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THE NIL-DEGREE OF A TORSION-FREE ABELIAN GROUP

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Abstract: Recently Webb, Acta Sci. Math. Szeged 39 (1977), 185-188 provided a bound for the nil-degree (if it is finite) of a torsion free group of finite rank. In this paper we extend Webb's result to torsion-free groups A, not necessarily of finite rank, but with certain finiteness conditions on the rank of A/pA for each prime p. We also prove an associative ring on such a group is nilpotent exactly if it is nil.

Key words: Ring, (strong) nil-degree, p-adic module.

Classification: 20K20

All groups that we consider here are abelian groups, and all rings are not necessarily associative rings. A <u>ring on a group</u> A is a ring whose additive group is (isomorphic to) A. We write $(A,\cdot)$ for a ring on A and say that A <u>supports</u> $(A,\cdot)$. The rank of A is denoted by $r(A)$. We use the standard notation Z, and for a prime p, J_p for the group of integers and the group of p-adic integers, respectively.

Szele [9] defined the <u>nil degree</u> (<u>nilstufe</u>) of a group A to be ∞ or the largest integer n (if one exists) such that there is an associative ring $(A,\cdot)$ on A with $(A,\cdot)^n \neq 0$. Gard-

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ner [8] defined the strong nil-degree of A similarly, where for the non-associative ring $(A,\cdot)$ on A, $(A,\cdot)^n$ is the subring of $(A,\cdot)$ generated by all products of the form $(\ldots((a_1\cdot a_2)\cdot a_3)\cdot\ldots)\cdot a_n$. Throughout this paper $(A,\cdot)^n$ will always have this meaning. Feigelstock [4] has introduced a concept very similar to the strong nil-degree of a torsion-free group. Following Feigelstock we define the extra strong nil-degree (strong nilstufe) of the torsion-free group A to be the positive integer n such that there is a ring on A with a non-zero product of length n (all possible bracketings considered), but no ring on A with non-zero products of length greater than n. If no such n exists then the extra strong nil-degree is defined to be ∞ . For a torsion-free group A we let N(A), $N_S(A)$ and $N_E(A)$ respectively denote the nil-degree, the strong nil-degree and the extra strong nil-degree of A. A group is called nil if it has nil-degree 1.

Feigelstock [5] has claimed that if A is a torsion-free group of rank two then $N_E(A)$ is 1, 2, or ∞ , but appears to have only shown that N(A) is 1, 2, or ∞ ; his proof relies on Lemma 1 of Beaumont and Wisner [3] that requires consideration of associative rings. Feigelstock has also conjectured that if A is a torsion-free group of finite rank n then $N_E(A)$ is 1, 2,...,n or ∞ .

Recently Webb [10] has shown that if A is a torsion-free group of rank n then N(A) is 1,2,...,n or ∞ and $N_E(A)$ is $1,2,\ldots,2^{n-1}$ or ∞ . Also, he has provided an example of a torsion-free group A of rank three for which $N_E(A) = 4$. Thus Feigelstock's conjecture is not true. However, if we replace $N_E(A)$ with $N_S(A)$, the conjecture can be proved.

Theorem 1. Let $(A,\cdot)$ *be a ring on a torsion-free group* A *of finite rank* n. *If* $(A,\cdot)^m = 0$ *for some positive integer* m *then* $(A,\cdot)^{n+1} = 0$.

Proof: Suppose $(A,\cdot)^m = 0$ for some positive integer m, and k is a positive integer for which $(A,\cdot)^{k+1} \neq 0$. We show $(A,\cdot)^k/(A,\cdot)^{k+1}$ is not a torsion group.

Indeed, suppose $(A,\cdot)^k/(A,\cdot)^{k+1}$ is torsion. If we choose a non-zero element $a \in (A,\cdot)^k$ then there is an integer $n_1 \neq 0$ such that $n_1 a \in (A,\cdot)^{k+1}$. Thus

$$0 \neq n_1 a = a_1' \cdot a_1 + a_{1_2}' \cdot a_{1_2} + a_{1_3}' \cdot a_{1_3} + \ldots + a_{1_{n(1)}}' \cdot a_{1_{n(1)}},$$

where a_1 and a_{1_i} are in A, and a_1' and a_{1_i}' are in $(A,\cdot)^k$, for each $i \in \{2,3,\ldots,n(1)\}$. Without loss of generality we can assume $a_1' \cdot a_1 \neq 0$.

Since $a_1' \in (A,\cdot)^k$ it is possible to choose a non-zero integer n_2 such that $n_2 a_1' \in (A,\cdot)^{k+1}$. Hence

$$0 \neq n_2(a_1' \cdot a_1) = (a_2' \cdot a_2 + a_{2_2}' \cdot a_{2_2} + a_{2_3}' \cdot a_{2_3} + \ldots + a_{2_{n(2)}}' \cdot a_{2_{n(2)}}) \cdot a_1,$$

where a_2 and a_{2_i} are in A, and a_2' and a_{2_i}' are in $(A,\cdot)^k$, for each $i \in \{2,3,\ldots,n(2)\}$. Again we can assume $(a_2' \cdot a_2) \cdot a_1 \neq 0$.

If we repeat this procedure we can obtain elements a_1, $a_2,\ldots,a_{m-k}$ in A, and an element a_{m-k}' in $(A,\cdot)^k$ such that

$$(\ldots((a_{m-k}' \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \ldots) \cdot a_1 \neq 0.$$

Clearly

$$(\ldots((a_{m-k}' \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \ldots) \cdot a_1 \in (A,\cdot)^m$$

contradicting the fact that $(A,\cdot)^m = 0$. We conclude that

$(A,\cdot)^k/(A,\cdot)^{k+1}$ cannot be a torsion group.

Consequently, for each positive integer k for which $(A,\cdot)^{k+1} \neq 0$, $(A,\cdot)^k/(A,\cdot)^{k+1}$ has torsion-free rank greater than zero. That is, $r((A,\cdot)^k)$ is strictly greater than $r((A,\ )^{k+1})$. Since A has finite rank n, $(A,\cdot)^{n+1} = 0$.

Corollary 2. If A is a torsion-free group of rank n then $N_S(A)$ is $1,2,\ldots,n$ or ∞.

It is not difficult to find torsion-free groups A of rank n for which the bound of n for $N_S(A)$ in Corollary 2 is actually attained. Indeed consider $A = \bigoplus_{i=1}^{n} A_i$ where each A_i is a rational group with type $(2i,2i,\ldots,2i,\ldots)$. A reference to Theorem 4.2 of Gardner [8] shows $N_S(A) = n$.

The remainder of this paper is concerned with extending the associative case od Corollary 2 (that is, Webb's Theorem) to other classes of torsion-free groups. Our aim is two-fold: we wish to find some infinite rank torsion-free groups whose nil degrees, if finite, are bounded, and we would also like under certain circumstances to lower the bound on the finite nil-degrees mentioned in the Corollary. We concentrate our attention on torsion-free groups A with the property that for each prime p, $r(A/pA)$ is bounded by some positive integer n (not depending on p). This amounts to considering torsion-free groups whose p-basic subgroups all have rank $\leq$ n. Clearly a torsion-free group of rank n satisfies this property.

For a group A and a prime p let $\hat{A}_{(p)} = \varprojlim (A/p^kA)$ denote the p-adic completion of A. If A is torsion-free and p-reduced then clearly $\hat{A}_{(p)}$ is torsion-free. Also, $\hat{A}_{(p)}$ can be made into a module over the ring of p-adic integers Q_p^* by defining, for $j = s_0 + s_1 p + \dots + s_k p^k + \dots$ in Q_p^* and $(a_1 + pA, a_2 + p^2A, \dots, a_k + p^kA, \dots)$ in $\hat{A}_{(p)}$,

$$j(a_1 + pA, a_2 + p^2A, \dots, a_k + p^kA, \dots)$$
$$= (j^{(1)}(a_1 + pA), j^{(2)}(a_2 + p^2A), \dots, j^{(k)}(a_k + p^kA), \dots)$$

where $j^{(k)} = s_0 + s_1 p + \dots + s_{k-1}p^{k-1}$ for each positive integer k.

The next result enables us to extend rings on certain groups to rings on their p-adic completions.

<u>Proposition 3</u>. <u>Suppose</u> A <u>is a group with no elements of infinite</u> p<u>-height for some prime</u> p, <u>and</u> $(A,\cdot)$ <u>is a ring on</u> A. <u>Then there is exactly one ring structure</u> $(\hat{A}_{(p)},\cdot)$ <u>on</u> $\hat{A}_{(p)}$ <u>which extends that of</u> $(A,\cdot)$, <u>and this preserves associativity and commutativity in</u> $(A,\cdot)$.
<u>Furthermore</u> $(\hat{A}_{(p)},\cdot)$ <u>becomes</u> a <u>Q_p^*-algebra</u>.

Proof: The proof of the Proposition is analogous to the proof of Corollary 119.4 of Fuchs [7]. The only statements that require verification are that the extension $(\hat{A}_{(p)},\cdot)$ of $(A,\cdot)$ is unique, and that $(\hat{A}_{(p)},\cdot)$ becomes a Q_p^*-algebra. Since A can be regarded as a p-pure and p-dense subgroup of the p-reduced group $\hat{A}_{(p)}$ the proof of Lemma 119.2 of Fuchs [7] applies to show that $(\hat{A}_{(p)},\cdot)$ is unique. That $(\hat{A}_{(p)},\cdot)$ is a Q_p^*-algebra follows at once from the definition of the Q_p^*-module $\hat{A}_{(p)}$ given prior to the Proposition.

The following well known result is required.

(4) (Fuchs [6], p. 166). *Let* $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ *be a* p*-pure exact sequence. Then the sequence*

$$0 \to \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \xrightarrow{\hat{\beta}} \hat{C}_{(p)} \to 0$$

is splitting exact.

Lemma 5. Suppose A *is a torsion-free group and* B *is a* p*-basic subgroup of* A. *Then* $\hat{A}_{(p)}$ *and* $\hat{B}_{(p)}$ *are isomorphic* p*-adic modules. Furthermore,* $\hat{A}_{(p)}$ *has finite rank over* Q_p^* *if and only if* B *has finite rank over* Z, *and in this case the* Q_p^**-rank of* $\hat{A}_{(p)}$ *and* the Z*-rank of* B *coincide.*

Proof: Consider the p-pure exact sequence

$$0 \to B \xrightarrow{\alpha} A \to A/B \to 0$$

where α is the inclusion map. (4) shows that the sequence

$$0 \to \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \to (\widehat{A/B})_{(p)} \to 0$$

is splitting exact, so $\hat{A}_{(p)} \cong \operatorname{Im} \hat{\alpha} \oplus (\widehat{A/B})_{(p)}$. Since A/B is p-divisible, $(\widehat{A/B})_{(p)} = 0$, whence $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ (as groups).

Next let $(b_1 + pB, b_2 + p^2B, \ldots, b_k + p^kB, \ldots)$ be an arbitrary element of $\hat{B}_{(p)}$, and let j be a p-adic integer. Then

$$\begin{aligned} &\hat{\alpha}(j(b_1 + pB, b_2 + p^2B, \ldots, b_k + p^kB, \ldots)) \\ &= \hat{\alpha}(j^{(1)}b_1 + pB, j^{(2)}b_2 + p^2B, \ldots, j^{(k)}b_k + p^kB, \ldots) \\ &= (j^{(1)}b_1 + pA, j^{(2)}b_2 + p^2A, \ldots, j^{(k)}b_k + p^kA, \ldots) \\ &= j(b_1 + pA, b_2 + p^2A, \ldots, b_k + p^kA, \ldots) \\ &= j(\hat{\alpha}(b_1 + pB, b_2 + p^2B, \ldots, b_k + p^kB, \ldots)), \end{aligned}$$

so $\hat{A}_{(p)}$ and $\hat{B}_{(p)}$ are isomorphic Q_p^*-modules.

Suppose now the rank of B is finite. A trivial induction argument together with (4) show that the rank of $\hat{B}_{(p)}$ over Q_p^* is precisely the rank of B. Thus the Q_p^*-rank of $\hat{A}_{(p)}$ is the rank of B. To prove the converse suppose $\hat{A}_{(p)}$ has finite rank n over Q_p^*, and B has rank strictly greater than n. Then B contains a p-pure free summand of rank (n+1), so (4) shows that $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ contains a summand isomorphic to the direct sum of (n+1) copies of J_p. This is clearly impossible.

Suppose A is a torsion-free group and $\alpha : A \longrightarrow \hat{A}_{(p)}$ is the canonical map from A into its p-adic completion. If a is an arbitrary element of A then let $\hat{a}$ denote the image of a under the map α. Similarly if B is a p-basic subgroup of A and $\beta : B \longrightarrow \hat{B}_{(p)}$ is the canonical map from B into its p-adic completion, then let $\bar{b}$ denote the image of $b \in B$ under the map β. We can now improve the final assertion in Lemma 5.

Lemma 6. Let A be a torsion-free group with finite rank p-basic subgroup $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_n \rangle$. Then the elements $\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n$ of $\hat{A}_{(p)}$ form a basis of $\hat{A}_{(p)}$ over Q_p^*.

Proof: From Lemma 5 it suffices to show that the set $S = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\}$ of elements of $\hat{B}_{(p)}$ form a basis of $\hat{B}_{(p)}$ over Q_p^*.

First we show that S is independent over Q_p^*. Indeed suppose

(*) $$j_1\bar{b}_1 + j_2\bar{b}_2 + \ldots + j_n\bar{b}_n = \bar{0}$$

for some p-adic integers $j_1, j_2, \ldots, j_n$. With $j_i^{(k)}$ defined as usual for $i \in \{1,2,\ldots,n\}$ and $k \in \{1,2,\ldots\}$, (*) becomes

$$(j_1^{(1)}b_1 + pB,\ j_1^{(2)}b_1 + p^2B, \dots, j_1^{(k)}b_1 + p^kB, \dots) +$$
$$+ (j_2^{(1)}b_2 + pB,\ j_2^{(2)}b_2 + p^2B, \dots, j_2^{(k)}b_2 + p^kB, \dots) + \dots +$$
$$+ (j_n^{(1)}b_n + pB,\ j_n^{(2)}b_n + p^2B, \dots, j_n^{(k)}b_n + p^kB, \dots)$$
$$= (pB,\ p^2B, \dots, p^kB, \dots).$$

Thus

$$j_1^{(k)}b_1 + j_2^{(k)}b_2 + \dots + j_n^{(k)}b_n \in p^kB$$

for each $k \in \{1,2,\dots\}$. Hence for every $k \in \{1,2,\dots\}$ there are integers $\ell_1^{(k)},\ \ell_2^{(k)}, \dots, \ell_n^{(k)}$ such that

$$j_1^{(k)}b_1 + j_2^{(k)}b_2 + \dots + j_n^{(k)}b_n = \ell_1^{(k)}p^kb_1 + \ell_2^{(k)}p^kb_2 + \dots + \ell_n^{(k)}p^kb_n.$$

Consequently $j_i^{(k)} = \ell_i^{(k)}p^k$ for each $i \in \{1,2,\dots,n\}$. But then

$$j_i\bar{b}_i = (j_i^{(1)}b_i + pB,\ j_i^{(2)}b_i + p^2B, \dots, j_i^{(k)}b_i + p^kB, \dots)$$
$$= (\ell_i^{(1)}pb_i + pB,\ \ell_i^{(2)}p^2b_i + p^2B, \dots, \ell_i^{(k)}p^kb_i + p^kB, \dots)$$
$$= \bar{0},$$

for each $i \in \{1,2,\dots,n\}$. Since $\hat{B}_{(p)}$ is torsion-free as a Q_p^*-module, we conclude that S is independent over Q_p^*.

Next we show that S generates $\hat{B}_{(p)}$. Let

$$(b^{(1)} + pB,\ b^{(2)} + p^2B, \dots, b^{(k)} + p^kB, \dots)$$

be an arbitrary element of $\hat{B}_{(p)}$. Then for each $k \in \{1,2,\dots\}$ there are suitable integers $m_i^{(k)}$, $i \in \{1,2,\dots,n\}$, such that

$$b^{(k)} + p^kB = (m_1^{(k)}b_1 + m_2^{(k)}b_2 + \dots + m_n^{(k)}b_n + p^kB,$$

and $0 \leq m_i^{(k)} < p^k$. Now for each $k \in \{1,2,\dots\}$

$$b^{(k+1)} + p^kB = b^{(k)} + p^kB,$$

so $b^{(k+1)} - b^{(k)} \in p^kB$. It follows that for each $i \in \{1,2,\ldots,n\}$ and each $k \in \{1,2,\ldots\}$, $(m_i^{(k+1)} - m_i^{(k)})b_i \in p^k \langle b_i \rangle$. Thus for each $i \in \{1,2,\ldots,n\}$, the sequence $m_i^{(1)}, m_i^{(2)}, \ldots, m_i^{(k)}, \ldots$ has the property that $m_i^{(k+1)} \equiv m_i^{(k)} \pmod{p^k}$, for each $k \in \{1,2,\ldots\}$. Hence $m_i^{(1)}, m_i^{(2)}, \ldots, m_i^{(k)}, \ldots$ determines a p-adic integer j_i for which $j_i^{(k)} = m_i^{(k)}$ for each $k \in \{1,2,\ldots\}$. But then

$$\begin{aligned}
& j_1\bar{b}_1 + j_2\bar{b}_2 + \ldots + j_n\bar{b}_n \\
&= (m_1^{(1)}b_1 + pB,\ m_1^{(2)}b_1 + p^2B, \ldots, m_1^{(k)}b_1 + p^kB, \ldots) + \\
&+ (m_2^{(1)}b_2 + pB,\ m_2^{(2)}b_2 + p^2B, \ldots, m_2^{(k)}b_2 + p^kB, \ldots) + \ldots + \\
&+ (m_n^{(1)}b_n + pB,\ m_n^{(2)}b_n + p^2B, \ldots, m_n^{(k)}b_n + p^kB, \ldots) \\
&= ((m_1^{(1)}b_1 + m_2^{(1)}b_2 + \ldots + m_n^{(1)}b_n) + pB, \\
&\qquad (m_1^{(2)}b_1 + m_2^{(2)}b_2 + \ldots + m_n^{(2)}b_n) + p^2B, \ldots \\
&\qquad \ldots, (m_1^{(k)}b_1 + m_2^{(k)}b_2 + \ldots + m_n^{(k)}b_n) + p^kB, \ldots) \\
&\qquad = (b^{(1)} + pB,\ b^{(2)} + p^2B, \ldots, b^{(k)} + p^kB, \ldots),
\end{aligned}$$

so S indeed generates $\hat{B}_{(p)}$.

A consequence of Lemma 6 and Proposition 3 is the following.

<u>Proposition 7</u>. <u>Suppose</u> A <u>is a torsion-free group with no elements of infinite</u> p<u>-height for some prime</u> p, <u>and</u> $r(A/pA)$ <u>is finite</u>. <u>Then any ring</u> $(A,\cdot)$ <u>on</u> A <u>is completely determined by its effect upon any</u> p<u>-basic subgroup of</u> A.

If A *has finite rank and* $r(A) = r(A/pA)$, *then it is possible to choose a* p*-basic subgroup of* A *that is also a subring of* $(A,\cdot)$.

Proof: Let $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \dots \oplus \langle b_n \rangle$ be a p-basic subgroup of A. If $\hat{A}_{(p)}$ is the p-adic completion of A then Proposition 3 shows that $(A,\cdot)$ may be viewed as a subring of $(\hat{A}_{(p)},\cdot)$. Lemma 6 now shows that the ring $(\hat{A}_{(p)},\cdot)$, and hence the ring $(A,\cdot)$, is determined by the effect of $(A,\cdot)$ on the set $\{b_1,b_2,\dots,b_n\}$.

To prove the final assertion of the Proposition we use an argument similar to the proof of Lemma 4.3 of Beaumont and Pierce [2]. Suppose $r(A) = r(A/pA) = n$. Then $\{b_1,b_2,\dots,b_n\}$ is a maximal independent set of elements of A, so for all i and $j \in \{1,2,\dots,n\}$ there exists an integer m with $(m,p) = 1$, and integers $m_1,m_2,\dots,m_n$ such that

$$m(b_i \cdot b_j) = m_1b_1 + m_2b_2 + \dots + m_nb_n .$$

Consequently $(mB,\cdot) = (\langle mb_1,mb_2,\dots,mb_n\rangle,\cdot)$ is a subring of $(A,\cdot)$. Finally since B is p-pure in A and $(m,p) = 1$ it follows that mB is a p-basic subgroup of A.

The partial similarity of Proposition 7 with Theorem 120.1 of Fuchs [7] cannot be strengthened. To demonstrate this simply let A be a rational group with non-idempotent type. It is clear that there is a prime p for which A satisfies the conditions of Proposition 7. However since A is a nil group and every p-basic subgroup of A is cyclic, not every partial multiplication on a p-basic subgroup of A will extend to a ring on A.

Suppose now A is a torsion-free group with no elements of intinite p-height, for some prime p, and $(A,\cdot)$ is an associative ring on A. Proposition 3 shows that $(A,\cdot)$ can be viewed as a subring of an associative ring $(\hat{A}_{(p)},\cdot)$ on $\hat{A}_{(p)}$. If we let K denote the quotient field of Q_p^*, then $K \otimes_{Q_p^*} \hat{A}_{(p)}$ can be made into an associative algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)},\cdot)$ over K by defining, for k_1, k_2 in K and $\hat{a}_1$, $\hat{a}_2$ in $\hat{A}_{(p)}$,

$$(k_1 \otimes \hat{a}_1)\cdot(k_2 \otimes \hat{a}_2) = (k_1 k_2) \otimes (\hat{a}_1 \cdot \hat{a}_2)$$

and

$$k_1(k_2 \otimes \hat{a}_1) = (k_1 k_2) \otimes \hat{a}_1.$$

It is clear that if $\hat{A}_{(p)}$ has finite rank over Q_p^* then $K \otimes_{Q_p^*} \hat{A}_{(p)}$ will have finite dimension over K. Also the map $\hat{a} \rightarrow 1 \otimes \hat{a}$ for each $\hat{a} \in \hat{A}_{(p)}$ is an embedding of $(\hat{A}_{(p)},\cdot)$ in $(K \otimes_{Q_p^*} \hat{A}_{(p)},\cdot)$, so $(A,\cdot)$ can be viewed as a subring of the algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)},\cdot)$.

These comments form the basis for the proof of our next result.

<u>Proposition 8</u>. <u>Let</u> A <u>be a torsion-free group with no elements of infinite</u> p<u>-height for some prime</u> p, <u>and suppose</u> $r(A/pA) = n$. <u>If</u> $(A,\cdot)$ <u>is a nil ring then</u> $(A,\cdot)^{n+1} = 0$.

Proof: $(A,\cdot)$ can be embedded in the associative algebra $(K \otimes_{Q_p^*} \hat{A}_{(p)},\cdot)$ over the field K. If B is a p-basic subgroup of A then there exist elements $b_1, b_2, \ldots, b_n$ of A such that $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_n \rangle$. Lemma 6 shows that $\{\hat{b}_1, \hat{b}_2, \ldots \ldots, \hat{b}_n\}$ is now a basis of $\hat{A}_{(p)}$ over Q_p^*, so $\{1 \otimes \hat{b}_1, 1 \otimes \hat{b}_2, \ldots, 1 \otimes \hat{b}_n\}$ is a basis of $K \otimes_{Q_p^*} \hat{A}_{(p)}$ over K.

For each $i \in \{1,2,\ldots,n\}$, b_i is a nilpotent element of $(A,\cdot)$, so $1 \otimes \hat{b}_i$ is a nilpotent element of

$(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$. Since $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)$ has finite dimension n over K, a reference to Abian [1], p. 155, now shows $(K \otimes_{Q_p^*} \hat{A}_{(p)}, \cdot)^{n+1} = 0$. Thus $(A, \cdot)^{n+1} = 0$, as desired.

Now for the main results.

Theorem 9. Suppose $A = D \oplus R$ is a torsion-free group, where D is a divisible group and R is a reduced group. Suppose further that D has finite rank d and the rank of A/pA is bounded by the integer n, for every prime p. If $(A, \cdot)$ is a nil ring on A then $(A, \cdot)^{(d+1)(n+1)} = 0$.

Proof: Let $(A, \cdot)$ be a nil ring on A. If there is a prime p for which A has no elements of infinite p-height, then Proposition 8 shows $(A, \cdot)^{n+1} = 0$. Hence we can assume that A has elements of infinite p-height for every prime p.

Consider a fixed prime p. It is readily checked that $A/p^{\omega}A$ is a torsion-free group with no elements of infinite p-height such that $r((A/p^{\omega}A)/p(A/p^{\omega}A)) \leq n$. Also, since $p^{\omega}A$ is a fully invariant subgroup of A, the nil ring $(A, \cdot)$ on A yields a nil ring $(A/p^{\omega}A, \cdot)$ on $A/p^{\omega}A$. Thus Proposition 8 implies $(A/p^{\omega}A, \cdot)^{n+1} = 0$. Since this is true for every prime p, $(A, \cdot)^{n+1} \subseteq \bigcap_p p^{\omega}A = D$.

Now $(D, \cdot)$ is an ideal of $(A, \cdot)$, so $(D, \cdot)$ is also a nil ring. Since $(D, \cdot)$ can be made into a finite dimensional algebra over the field of rationals Q, $(D, \cdot)$ is a nilpotent ring. Theorem 1 now shows $(D, \cdot)^{d+1} = 0$, so $(A, \cdot)^{(n+1)(d+1)} = 0$, as required.

Corollary 10. Let A be a reduced torsion-free group with the property that r(A/pA) is bounded by the positive integer n, for every prime p. Then N(A) is 1,2,...,n or ∞ .

We conclude by noting that certain results in Webb [10] enable us to give the non-associative analogues of the previous Theorem and its Corollary. The proofs are omitted since they are direct consequences of the non-associative results in Webb's work and the arguments used to prove Theorem 9.

Theorem 11. Let $A = D \oplus R$ be a torsion-free group where D is a divisible group and R is a reduced group. Suppose D has finite rank d and the rank of A/pA is bounded by the integer n, for every prime p. If $(A,\cdot)$ is a ring on A for which there is a positive integer m such that every product of length m is zero, then every product of length $(2^{n-1} + 1)(2^{d-1} + 1)$ is zero.

Corollary 12. Suppose A is a reduced torsion-free group with the property that r(A/pA) is bounded by the positive integer n, for every prime p. Then $N_E(A)$ is $1,2,\dots,2^{n-1}$ or ∞.

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