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REMARKS ON PERIODIC SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: We prove the existence of periodic solutions for nonlinear ordinary differential equations of the Liénard type under the various conditions upon the nonlinear part of the considered differential operator.

Key words: Nonlinear ordinary differential equations, periodic problems.

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1. Introduction. Let n be a positive integer and let $T > 0$. Denote C_T^n the (Banach) space of all n - times continuously differentiable and T - periodic functions defined on the real line R with the norm

$$\|u\|_{C_T^n} = \sum_{i=0}^{n} \max_{x \in R} |u^{(i)}(x)| .$$

The following theorem is proved in [1].

In the sequel k denotes a positive integer.

Theorem 1. Let $a_1, \ldots, a_{2k-1}$ be real numbers such that

(1) $(-1)^j \; a_{2k-2j} \leq 0$

for $j = 1, \ldots, k-1$. Let f and g be continuous real valued functions,

(2) $\sup_{\xi \in R} |g(\xi)| = M < \infty$

Suppose that there exists $r > 0$ such that

$$g(\xi)\xi > 0, \quad \xi \in R, \ |\xi| > r$$

or

$$g(\xi)\xi < 0, \quad \xi \in R, \ |\xi| > r .$$

Then for arbitrary $y \in C^0_T$ such that

(3) $$\int_0^T y(t)\, dt = 0$$

there exists $u \in C^{2k}_T$ verifying the Liénard equation

(4) $$-(-1)^k u^{(2k)}(x) + a_1 u^{(2k-1)}(x) + \dots + a_{2k-1} u'(x) + f(u(x))u'(x) + g(u(x)) = y(x) .$$

Moreover, arbitrary solution $u \in C^{2k}_T$ satisfies

(5) $$\|u\|_{C^0_T} < \gamma$$

where

$$\gamma = r + 3^{-\frac{1}{2}} T^2 (2\pi)^{-k} (M + \|y\|_{C^0_T}) .$$

From this result it immediately follows :

Theorem 2. Let $f, g : R \to R$ be continuous functions, suppose (1), (2). Then for arbitrary $y \in C^0_T$ with

(6) $$\limsup_{\xi \to -\infty} g(\xi) < \frac{1}{T} \int_0^T y(t)\, dt < \liminf_{\xi \to +\infty} g(\xi)$$

there exists at least one solution $u \in C_T^{2k}$ of (4).

Moreover, arbitrary solution $u \in C_T^{2k}$ satisfies (5) with

$$\gamma = r_1 + 3^{-\frac{1}{2}} T^2 (2\pi)^{-k} \left(\sup_{\xi \in R} |\tilde{g}(\xi)| + \|\tilde{y}\|_{C_T^0} \right)$$

where

$$\tilde{g}(\xi) = g(\xi) - \frac{1}{T} \int_0^T y(t)\, dt ,$$

$$\tilde{y}(x) = y(x) - \frac{1}{T} \int_0^T y(t)\, dt ,$$

and for $r_1 > 0$ it is

$$\tilde{g}(\xi)\xi > 0 \qquad \text{if } |\xi| > r_1 .$$

Theorem 3. Assume (1). Let $f, g: R \longrightarrow R$ be continuous and suppose that there exist finite limits

$$\lim_{\xi \to +\infty} g(\xi) = g(+\infty) , \qquad \lim_{\xi \to -\infty} g(\xi) = g(-\infty) .$$

Moreover, suppose

(7) $$\min \{ g(-\infty), g(+\infty) \} < g(\xi) < \max \{ g(-\infty), g(+\infty) \}$$

for each $\xi \in R$.

Then the equation (4) has at least one T-periodic solution for $y \in C_T^0$ if and only if

(8) $$\min \{ g(-\infty), g(+\infty) \} < \frac{1}{T} \int_0^T y(x)\, dx < \max \{ g(-\infty), g(+\infty) \} .$$

For the proof of Theorem 1 the following property (F) of the differential equation (4) is essential:

(F) If $y \in C_T^o$, then the set of all T - periodic solutions of the equation (4) is a bounded subset of the space C_T^o .

Now let's prove that the (F) is in a certain sense necessary and sufficient for the validity of the assertion of Theorem 3.

<u>Theorem 4.</u> Let f , g : $R \to R$ be continuous. Suppose (2) and denote

$$\bar{G} = \sup_{\xi \in R} g(\xi) , \qquad \underline{G} = \inf_{\xi \in R} g(\xi) ,$$

$$\mathcal{P} = \{ y \in C_T^o \text{ ; there exists a T-periodic solution of (4)} \}$$

$$\mathcal{M} = \{ y \in C_T^o \text{ ; } \underline{G} < \frac{1}{T} \int_0^T y(x)\, dx < \bar{G} \} ,$$

$$\mathcal{N} = \{ y \in C_T^o \text{ ; } \underline{G} \leq \frac{1}{T} \int_0^T y(x)\, dx \leq \bar{G} \}$$

Assume that the sets $g^{-1}(\underline{G})$, $g^{-1}(\bar{G})$ do not contain a nondegenerated interval.

Suppose that the differential equation (4) has the property (F).

i) Then there exist finite limits $g(+\infty)$, $g(-\infty)$ and it is

$$(\mathcal{P} \cap \mathcal{N}) \setminus \mathcal{M} \subseteq \{\overline{G}, \underline{G}\} .$$

ii) If $(\mathcal{P} \cap \mathcal{N}) \setminus \mathcal{M} = \{\overline{G}\}$, then we have

$$g(+\infty) = \underline{G} \quad \text{or} \quad g(-\infty) = \underline{G} .$$

iii) If $(\mathcal{P} \cap \mathcal{N}) \setminus \mathcal{M} = \{\underline{G}\}$, then we have

$$g(+\infty) = \overline{G} \quad \text{or} \quad g(-\infty) = \overline{G} .$$

(iv) If $(\mathcal{P} \cap \mathcal{N}) \setminus \mathcal{M} = \phi$, then it is

$$\begin{cases} g(-\infty) = \underline{G}, \quad g(+\infty) = \overline{G} \\ g(-\infty) < g(\xi) < g(+\infty), \quad \xi \in R \end{cases}$$

or

$$\begin{cases} g(-\infty) = \overline{G}, \quad g(+\infty) = \underline{G} \\ g(+\infty) < g(\xi) < g(-\infty), \quad \xi \in R \end{cases}$$

<u>Proof:</u> i) Suppose that $g(+\infty)$ does not exist. Denote

$$A = \limsup_{\xi \to +\infty} g(\xi) , \qquad B = \liminf_{\xi \to +\infty} g(\xi)$$

Then $B < A$, and there exists a sequence

$\{\xi_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \xi_n = +\infty$ such that

$$g(\xi_n) = \frac{A+B}{2} .$$

Thus $\frac{A+B}{2}$, and the constant functions ξ_n are the T - periodic solutions of (4) .

This is in contradiction with the property (F).

Analogously we prove the existence of $g(-\infty)$.

Let $y \in (\mathcal{P} \cap \mathcal{N}) \setminus \mathcal{M}$ and let, for example,

$$\frac{1}{T} \int_0^T y(x)\, dx = \bar{G} .$$

If $u_0 \in C_T^{2k}$ is a solution of (4), then we have

$$\int_0^T (\bar{G} - g(u_0(x)))\, dx = 0$$

and since $g^{-1}(\bar{G})$ does not contain a nondegenerated interval, u_0 is a constant function, thus $y = \bar{G}$.

The proofs of parts (ii) - (iv) are the consequences of the part (i) .

From the Theorems 3 and 4 it immediately follows :

<u>Theorem 5.</u> Suppose that $f, g : R \longrightarrow R$ are the continuous functions, let g be bounded and let $g^{-1}(\bar{G})$, $g^{-1}(\underline{G})$ do not contain a non degenerated interval.

Assume (1).

Then the equation (4) has the property (F) and $\mathcal{P} = \mathcal{M}$ if and only if there exist $g(+\infty)$, $g(-\infty)$ and if (7) is fulfilled .

In this note we shall deal with the T-periodic solvability of (4) in the case of nonexistence of Limits $g(+\infty)$, $g(-\infty)$.

2. Expansive nonlinearities. The following definition is a generalization of the so-called expansive function introduced in [3].

Definition. Let $g: R \to R$ be a nontrivial bounded continuous function. For p, q such that

$$\inf_{\xi \in R} g(\xi) < q \leq p < \sup_{\xi \in R} g(\xi) \tag{9}$$

we put

$$\begin{aligned} M_{p,q} = \{ & d \in R, \exists \eta_1, \eta_2 \in R, \quad 0 \leq \eta_1 < \eta_2 \text{ such that for} \\ & \xi \in \langle \eta_1, \eta_2 \rangle \text{ it is } g(\xi) > p \text{ and for} \\ & \xi \in \langle -\eta_2, -\eta_1 \rangle \text{ it is } g(\xi) < q \text{ and } d \leq \eta_2 - \eta_1 \} \cup \\ \cup \{ & d \in R, \exists \eta_1, \eta_2 \in R, \quad 0 \leq \eta_1 < \eta_2 \text{ such that for} \\ & \xi \in \langle \eta_1, \eta_2 \rangle \text{ it is } g(\xi) < q \text{ and for} \\ & \xi \in \langle -\eta_2, -\eta_1 \rangle \text{ it is } g(\xi) > p \text{ and } d \leq \eta_2 - \eta_1 \}. \end{aligned}$$

Denote $\quad \mathcal{E}_{p,q}(g) = \sup M_{p,q}$.

If $\mathcal{E}_{p,q}(g) = \infty$ for each p, q from (9), then g is called the expansive function.

Examples. (i) $\mathcal{E}_{p,q}(\sin) = \min\{\pi - 2|\arcsin p|, \pi - 2|\arcsin q|\}$

for each $-1 < q \leqslant p < 1$.

(ii) The function

$$g : \xi \longmapsto \sin\left(\xi^{\frac{2k-1}{2k+1}}\right)$$

(k is a positive integer) is expansive.

(iii) The function

$$g : \xi \longmapsto \operatorname{arctg} \xi^2 . \sin\left(\xi^{\frac{2k-1}{2k+1}}\right)$$

is expansive.

The main result of this section is the following.

Theorem 6. Suppose (1) and let $f, g : R \longrightarrow R$ be continuous, g bounded. Then the equation (4) has at least one T - periodic solution if

a) $$y \in C_T^o, \quad q \leqslant \frac{1}{T}\int_0^T y(x)\,dx \leqslant p$$

b) $$3^{-\frac{1}{2}} T^2 (2\pi)^{-k} \left(\sup_{\xi \in R} \left|g(\xi) - \frac{1}{T}\int_0^T y(x)\,dx\right| + \left\|y - \frac{1}{T}\int_0^T y(x)\,dx\right\|_{C_T^o}\right) < \mathcal{E}_{p,q}(g).$$

Proof. Suppose

$$g(\xi) > p \quad \text{for } \xi \in \langle \eta_1, \eta_2 \rangle ,$$

$$g(\xi) < q \quad \text{for } \xi \in \langle -\eta_2, -\eta_1 \rangle ,$$

$$3^{-\frac{1}{2}}T^2(2\pi)^{-k}\left(\sup_{\xi\in R}\left|g(\xi)-\frac{1}{T}\int_0^T y(x)\,dx\right|+\left\|y-\frac{1}{T}\int_0^T y(x)\,dx\right\|_{C_T^o}\right)<\eta_2-\eta_1 .$$

Define the function $g_{p,q}$ by

$$g_{p,q} : \xi \longmapsto \begin{cases} g_{p,q}(\xi) = g(\xi) & \text{if} \quad \xi \in \langle -\eta_2, \eta_2 \rangle , \\ g_{p,q}(\xi) = g(-\eta_2) & \text{if} \quad \xi \in (-\infty, -\eta_2 \rangle , \\ g_{p,q}(\xi) = g(\eta_2) & \text{if} \quad \xi \in \langle \eta_2, +\infty) . \end{cases}$$

Obviously $\qquad g_{p,q}(+\infty) > p \quad$ and $\quad g_{p,q}(-\infty) < q$.

According to Theorem 2 there exists at least one T - periodic solution u_o of the equation

$$(10) \; -(-1)^k u^{(2k)}(x) + a_1 u^{(2k-1)}(x) + \ldots + a_{2k-1} u'(x) + f(u(x))\,u'(x) + g_{p,q}(u(x)) = y(x) .$$

Moreover, from (5) it follows

$$\|u_o\|_{C_T^o} < \eta_1 + 3^{-\frac{1}{2}}T^2(2\pi)^{-k}\left(\sup_{\xi\in R}\left|g(\xi)-\frac{1}{T}\int_0^T y(x)\,dx\right| + \right.$$

$$\left. + \left\|y-\frac{1}{T}\int_0^T y(x)\,dx\right\|_{C_T^o}\right) < \eta_1 + \eta_2 - \eta_1 = \eta_2 .$$

From this we have

$$g(u_o(x)) = g_{p,q}(u_o(x)) , \quad x \in R$$

and thus the function u_o is a T-periodic solution of (4) .

As a corollary we obtain immediately :

Theorem 7. Suppose (1) and let $f, g : R \to R$ be continuous and let g be expansive.

Then $$\mathcal{M} \subset \mathcal{P} \subset \mathcal{N}.$$

If, moreover, the sets $\overset{-1}{g}(\underline{G})$, $\overset{-1}{g}(\overline{G})$ are non empty and do not contain nondegenerated intervals then

$$\mathcal{P} = \mathcal{M} \cup \{\overline{G}, \underline{G}\}$$

If the sets $\overset{-1}{g}(\overline{G})$, $\overset{-1}{g}(\underline{G})$ are empty then

$$\mathcal{P} = \mathcal{M}.$$

3. Mathematical pendulum equation. From the previous section it follows that the equation

$$u''(x) + \sin\left(u^{\frac{2k-1}{2k+1}}(x)\right)' = y(x)$$

possesses at least one T-periodic solution if and only if

$$y \in C_T^0, \quad -1 < \frac{1}{T}\int_0^T y(x)\,dx < 1 \quad \text{or} \quad y = \pm 1 .$$

Now we shall consider the equation

(11) $$u''(x) + g(u(x)) = y(x),$$

where g is a τ-periodic continuous function. The model for such equation is

(12) $$u''(x) + \sin u(x) = f(x)$$

which has at least one T-periodic solution, provided there exists $p \in (0,1)$ such that

$$-p \leqslant \frac{1}{T}\int_0^T y(x)\,dx \leqslant p$$

and

$$3^{-\frac{1}{2}}T^2\left(1+\frac{1}{T}\left|\int_0^T y(x)dx\right| + \left\| y - \frac{1}{T}\int_0^T y(x)\,dx\right\|_{C^0_T}\right) < \pi - 2\arcsin p$$

(see Theorem 6 and Example (i)).

In the sequel let's suppose :

a) $g \in C^1_\tau$, $|g'(\xi)| \leqslant 1$ for $\xi \in R$,

b) the sets $g^{-1}(\bar{G})$, $g^{-1}(\underline{G})$ do not contain nondegenerated intervals.

It is obvious to see that

$$\mathcal{P} \subsetneq \mathcal{N}$$

$$(\mathcal{N} \setminus \mathcal{M}) \cap \mathcal{P} = \{\bar{G}, \underline{G}\},$$

$\mathcal{P}$ is unbounded subset of C^0_T ,

$\mathcal{P}$ is closed in C^0_T ,

(the last assertion follows from the fact that we can consider only such solutions of (11) for which $|u(0)|\leqslant\tau$) .

<u>Theorem 8.</u> Let $T\in(0,\pi)$, $y\in C_T^o$ and $x_o\in R$. If u_1 , u_2 are T-periodic solutions of (11) such that

$$u_1(x_o) = u_2(x_o) = c$$

then u_1 and u_2 coincide on R .

<u>Proof.</u> Denote $v_i(x) = u_i(x) - c$. Then $v_i\in C_T^2$ $(i=1,2)$ satisfy the equation

$$v'' + g(c+v) = y .$$

There exists a function $\xi : R\longrightarrow R$ such that

$$v_1''(x)-v_2''(x) = g(c+v_2(x)) - g(c+v_1(x)) = g'(\xi(x))(v_2(x)-v_1(x))$$

and, moreover, the function

$$g'(\xi(\cdot))(v_2(\cdot)-v_1(\cdot))$$

is continuous.

Thus

$$\int_{x_o}^{x_o+T}(v_1'(x)-v_2'(x))^2\,dx = \left|\int_{x_o}^{x_o+T}(v_1''(x)-v_2''(x))(v_1(x)-v_2(x))\,dx\right| \leqslant$$

$$\leqslant \left| \int_{x_o}^{x_o+T} g'(\xi(x))\,(v_1(x) - v_2(x))^2\,dx \right| \leqslant \int_{x_o}^{x_o+T} (v_1(x)-v_2(x))^2 dx \leqslant$$

$$\leqslant \frac{T^2}{\pi^2} \int_{x_o}^{x_o+T} (v_1'(x)-v_2'(x))^2\,dx$$

from which it follows that

$$\int_{x_o}^{x_o+T} (v_1(x)-v_2(x))^2\,dx = 0 .$$

This completes the proof.

Theorem 9. Let $T \in (0,\pi)$ and $y \in C_T^o$.

Then the Dirichlet problem

$$(13)\qquad \begin{cases} u''(x)+g(c+u(x)) = y(x), & x \in (0, T) \\ u(0) = u(T) = 0 \end{cases}$$

has a unique solution for arbitrary $c \in R$.

(The existence of at least one weak solution is possible to prove on the basis of the theorem on surjectivity of pseudomonotone operators - see [2] . By means of standard regularity arguments we obtain that arbitrary weak solution of (13) is classical. The unicity may be proved in the same way as in Theorem 8.).

In the sequel we shall suppose $T \in (0, \mathcal{T})$. Denote by $\tilde{u}_{c,y}$ the solution of (13) put

$$u_{c,y}(x) = c + \tilde{u}_{c,y}(x - kT)$$

for $x \in \langle kT, (k+1)T \rangle$ (k is an integer).

It is easy to see that $u_{c,y}$ is a T-periodic solution of (11) if and only if

$$\int_0^T g(u_{c,y}(x))\, dx = \int_0^T y(x)\, dx .$$

Define the mapping $\Phi : R \times C_T^0 \longrightarrow C_T^0$ by

$$\Phi(c,y) = u_{c,y} .$$

It is possible to prove that $\Phi(\cdot,\cdot)$ is continuous .

Let $y \in C_T^0$ be fixed. Define

$$\Phi_y : c \longmapsto \int_0^T (g \circ \Phi(c,y))(x)\, dx .$$

Then $\Phi_y : R \longrightarrow R$ is continuous $\mathcal{T}$- periodic function. Put

(14) $$\Gamma(y) = \max_{c \in R} \Phi_y(c) ,$$

(15) $$\gamma(y) = \min_{c \in R} \Phi_y(c) .$$

The mapping $\Gamma : C_T^0 \longrightarrow R$ is upper continuous and $\gamma : C_T^0 \longrightarrow R$ is lower continuous .

The main result is the following theorem the proof of which follows immediately from the previous considerations.

Theorem 10. Consider the differential equation (11) and let $T \in (0, \pi)$. Then

$$\mathcal{P} = \left\{ y \in C_T^o ;\ \gamma(y) \leq \int_0^T y(x)\,dx \leq \Gamma(y) \right\},$$

where Γ, γ are defined by the relations (14), (15).

Remark. It seems that the better characterization of the set $\mathcal{P}$ for the equation (11) is an open problem up to now. In this direction it will be interesting to give some further properties of the functions γ, Γ, e.g. $\gamma(y) \leq 0 \leq \Gamma(y)$.

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