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FUNCTION LIPSCHITZIAN MAPPINGS ON CONVEX METRIC SPACES

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Abstract: A lipschitzianness test for closed mappings acting on a (metrically) convex metric space, together with an application to contractive semidynamical systems is given.

Key words: Sequential maximality principle, closed mapping, metrically convex metric space, lipschitzianness test, contractive semidynamical system.

Classification: Primary 54F05, 54C10
Secondary 54H20

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0. Introduction. An important problem concerning a wide class of mappings acting on certain subsets of a metric space is that of finding sufficient conditions in order that a "local" Lipschitz property (in a sense precised by an appropriate context) should imply a "global" one on that subset. A first lot of results in this direction begins with the 1977 Kirk-Ray's lipschitzianness test [22] proved - in a normed framework - by a "local" method, involving the transfinite induction principle; later, in his 1978 paper, F.H. Clarke [13] initiated a second lot of results of this kind giving - in a metric framework - a differential lipschitzianness test based on Caristi's fixed point theorem. The main results of the present note belong to the first category of lipschitzi-

anness tests quoted above: more exactly, our main aim is to state and prove a "local" function lipschitzianness test for a class of closed mappings acting on a convex subset of a metrically convex metric space and taking values in a metrizable uniform space (extending in this way Kirk-Ray's result from a "functional" as well as a "metri able" point of view) the basic instrument in proving such an extension being a "sequential" maximality principle comparable with the classical Ekeland-Brøndsted's ones [17],[9]. As an application, a function lipschitzianness test for a class of contractive semidynamical systems is presented extending in this way a similar Crandall-Pazy's result [14] obtained by a direct method.

1. A "sequential" maximality principle. Concerning maximal elements in an abstract ordered set, a fundamental result obtained in this direction in the last few years is, undoubtedly, the so-called Brezis-Browder's ordering principle [8] (see also I. Ekeland [18]). However, in case of a "sequential" type additional structure, the above result seems to be - technically speaking - somehow difficult to be directly applicable, at least in its original form. It is the first objective of this section to formulate a "sequential" variant of Brezis-Browder's ordering principle; as a second objective, a "sequential" maximality principle will be stated and proved, extending in this way to metrizable uniform spaces the classical Ekeland-Brøndsted's contributions quoted in the Introduction, as well as those of J. Caristi [12] and W.A. Kirk [21].

Let X be a given (nonempty) set and let $\leq$ be an ordering on X (that is, a reflexive antisymmetric and transitive relation on X). A sequence $(x_n; n \in N)$ in X is said to be increasing iff $x_i \leq x_j$ whenever $i \leq j, i, j \in N$, and bounded above iff $x_n \leq y$, all $n \in N$, for some $y \in X$; also, a function $f: X \rightarrow R$ will be termed decreasing iff $x \leq y$ implies $f(x) \geq f(y)$, and bounded below iff $f(x) \geq b$, all $x \in X$, for some $b \in R$. Our main intention is now to establish the following "sequential" ordering principle on the abstract ordered set $(X, \leq)$.

Theorem 1. Under the above conventions, suppose the ordering $\leq$ on X and the denumerable family $(f_i; i \in N)$ of functions from X into R are such that

(i) any increasing sequence in X is bounded above,

(ii) f_i is monotone decreasing and bounded below for any $i \in N$.

Then, for every $x \in X$ there is an element $z \in X$ with $x \leq z$ and, moreover, for any $y \in X$ with $z \leq y$ we have $f_i(z) = f_i(y)$, all $i \in N$.

Proof. Let $x \in X$ be given. By the classical Brezis-Browder's ordering principle, there is an element $x_1 \geq x$ such that $y \in X$ and $x_1 \quad y$ imply $f_1(x_1) = f_1(y)$; furthermore, given $x_1 \in X$ there is, by the same ordering principle, an element $x_2 \geq x_1$ with the property $y \in X$ and $x_2 \leq y$ imply $f_2(x_2) = f_2(y)$, and so on. By induction, we get an increasing sequence $(x_n; n \in N)$ in X satisfying

(1) $n \in N$, $y \in X$ and $x_n \leq y$ imply $f_n(x_n) = f_n(y)$.

By (i), an element $z \in X$ may be found with $x_n \leq z$, all $n \in N$. We claim z is our desired element. Indeed, by the choice of our sequence, we evidently have $x \leq z$. Now, suppose $y \in X$ is such that $z \leq y$ then, clearly, $x_n \not\leq y$ all $n \in N$, and this gives, by (1), $f_n(x_n) = f_n(z) = f_n(y)$, all $n \in N$, completing the proof. Q.E.D.

Let X be an abstract (nonempty) set and let $D = (d_i; i \in \in N)$ be a denumerable and sufficient family of semimetrics on X (that is, $d_i(x,y) = 0$ for all $i \in N$ imply $x = y$). In this case, by a well-known construction, (X,D) appears as a metrizable uniform space. A sequence $(x_n; n \in N)$ in X is said to D - converge to $x \in X$ (and we write $x_n \xrightarrow{D} x$) iff it d_i - converges to x for any $i \in N$, and a D - Cauchy sequence iff it is a d_i - Cauchy sequence for any $i \in N$. Also, a function $f: X \longrightarrow R$ is called d_i - lower semicontinuous iff it is lower semicontinuous as a function from (X, d_i) into R (here $i \in N$ is an arbitrary fixed element). Finally, X' being another abstract (nonempty) set and $D' = (d'_i; i \in N)$ a denumerable and sufficient family of semimetrics on X', a mapping $T: X \longrightarrow X'$ will be termed closed iff for any sequence $(x_n; n \in N)$ in X and any couple $x \in X$, $x' \in X'$ with $x_n \xrightarrow{D} x$ and $Tx_n \xrightarrow{D'} x'$ as $n \longrightarrow \infty$ we have $Tx = x'$. Suppose in what follows (X,D) and (X', D') defined as above are complete metrizable uniform spaces (that is, every D (D') - Cauchy sequence in X (X') is a D (D') - convergent one) and let T be a closed mapping from X into X'. In such a situation, as an important application of Theorem 1, the following "sequential" maximality principle may be formulated.

<u>Theorem 2</u>. Suppose the denumerable families $(\varphi_i; i \in N)$ and $(\psi_i; i \in N)$ of functions from X into R are such that

(iii) φ_i and ψ_i are d_i - lower semicontinuous and bounded below, for any $i \in N$.

Then, for any $x \in X$ there is an element $z \in X$ such that the following conclusions hold

(a) $d_i(x,z) \leq \varphi_i(x) - \varphi_i(z)$, $d_i'(Tx,Tz) \leq \psi_i(x) - \psi_i(z)$, all $i \in N$

(b) for every $y \in X$, $y \neq z$, either $d_i(z,y) > \varphi_i(z) - \varphi_i(y)$ or $d_i'(Tz,Ty) > \psi_i(z) - \psi_i(y)$ for some element $i \in N$.

Proof. Let us define an ordering $\leq$ on X by

(2) $x \leq y$ iff $d_i(x,y) \leq \varphi_i(x) - \varphi_i(y)$ and $d_i'(Tx,Ty) \leq \psi_i(x) - \psi_i(y)$, all $i \in N$.

Firstly, φ_i and ψ_i are decreasing and, by hypothesis, bounded below, for any $i \in N$, proving (ii) holds. Secondly, let $(x_n; n \in N)$ be an increasing sequence in X, that is,

(3) $d_i(x_n,x_m) \leq \varphi_i(x_n) - \varphi_i(x_m)$, $d_i'(Tx_n,Tx_m) \leq \psi_i(x_n) - \psi_i(x_m)$, all $i \in N$, all $n,m \in N$, $n \leq m$.

It immediately follows $(\varphi_i(x_n); n \in N)$ and $(\psi_i(x_n); n \in N)$ are decreasing sequences in R hence (by the second part of (iii)) Cauchy sequences in R, for every $i \in N$ so that, by (3), $(x_n; n \in N)$ and $(Tx_n; n \in N)$ are D (D') - Cauchy sequences in X (X'). By completeness hypothesis, $x_n \xrightarrow{D} x$ and $Tx_n \xrightarrow{D'} x'$ as $n \to \infty$ for some $x \in X$, $x' \in X'$ and this gives (by closedness hypothesis) $Tx = x'$ that is, $Tx_n \xrightarrow{D'} Tx$ as $n \to \infty$. In such a case, taking the limit as $m \to \infty$ in (3) and remembering that first part of (iii) we get the evaluations

$d_i(x_n,x) \leq \varphi_i(x_n) - \varphi_i(x)$, $d'(Tx_n,Tx) \leq \psi_i(x_n) - \psi_i(x)$, all $i \in N$, all $n \in N$

that is, $x_n \leq x$, all $n \in N$, proving (i) holds, too. Consequently, Theorem 1 applies and this completes the proof. Q.E.D.

As an immediate application of Theorem 2, the following fixed point result on this class of metrizable uniform structures may be given.

<u>Theorem 3</u>. Under the same conditions of Theorem 2, suppose the mapping $U:X \longrightarrow X$ is such that

(4) $d_i(x,Ux) \preceq \varphi_i(x) - \varphi_i(Ux)$, $d_i'(Tx,TUx) \leq \psi_i(x) - \psi_i(Ux)$, all $i \in N$

then, for any $x \in X$ there is an element $z \in X$ such that conclusions (a) + (b) of Theorem 2 hold and, in addition, $z = Uz$ (z is a fixed point of U).

<u>Proof</u>. Let $x \in X$ be arbitrary fixed and let $z \in X$ be the element indicated by Theorem 2. By (4), $z \preceq Uz$ so that (taking into account (b)), we necessarily have $z = Uz$ and this ends the proof. Q.E.D.

As a particular case of our considerations, suppose the denumerable and sufficient families D and D' reduce to a single element (respectively, a single metric on X and X') then, Theorem 2 reduces to the author's result [26] while Theorem 3 to the Downing-Kirk's result [16] (see also D. Downing [15]). Moreover, in case $X = X'$, $T = I$ (the identity) and $\varphi = \psi$, from the corresponding variants of Theorem 2 and Theorem 3 we get Ekeland-Brøndsted's results quoted above (see also [18], [10] as well as J.P. Aubin and J. Siegel [2], E. Bishop and R.R. Phelps [6], M. Turinici [24], J.D. Weston [27]) and respectively, Caristi-Kirk-Browder's ones [12],[21],[11] (see also S.A. Husain and V.M. Sehgal [19], S. Kasahara [20], J. Siegel [23], M. Turinici [25], C.S. Wong [28]). A number of extensions to (non-metrizable) uniform spaces of the above theorems will be given in a forthcoming paper.

2. <u>The main results</u>. Let (X,d) be a given metric space. For every $x,y \in X$, let $[x,y]$ denote the metric segment between x and y (the subset of all $z \in X$ with $d(x,z) + d(z,y) = d(x,y)$) and put also $]x,y] = [x,y] \setminus \{x\}$, $]x,y[= [x,y] \setminus \{x,y\}$. Evidently, every segment is a nonempty bounded and closed subset of X; moreover, for every arbitrary fixed $x,y \in X$ and every $z \in [x,y]$ we have the inclusion $[x,z] \subset [x,y]$, $[z,y] \subset [x,y]$ (see, e.g., W.A. Kirk [21] for more details). A (nonempty) subset Y of X is said to be (metrically) convex iff for every $x,y \in Y$, the segment $[x,y]$ is contained in Y. Also, the ambient metric space (X,d) will be termed (metrically) convex in Menger's sense [7, ch. I] iff for any distinct $x,y \in X$, $]x,y[$ is not empty.

In what follows, (X,d) is a complete (metrically) convex metric space, Y a (nonempty) convex subset of X, (X',D') a (sequentially) complete metrizable uniform space defined as in the preceding section and $F = (f_i; i \in N)$ a denumerable family of functions from R_+ into itself. A mapping $T: Y \rightarrow X'$ is said to be directionally closed iff for any couple $x,y \in Y$, the restriction $T/[x,y]$ is a closed mapping from $([x,y],d)$ into (X',D'). In the same context, T will be termed directionally F - lipschitzian iff for any distinct $x,y \in Y$ there is an element $u \in]x,y]$ satisfying $d_i'(Tx,Tu) \leq f_i(d(x,u))$, all $i \in N$, and globally F - lipschitzian iff $d_i'(Tx,Ty) \leq f_i(d(x,y))$, all $i \in N$, all $x,y \in Y$. Finally, a function $f: R_+ \rightarrow R_+$ is said to be super-additive iff $f(t+s) \geq f(t)+f(s)$ for all $t,s \in R_+$.

From the above definitions it trivially follows that every globally F - lipschitzian mapping is also directionally F - lipschitzian but the converse is not in general true so that, it is justified to look for an answer to the following question: under what (supplementary) conditions does a directionally F - lipschitzian mapping become a globally F - lipschitzian one? In this direction, the main result of the present note is

<u>Theorem 4</u>. Let (X,d), (X',D') and Y be as before and suppose the mapping $T:Y \longrightarrow X'$ and the family $F = (f_i; i \in N)$ of functions from R_+ into itself are such that

(iv) T is directionally F - lipschitzian

(v) T is directionally closed

(vi) f_i is super-additive and lower semicontinuous, for every $i \in N$.

Then, necessarily, T is globally F - lipschitzian (on Y).

<u>Proof</u>. Let $x,y \in Y$, $x \neq y$ be arbitrary. Define a denumerable family of functions $\varphi : X \longrightarrow R_+$ and $\psi_i : X \longrightarrow R_+$ $(i \in N)$ by the convention

(5) $\varphi(u) = d(u,y)$, $\psi_i(u) = f_i(d(u,y))$, $(i \in N)$, $u \in X$.

Firstly, by the second part of (vi), φ is continuous and ψ_i lower semicontinuous for any $i \in N$ and the same conclusion is valid for the restrictions $\varphi/[x,y]$ and $\psi_i/[x,y]$ $(i \in N)$. Secondly, by (v), the restriction $T/[x,y]$ (denoted also by T in what follows) is a closed mapping from $[x,y]$ into X'. This shows that Theorem 2 applies (with (X,D) replaced by $([x,y],d)$) so that, for $x \in [x,y]$ there is an element $z \in [x,y]$ satisfying conclusions

(a)' $d(x,z) \leq \varphi(x) - \varphi(z)$ and $d_i'(Tx,Tz) \leq \psi_i(x) - \psi_i(z)$, all $i \in N$

(b)' for every $u \in [x,y]$, $u \neq z$, either $d(z,u) > \varphi(z) - \varphi(u)$ or $d_i'(Tz,Tu) > \psi_i(z) - \psi_i(u)$, for some $i \in N$.

Suppose $z \neq y$. For every $u \in]z,y] \subset [x,y]$, the first relation of the conclusion (b)$'$

$$d(z,u) > \varphi(z) - \varphi(u) = d(z,y) - d(u,y) = d(z,u)$$

is impossible, so we must have (taking also into account the first part of (vi))

$$d_i'(Tz,Tu) > \psi_i(z) - \psi_i(u) = f_i(d(z,y)) - f_i(d(u,y)) \geq$$
$$f_i(d(z,y) - d(u,y)) = f_i(d(z,u)) \text{ for some } i \in N$$

which is also impossible, because of (iv). Consequently, $z =$ $= y$ and then, by the conclusion (a)$'$,

$$d_i'(Tx,Ty) \leq \psi_i(x) - \psi_i(y) = f_i(d(x,y)), \text{ all } i \in N$$

and since $x,y \in Y$ were arbitrary, our proof is complete. Q.E.D.

Concerning condition (vi) of the main result, it should be noted that an important example of functions from R_+ into itself satisfying that condition is offered by the choice

(6) $f(t) = k\, t^r$, $t \in R_+$

$k \geq 0$ and $r \geq 1$ being arbitrary fixed elements. In the same time, concerning condition (v), it is almost evident that it is automatically fulfilled by any mapping T from Y into X' closed in Altman's sense [1] (that is, for any sequence (x_n; $n \in$ $\in N$) in Y and any couple $x \in X$, $x' \in X'$ with $x_n \xrightarrow{d} x$ and $Tx_n \xrightarrow{D'} x'$ as $n \to \infty$ we have $x \in Y$ and $Tx = x'$). Finally, let $(X,d),(X',D')$ and Y be as before and let $K = (k_i; i \in N)$ be a denumerable family of positive numbers. By convention, a mapping $T: Y \to X'$ will be termed directionally (globally) K - lipschitzian iff it is directionally (globally) F - lipschitzian, F being the denumerable family $(f_i; i \in N)$ of functions from R_+ into itself defined by: for any $i \in N$, f_i is that expressed by (6) with $r = 1$ and $k = k_i$. In such a case, as a direct consequence of the main result, the following (ordinary) lipschitzianness test on (metrically) convex metric spaces may be formulated.

Theorem 5. Under the same general conventions, suppose the mapping $T:Y \longrightarrow X'$ is directionally K - lipschitzian in th above sense and closed in Altman's sense. Then, necessarily, T is globally K - lipschitzian (on Y).

It should be noted that in case D' reduces to a single element (a single metric d' on X') the above theorem reduces in fact to Kirk-Ray's result quoted in the Introduction (see also the author's paper [24] as well as S.A. Husain and V.M. Sehgal [19]).

3. Applications to semidynamical systems. Let (X,D) be a given complete metrizable uniform space and let $\Omega = (\omega_i;$ $i \in N)$ be a denumerable family of real numbers. By an Ω-contractive semidynamical system on X we mean a mapping $(t,x) \mapsto$ $\mapsto S(t,x) = S(t)x$ from $R_+ \times X$ into X satisfying

(vii) $S(0)x = x$ for all $x \in X$

(viii) $S(t+s)x = S(t)S(s)x$, all $t,s \in R_+$, all $x \in X$

(ix) $d_i(S(t)x,S(t)y \leq (\exp(\omega_i t))\, d_i(x,y)$, for all $t \in R_+$, $x,y \in Y$ and $i \in N$.

(Of course, the notion of contractive semidynamical system may be compared with that of semidynamical system in Bajaj's sense [3] (see also N.P. Bhatia and G.P. Szegö [5, ch. I]) or, equivalently, with that of contractive semigroup in Brezis-Browder's sense [8]). An important problem concerning this class of semidynamical systems is that regarding (function) Lipschitz properties with respect to the temporal variable.

In order to give an efficient answer in this direction, suppose the considered Ω-contractive semidynamical system S on X satisfies the following closedness property at every point $x \in X$

(x) for any sequence $(t_n; n \in N)$ in R_+ and any couple $t \in R_+$, $y \in X$ with $t_n \to t$ and $S(t_n)x \xrightarrow{D} y$ as $n \to \infty$ we have $S(t)x = y$

and let the denumerable family of functions from R_+ into itself $F = (f_i; i \in N)$ be such that condition (vi) of the preceding section holds. Denote by $X(S,F)$ the subset of all $x \in X$ satisfying

(xi) for any $\varepsilon > 0$ there is a number $0 < \sigma < \varepsilon$ such that $d_i(x, S(\sigma)x) \leq f_i(\sigma)$, all $i \in N$

and, for the sake of simplicity denote also

(7) $\beta_i(t) = \max(\exp(\omega_i t), 1)$, $t \in R_+$, $i \in N$

In such a case, let $x \in X(S,F)$ and $a > 0$ be arbitrary fixed. Given two positive numbers $s, t \in R_+$ with $0 \leq s < t \leq a$, there is, by (xi), a positive $\sigma < t-s$ such that $d_i(x, S(\sigma)x) \leq f_i(\sigma)$, all $i \in N$ so (denoting $r = s + \sigma$), we get by (viii) + (ix) and the notation (7), the evaluation

$$d_i(S(s)x, S(r)x) \leq (\exp(\omega_i s)) d_i(x, S(\sigma)x) \leq \beta_i(s) f_i(\sigma) \leq$$
$$\leq \beta_i(a) f_i(\sigma) = \beta_i(a) f_i(r-s), \text{ all } i \in N$$

proving the mapping $t \mapsto S(t)x$ is directionally G - lipschitzian on the interval $[0,a]$, the denumerable family of functions from R_+ into itself $G = (g_i; i \in N)$ being defined by the convention $g_i = \beta_i(a) f_i$, all $i \in N$. Consequently, the main result applies (with $X = R_+$, $Y = [0,a]$ and X' = the ambient metrizable uniform space of the considered semidynamical system) so that we proved

Theorem 6. Under the conventions stated above, for any $x \in X(S,F)$ and any $a > 0$, the mapping $t \mapsto S(t)x$ is necessarily globally G - lipschitzian on $[0,a]$ that is,

$$d_i(S(t)x,S(s)x) \leq \beta_i(a)f_i(t-s), \; 0 \leq s \leq t \leq a, \; i \in N.$$

An important particularization of the above theorem corresponds to the case when D reduces to a single metric d on X (and, correspondingly, Ω reduces to a single real number ω). In such a case, let us denote

(8) $$L(x) = \liminf_{t \to 0_+} (1/t)\, d(x,S(t)x), \; x \in X$$

and let the function $h: R_+ \to R_+$ be defined by

(9) $$h(t) = \begin{cases} (1/\omega)(\exp(\omega t) - 1), & t \in R_+, \; \omega \neq 0 \\ t & , t \in R_+ \; \omega = 0 \end{cases}$$

Now, X(S) denoting the subset of all points $x \in X$ with $L(x) < +\infty$, it is a simple matter to verify condition (x) will be satisfied by any mapping $f = Mh$, $M > L(x)$ being arbitrary fixed, in which case, as an important consequence of Theorem 6 we have

Theorem 7. Under the particular cases expressed above, for any $x \in X(S)$ and any $a > 0$, the mapping $t \mapsto S(t)x$ is necessarily globally $\beta(a)L(x)h$ - lipschitzian on $[0,a]$ that is,

$$d(S(t)x,S(s)x \leq \beta(a)L(x)\, h(t-s), \; 0 \leq s \leq t \leq a.$$

It should be noted that the above result proved - in case of a Banach space - by M.G. Crandall and A. Pazy [14] (see also V. Barbu [4, ch. III]) has a number of important applications to nonlinear contraction semigroups theory; we refer especially to the above quoted Barbu's work for more details and concrete discussions.

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