

Proof. Follows immediately from Theorem 3.1.

For a recent general theorem on the existence of maximal elements see Hadžić [1986, Proposition].

We turn now to the problem of the existence of equilibria of qualitative games or abstract economies.

Let X_i be a nonempty set for $i \in I$, where I is an index set. An <u>abstract economy</u> or <u>qualitative game</u> E is defined by a family of ordered triples (X_i, A_i, P_i) where $A_i : \prod_{j \in I} X_j \longrightarrow X_i$ and $P_i :$ $: \prod_{j \in I} X_j \longrightarrow X_i$ are multifunctions. An <u>equilibrium</u> for E is an

$x^* \in \prod_{i\in I} X_j$ such that for each i:

(i) $x_i^* \in \text{cl}\, A_i(x^*)$,

(ii) $P_i(x^*) \cap A_i(x^*) = \emptyset$.

We now prove the following generalization of the main theorem of Yannelis and Prabhakar [1983].

Theorem 4.3. Let $E=(X_i, A_i, P_i)$ be an abstract economy satisfying for each i in some countable index set I.

(i) X_i is nonempty, compact, convex, metrizable subset of a locally convex linear topological space;

(ii) $A_i(x)$ is convex and nonempty for all $x \in X$;

(iii) the correspondence $\bar{A}_i : X \longrightarrow 2^{X_i}$ defined by $\bar{A}_i(x) = \text{cl}\, A_i(x)$ for all $x \in X$ is upper semicontinuous.

(iv) the set $U_i = \{x \in X : \Phi_i(x) \neq \emptyset\}$ is open in X;

(v) for each $x \in U_i$, there exists y_i in X_i such that $x \in \text{int}\, \Phi_i^{-1}(y_i)$, where $\Phi_i(x) = A_i(x) \cap \text{co}\, P_i(x)$ for all $x \in X$;

(vi) $x_i \notin \text{co}\, P_i(x)$ for all $x \in X$.

then E has an equilibrium.

Proof. Consider the correspondence $\Phi_i : U_i \longrightarrow 2^{X_i}$. Assumptions (ii) and (iv) imply that Φ_i is nonempty and convex valued for each $x \in U_i$. Assumption (v) implies that assumption (ii) of Theorem 2.2, is satisfied. Hence, as the proof of Theorem 2.2, shows, there exists a continuous function $f_i : U_i \longrightarrow X_i$ such that $f_i(x) \in \Phi_i(x)$ for all $x \in U_i$, since U_i is paracompact as a subset of the metrizable space $\prod_{j\in I} X_j$.

The rest of the proof is based on an idea used by Gale and Mas-Colell [1975, p. 10] and is carried out as in Flam [1979] or Yannelis and Prabhakar [1983]. For completeness we give it here.

Define the correspondence $F_i : X \longrightarrow 2^{X_i}$ by

$$F_i(x) = \{f_i(x)\} \text{ if } x \in U_i,$$

$$F_i(x) = \bar{A}_i(x) \text{ if } x \notin U_i.$$

F_i is easily seen to be upper semi-continuous. Define $F : X \longrightarrow 2^X$ by $F(x) = \prod_{i\in I} F_i(x)$. It is easily verified that F is an upper semi-continuous multifunction with non-empty, closed and convex values. Hence, by Himmelberg's theorem [1972], F has a fixed point x^*. For each i, $x^* \notin U_i$ since if $x^* \in U_i$, then $x_i^* = f_i(x^*) \in$

$\in \Phi_i(x^*) \subseteq \text{co } P_i(x^*)$, a contradiction to assumption (vi). Hence, $x^* \notin U_i$ and it is easily checked that x^* is an equilibrium point of the economy E.

q.e.d.

Remark. Yannelis and Prabhakar [1983] assume that A_i and P_i have open inverse images, i.e. $A_i^{-1}(y)$ and $P_i^{-1}(y)$ are open for each y. It is easily verified that these assumptions imply that conditions (iv) and (v) of Theorem 4.3 are satisfied. Consequently Theorem 4.3 is a generalization of the main theorem of Yannelis and Prabhakar [1983, Theorem 6.1].

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(Oblatum 31.3. 1987)