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SIMULTANEOUS INTEGRABILITY OF TWO J-RELATED ALMOST TANGENT STRUCTURES

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Abstract: Let M be a differentiable manifold provided with two almost tangent structures f, g, such that 1. $\operatorname{Ker} f = \operatorname{Im} f$, $\operatorname{Ker} g = \operatorname{Im} g$, 2. $fg = gf = 0$, 3. f and g induce a complex structure J on $\operatorname{Ker} f$. We shall associate with the couple f, g in a natural manner a G-structure on M and give necessary and sufficient conditions for its integrability. An example of the above mentioned structure will be given.

Key words: Lie group, G-structure, distribution, local coordinates, Nijenhuis tensor.

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0. All differentiable structures considered in this paper are supposed to be of class C^∞.

Let M be a differentiable manifold of dimension $2n$ endowed with the couple f, g of almost tangent structures, i.e. tensor fields of type (1,1) such that $f^2 = 0$, $g^2 = 0$. Let us suppose

(i) $\operatorname{Ker} f = \operatorname{Im} f$, $\operatorname{Ker} g = \operatorname{Im} g$, (ii) $fg = gf = 0$.

It is easy to see that (i) and (ii) imply $\operatorname{Ker} f = \operatorname{Ker} g$. We shall denote $\operatorname{Ker} f = D$. Then D is a differentiable distribution on M, $\dim D = n$.

For arbitrary $u \in M$, let us define an isomorphism J_u: $D_u \longrightarrow D_u$ by means of the commutative diagram

(1)
$$\begin{tikzcd} & D_u \arrow[d, "J_u"] \\ T_u/D_u \arrow[ur, "\tilde{f}_u"] \arrow[r, "\tilde{g}_u"'] & D_u \end{tikzcd}$$

where $T_u = T_u(M)$ is the tangent space at u, $\tilde{f}_u$ and $\tilde{g}_u$ are the isomorphisms induced by f_u and g_u, respectively.

Let us suppose that

(iii) $(J_u)^2 = -\,\mathrm{id}$ for every $u \in M$, i.e. that J_u is a complex structure on D_u. It is the well known fact that $n = \dim D$ must be even. We shall write $n = 2p$.

1. We shall call adapted basis at $u \in M$ every basis $X_1,\dots,X_{2p},Y_1,\dots,Y_{2p}$ of T_u with respect to which f has the matrix expression $\tilde{I}$ and g has the matrix expression $\tilde{H}$,

$$I = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} O_n & O_n \\ H_n & O_n \end{pmatrix}, \quad H_n = \begin{pmatrix} O_p & I_p \\ -I_p & O_p \end{pmatrix},$$

where O_n and I_n is the zero and unit matrix of type $n \times n$ and similarly for O_p and I_p.

In such a way there is

$$\begin{aligned} &f X_a = Y_a,\ fY_a = 0, \\ &g X_i = -Y_{i+p},\ gX_{i+p} = Y_i,\ gY_a = 0, \\ &J Y_i = -Y_{i+p},\ JY_{i+p} = Y_i, \\ &a = 1,\dots,2p,\ i = 1,\dots,p \end{aligned} \tag{2}$$

in terms of the adapted basis $X_1,\dots,X_{2p},Y_1,\dots,Y_{2p}$.

It can be easily proved that the set $\mathbb{G}$ of all matrices $A \in GL(2n,R)$ such that

$$A\,\tilde{I} = \tilde{I}\,A,\ A\,\tilde{H} = \tilde{H}\,A \tag{3}$$

is a Lie subgroup of $GL(2n,R)$. An easy computation shows that

$$(4) \quad \mathbb{G} = \left\{ \left(\begin{array}{c|c} \begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix} & 0_n \\ \hline C & \begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix} \end{array} \right) \;\middle|\; \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \text{ is regular}, \alpha \text{ is of type } p \times p \right\}.$$

<u>Lemma 1</u>. The set $B_{\mathbb{G}}$ of all adapted bases of $T_u(M)$, $u \in M$, is a $\mathbb{G}$-structure.

<u>Proof</u>: In a neighbourhood of an arbitrarily chosen point of M we can choose a local basis $Y_1,\dots,Y_{2p}$ of the distribution D with respect to which the matrix expression of J is H_n. Then it is possible to find vector fields $X_1,\dots$ $\dots,X_{2p}$ linearly independent over D such that $fX_a = Y_a$, $a = 1,\dots,2p$. Apparently $X_1,\dots,X_{2p},Y_1,\dots,Y_{2p}$ is a local section of $B_{\mathbb{G}}$. Other details of the proof are left to the reader. Q.E.D.

When speaking about simultaneous integrability of f and g, we shall always mean the integrability of this $\mathbb{G}$-structure.

2. We shall start this paragraph with the following definition.

Given two tensor fields h, k of type $(1,1)$ satisfying $hk = kh$, we can define a tensor field $\{h,k\}$ of type $(1,2)$ by the formula

$$(5) \quad \{h,k\}(X,Y) = [hX,kY] + hk[X,Y] - h[X,kY] - k[hX,Y],$$

where X, Y are vector fields, $[\ ,\ ]$ is the Lie bracket. This definition was given by Nijenhuis and $\{h,k\}$ is called the Nijenhuis torsion of h, k.

The Nijenhuis torsion tensor is widely used in the theory of G-structures. We shall recall here two well known results on the integrability of an almost complex structure and the integrability of an almost tangent structure.

Theorem A. An almost complex structure J on a differentiable manifold M is integrable if and only if $\{J,J\} = 0$.

For the proof see [1], Chapter IX, p. 124.

Theorem B. An almost tangent structure f on a differentiable manifold M is integrable if and only if $\{f,f\} = 0$ and Ker f is involutive. (We don't suppose here Ker f = Im f.)

For the proof see [2].

It is not difficult to see that the conditions $\{f,f\} = 0$, $\{f,g\} = 0$, $\{g,g\} = 0$ are necessary for f and g to be simultaneously integrable.

Lemma 2. If $\{f,f\} = 0$, $\{f,g\} = 0$, then $\{g,g\} = 0$.

Proof: It is easy to see that $\{f,f\} = 0$ implies that the distribution Im f is involutive. We have Ker f = Im f, so that due to Theorem B there locally exist coordinates (x,y), $x = (x_1,\dots,x_{2p})$, $y = (y_1,\dots,y_{2p})$ such that

$$(6) \qquad f\frac{\partial}{\partial x_a} = \frac{\partial}{\partial y_a}, \quad f\frac{\partial}{\partial y_a} = 0, \quad a = 1,\dots,2p.$$

We shall call such local coordinates f-adapted.

Evidently $\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{2p}}$ is a local basis of D and integral manifolds of D are of the form (x_0,y), $x_0 \in \mathbb{R}^{2p}$.

Let us now write

$g\frac{\partial}{\partial x_a} = \gamma_a^b\frac{\partial}{\partial y_b}$. We have $\{f,g\}\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}\right) = \left[\frac{\partial}{\partial y_a}, \gamma_b^c\frac{\partial}{\partial y_c}\right] =$

$= \dfrac{\partial \gamma_b^c}{\partial y_a} \dfrac{\partial}{\partial y_c} = 0$, and so the matrix function $\gamma = \gamma_a^b$, a, b = 1,...,2p, does not depend on y.

We shall introduce local coordinates (x', y') by the formulas

$$x'_a = x_a$$

$$y'_a = \wp_b^a(x) y_b,$$

where $\wp = \gamma^{-1}$.

Then

$$g \frac{\partial}{\partial x_a} = \gamma_a^b(x)\, \wp_b^c(x) \frac{\partial}{\partial y_c} = \frac{\partial}{\partial y'_a} \quad . \tag{7}$$

The equalities (7) together with Ker g = Im g are equivalent with $\{g,g\} = 0$. Q.E.D.

We are going to make further considerations and constructions under the assumption that

(iv) $\{f,f\} = 0$, $\{f,g\} = 0$.

Let us consider the factor bundle $^T/D$. We shall define for every $u \in M$ an endomorphism $\bar{J}_u$ of $(^T/D)_u$ by means of the following diagram

$$\begin{tikzcd}[column sep=huge, row sep=large]
(^T/D)_u \arrow[r, "\tilde{f}_u"] \arrow[d, "\bar{J}_u"'] \arrow[dr, "\tilde{g}_u"] & D_u \arrow[d, "J_u"] \\
(^T/D)_u & D_u \arrow[l, "\tilde{f}_u^{-1}"']
\end{tikzcd} \tag{8}$$

(compare with the diagram (1)).

An explicit formula for J_u can be given in the form:

(9) $\bar{J}_u \bar{Z} = \bar{V}$ if and only if $f_u V = g_u Z$,

where $V, Z \in T_u$, $\bar{V}, \bar{Z}$ are the elements of $({}^T/D)_u$ determined by V and Z.

It can be easily seen that $(\bar{J}_u)^2 = -\mathrm{id}$, $u \in M$, so we have got a complex structure on the factor-bundle ${}^T/D$.

We shall say that a vector field X on M is an infinitesimal automorphism of D (abbreviated IA) if the local 1-parameter group of local transformations φ_t generated by X leaves D invariant. In terms of the Lie derivative it means that $L_X Y \in D$ whenever the vector field $Y \in D$.

<u>Lemma 3</u>. A vector field X is an IA of D if and only if the expression of X in f-adapted local coordinates (x,y) is

$$X = A^a(x)\frac{\partial}{\partial x_a} + B^a(x,y)\frac{\partial}{\partial y_a}.$$

<u>Proof</u>: Let us write $X = A^a(x,y)\frac{\partial}{\partial x_a} + B^a(x,y)\frac{\partial}{\partial y_a}$. For any vector field $Y = C^a(x,y)\frac{\partial}{\partial y_a}$ it has to be $[X,Y] \in D$. An easy computation shows that this is equivalent with
$\frac{\partial A^a}{\partial y_b} = 0$, $a,b = 1,\dots,2p$. Q.E.D.

<u>Remark</u>. As a matter of fact, in Lemma 3 there is sufficient to use only D-adapted local coordinates, i.e. such ones that $\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{2p}}$ is a local basis of D. But our aim is to study the simultaneous integrability of f and g, which includes the integrability of f. That's why we shall in the following text always use f-adapted local coordinates.

<u>Lemma 4</u>. If X is an IA of D, then any vector field X_1 satisfying $\bar{J}\bar{X} = \bar{X}_1$ is also an IA of D.

<u>Proof</u>: Let Y be a vector field, $Y \in D$. We shall show that

$L_{X_1}Y \in D$, i.e. that $f(L_{X_1}Y) = 0$. Let Y' be a vector field such that $gY' = Y$. We have $gX = fX_1$, $\{f,f\} = 0$, $\{f,g\} = 0$. Therefore

$$f(L_{X_1}Y) = f[X_1,Y] = f[X_1,gY'] = [fX_1,gY'] - g[fX_1,Y'] -$$
$$- \{f,g\}(X_1,Y') = [gX,gY] - g[gX,Y'] =$$
$$= \{g,g\}(X,Y') + g[X,gY'] = g[X,gY'] \in D.$$

Q.E.D.

We shall say that a section $\mathcal{X}$ of T/D has a property (IA) if there exists an infinitesimal automorphism X of D such that $\mathcal{X} = \bar{X}$. Given two sections $\mathcal{X}, \mathcal{Y}$ of T/D with a property (IA), it is possible to define the Lie bracket of $\mathcal{X}, \mathcal{Y}$ as follows: $[\mathcal{X},\mathcal{Y}] = [\overline{X,Y}]$, where X, Y are IA of D such that $\mathcal{X} = \bar{X}$, $\mathcal{Y} = \bar{Y}$. It can be easily verified that the definition does not depend on the choice of X and Y.

Now we shall construct an analogue of the Nijenhuis torsion of the tensor $\bar{J}$ on the factor-bundle T/D. Let $u_o \in M$ be an arbitrary point and let $\mathcal{V}$, $\mathcal{W}$ be two elements of $(T/D)_{u_o}$. Let us choose vectors V_o, $W_o \in T_u(M)$ such that $\bar{V}_o = \mathcal{V}$, $\bar{W}_o = \mathcal{W}$. There exist two vector fields V, W defined on a neighbourhood of u_o and satisfying

(a) $V(u_o) = V_o$, $W(u_o) = W_o$

(b) V, W are IA of D.

We shall define

$$\{\bar{J},\bar{J}\}(\mathcal{V},\mathcal{W}) = [\bar{J}\bar{V},\bar{J}\bar{W}]_{u_o} - [\bar{V},\bar{W}]_{u_o} - \bar{J}_{u_o}[\bar{J}\bar{V},\bar{W}]_{u_o} - \tag{10}$$
$$- \bar{J}_{u_o}[\bar{V},\bar{J}\bar{W}]_{u_o}.$$

We have to show that the definition (10) is correct, i.e. that the right side of the formula depends only on the values of $\mathcal{V}$ and $\mathcal{W}$.

Let us use f-adapted local coordinates (x,y) in a neighbourhood of u_0 and write

$$V(x,y) = v^a(x)\,\frac{\partial}{\partial x_a} + v^a(x,y)\,\frac{\partial}{\partial y_a}\,,$$

$$W(x,y) = w^a(x)\,\frac{\partial}{\partial x_a} + w^a(x,y)\,\frac{\partial}{\partial y_a}\,,$$

$$g\,\frac{\partial}{\partial x_a} = \gamma^{\,b}_{\,a}(x)\,\frac{\partial}{\partial y_b}\,, \text{ as usual } a,b = 1,\dots,2p.$$

(In the following text we shall no more emphasize that similar formulas represent a system of formulas, a.b.c.d.e running from 1 to 2p.)

For the further computation we may use the representation

$$\bar V = \overline{v^a(x)\,\frac{\partial}{\partial x_a}}\,, \quad \bar W = \overline{w^a(x)\,\frac{\partial}{\partial x_a}}\,,$$

$$J\bar V = \overline{v^a(x)\,\gamma^{\,b}_{\,a}(x)\,\frac{\partial}{\partial x_b}}\,, \quad J\bar W = \overline{w^a(x)\,\gamma^{\,b}_{\,a}(x)\,\frac{\partial}{\partial x_b}}\,.$$

It is easy to compute that

$$[J\bar V, J\bar W] = \Big(v^a\gamma^{\,b}_{\,a}\,\frac{\partial w^c}{\partial x_b}\,\gamma^{\,d}_{\,c} - w^a\gamma^{\,b}_{\,a}\,\frac{\partial v^c}{\partial x_b}\,\gamma^{\,d}_{\,c} + {}$$

$$+ v^a\,\gamma^{\,b}_{\,a}\,w^c\,\frac{\partial \gamma^d_c}{\partial x_b} - w^a\,\gamma^{\,b}_{\,a}\,v^c\,\frac{\partial \gamma^d_c}{\partial x_b}\Big)\,\overline{\frac{\partial}{\partial x_d}}\,,$$

$$[\bar V, \bar W] = \Big(v^a\,\frac{\partial w^d}{\partial x_a} - w^a\,\frac{\partial v^d}{\partial x_b}\Big)\,\overline{\frac{\partial}{\partial x_d}}\,,$$

$$J[J,\bar V,\bar W] = \Big(v^a\gamma^{\,b}_{\,a}\,\frac{\partial w^c}{\partial x_b}\;{}^{d}_{c} + w^a\,\frac{\partial v^b}{\partial x_a}\gamma^c_b\gamma^d_c - w^a v^b\frac{\partial \gamma^c_b}{\partial x_a}\gamma^{\,d}_{\,c}\Big)\overline{\frac{\partial}{\partial x_d}} =$$

$$= \Big(v^a\gamma^{\,b}_{\,a}\,\frac{\partial w^c}{\partial x_b}\gamma^{\,d}_{\,c} + w^a\,\frac{\partial v^d}{\partial x_a} - w^a v^b\,\frac{\partial \gamma^c_b}{\partial x_a}\,\gamma^{\,d}_{\,c}\Big)\overline{\frac{\partial}{\partial x_d}}$$

and similarly

$$\bar{J}[\bar{V},\bar{J},\bar{W}] = (- v^a\frac{\partial w^d}{\partial x_a} + v^a w^b\frac{\partial \gamma^c_b}{\partial x_a}\gamma^d_c - w^a\gamma^b_c - w^a\gamma^b_a\frac{\partial v^c}{\partial x_b}\gamma^d_c)\overline{\frac{\partial}{\partial x_d}} .$$

Therefore

$$\{J,J\}\ (\mathcal{V},\mathcal{W}) = [v^a(u_o)\gamma^b_a(u_o)w^c(u_o)\frac{\partial\gamma^d_c}{\partial x_b}\Big|_{u_o} -$$

$$- w^a(u_o)\gamma^b_a(u_o)v^c(u_o)\frac{\partial\gamma^d_c}{\partial x_b}\Big|_{u_o} + w^a(u_o)\frac{\partial\gamma^c_b}{\partial x_a}\Big|_{u_o}\gamma^d_c(u_o) -$$

$$- v^a(u_o)w^b(u_o)\frac{\partial\gamma^c_b}{\partial x_a}\Big|_{u_o}\gamma^d_c(u_o)]\overline{\frac{\partial}{\partial x_d}\Big|_{u_o}}$$

which depends only on $\mathcal{V}$, $\mathcal{W}$.

The independence on the choice of V_o, W_o is also apparent from the above computation.

Now we are ready to formulate the main theorem of this paper.

<u>Theorem</u>. f and g are simultaneously integrable if and only if

$$\{f,f\} = 0,\ \{f,g\} = 0,\ \{\bar{J},\bar{J}\} = 0.$$

<u>Proof</u>: Let (x,y) be f-adapted local coordinates. Let us write $g\frac{\partial}{\partial x_a} = \gamma^b_a(x)\frac{\partial}{\partial y_b}$. We are looking for f-adapted local coordinates (x',y') defined in the same domain satisfying

$$g\,\frac{\partial}{\partial x_a} = H^b_a\,\frac{\partial}{\partial y_b},\quad H = \begin{pmatrix} 0_p & I_p \\ -I_p & 0_p \end{pmatrix},$$

(11)

$$g\,\frac{\partial}{\partial y_a} = 0.$$

Local coordinates (x,y) and (x',y') have to be related by the formulas

$$x'_a = F_a(x)$$

$$y'_a = \frac{\partial F_a}{\partial x_b}\,y_b + \varphi_a(x).$$

It is

$$\frac{\partial}{\partial x_a} = \frac{\partial F_b}{\partial x_a}\frac{\partial}{\partial x'_b} + \Phi\frac{\partial}{\partial y'_b}$$

$$\frac{\partial}{\partial y_a} = \frac{\partial F_b}{\partial x_a}\frac{\partial}{y'_b},$$

where Φ is a certain function. It is easy to see that (11) is satisfied if and only if

$$\gamma^{b}_{a}(x)\frac{\partial F_c}{\partial x_b} = \frac{\partial F_b}{\partial x_a}H^c_b \tag{12}$$

In other words, we are looking for the solution of the system of partial differential equations (12). This system arises in the study of integrability of an almost complex structure. From Theorem A it follows that the system (12) has a solution if and only if

$$\gamma^b_a\frac{\partial\gamma^d_c}{\partial x_b} - \gamma^b_c\frac{\partial\gamma^d_a}{\partial x_b} + \frac{\partial\gamma^b_a}{\partial x_c}\gamma^d_b - \frac{\partial\gamma^b_c}{\partial x_a}\gamma^b_d = 0\ . \tag{13}$$

Let us now compute the value $\{\bar{J},\bar{J}\}$ on $(\overline{\frac{\partial}{\partial x_a}},\overline{\frac{\partial}{\partial x_c}})$. It is

$$\{J,J\}\,(\overline{\frac{\partial}{\partial x_a}},\overline{\frac{\partial}{\partial x_c}}) = [\overline{\gamma^b_a\frac{\partial}{\partial x_b},\ \gamma^d_c\frac{\partial}{\partial x_d}}] - \bar{J}[\overline{\frac{\partial}{\partial x_a},\ \gamma^b_c\frac{\partial}{\partial x_b}}] -$$

$$- \bar{J}[\overline{\gamma^b_a\frac{\partial}{\partial x_b},\frac{\partial}{\partial x_c}}] = (\gamma^b_a\frac{\partial\gamma^d_c}{\partial x_b} - \gamma^b_c\frac{\partial\gamma^d_a}{\partial x_b})\overline{\frac{\partial}{\partial x_d}} -$$

$$- \frac{\partial\gamma^b_c}{\partial x_a}\gamma^d_b\overline{\frac{\partial}{\partial x_d}} + \frac{\partial\gamma^b_a}{\partial x_c}\gamma^d_b\overline{\frac{\partial}{\partial x_d}} = 0.$$

Because of the linear independence of the elements $\overline{\frac{\partial}{\partial x_1}},\ldots$ $\ldots,\overline{\frac{\partial}{\partial x_{2p}}}$, the conditions (13) are satisfied and the theorem is proved, as the local coordinates (x',y') defined by formulas

$$x'_a = F_a(x),$$

$$y'_a = \frac{\partial F_a}{\partial x_b} y_b ,$$

where the functions $F_1(x),\dots,F_{2p}(x)$ solve (12), are f-adapted and satisfy (11).

Q.E.D.

4. We shall present here an example of the above described structure.

Let M be a differential manifold of dimension 2p provided with an almost complex structure $\mathcal{J}$. We shall denote T(M) the tangent bundle and π the usual projection $T(M)\longrightarrow M$. If $u\in T(M)$, then there exists a canonical isomorphism $i:T_{\pi(u)}(M)\to \longrightarrow T_u(T_{\pi(u)}(M))$. Now let us define two endomorphisms f_u, g_u: $:T_u(T(M))\longrightarrow T_u(T(M))$ by the formulas

$$f_u = i\circ\pi_* \ , \quad g_u = i\circ\mathcal{J}_{\pi(u)}\circ\pi_* \ .$$

Apparently $f^2 = 0$, $g_u^2 = 0$, Ker f_u = Ker g_u = Im f_u = Im g_u. Let us denote Ker $f_u = D_u$ and $T_u(T(M)) = \mathcal{T}_u$.

Let us consider the diagram (8). The isomorphism $\bar{J}$ in this case satisfies:

(14) $$\bar{J}\bar{Z} = \bar{V} \Longleftrightarrow gZ = fV$$

where $Z, V\in\mathcal{T}_u$, $\bar{Z}$, $\bar{V}$ are the corresponding classes from $\mathcal{T}_u/D_u$. The right side of (14) means $i(\mathcal{J}(\pi_* Z)) = i(\pi_* V)$, i.e. $\mathcal{J}(\pi_* Z) = \pi_* V$. Therefore

$$\bar{J}\bar{Z} = \bar{V} \Longleftrightarrow \mathcal{J}(\pi_* Z) = \pi_* V.$$

When we define $\bar{\pi}_u : \mathcal{T}_u/D_u \longrightarrow T_{\pi(u)}(M)$ by the formula $\bar{\pi}\bar{X} = \pi_* X$, $X\in\mathcal{T}_u$ (the definition is correct), we have the following commutative diagram

$$
(15)\qquad
\begin{tikzcd}
(\mathcal{T}/D)_u \arrow[r, "\bar{J}"] \arrow[d, "\bar{\pi}"] & (\mathcal{T}/D)_u \arrow[d, "\bar{\pi}\ \text{isom.}"] \\
T_{\pi(u)}(M) \arrow[r, "\mathcal{J}"] & T_{\pi(u)}(M)
\end{tikzcd}
$$

If $(x_1,\dots,x_{2p})$ are local coordinates in a domain $U\subset M$ and (x,y) are canonically induced local coordinates in $\pi^{-1}U\subset$ $\subset T(M)$, then there is

$$f\,\frac{\partial}{\partial x_a}\Big|_u = \frac{\partial}{\partial y_a}\Big|_u,\quad f\,\frac{\partial}{\partial y_a}\Big|_u = 0,$$

$$g\,\frac{\partial}{\partial x_a}\Big|_u = \gamma_a^b(x)\,\frac{\partial}{\partial y_b}\Big|_u,\quad g\,\frac{\partial}{\partial y_a}\Big|_u = 0,$$

$a,b = 1,\dots,2p$, $u\in\pi^{-1}U$, $u = (x,y)$, where $\gamma_a^b(x)$ are defined by $\mathcal{J}\,\frac{\partial}{\partial x_a}\Big|_{\pi u} = \gamma_a^b(x)\,\frac{\partial}{\partial x_b}\Big|_{\pi u}$. Now it is very easy to verify that $\{f,f\} = 0, \{f,g\} = 0$.

It can be also easily verified that

$$(16)\qquad (\{\bar{J},\bar{J}\}(\bar{V},\bar{Z})) = \{\mathcal{J},\mathcal{J}\}(\bar{\pi}\bar{V},\bar{\pi}\bar{Z}),\ \bar{V},\bar{Z}\in(\mathcal{T}/D)_u.$$

This formula shows us that f, g are simultaneously integrable if and only if the almost complex structure $\mathcal{J}$ on M is integrable.

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