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AMALGAMATING COMMUTATIVE REGULAR RINGS

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Abstract: The category of commutative regular rings with identity enjoys the amalgamation property and has transferable injections. It follows that the amalgamation class of the category of commutative semiprime rings with identity consists of those rings in which ann ann $J = \sqrt{J}$ for each finitely generated ideal J. Another consequence is that the variety of commutative Rickart rings has transferable injections, even though it possesses no non-trivial injective objects.

Key words: Amalgamation property, congruence extension property, transferable injections, regular ring, semiprime ring, Rickart ring.

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1. Regular rings and Rickart rings. By a category $\underline{K}$ of algebras, we mean a category whose objects are members of a class of universal algebras of some fixed type and whose morphisms are precisely all the homomorphisms between the objects. An embedding is a one-to-one morphism and an essential morphism $f: A \longrightarrow B$ is an embedding such that if $gf: A \longrightarrow C$ is an embedding then $g: B \longrightarrow C$ is necessarily an embedding. Following Grätzer and Lakser [11], we define the amalgamation class of $\underline{K}$ to be the class Amal($\underline{K}$) consisting of all $\underline{K}$-algebras A such that if $f: A \longrightarrow B$ and $g: A \longrightarrow C$ are embeddings into $\underline{K}$-algebras B and C then there exists a $\underline{K}$-al-

gebra D and embeddings $h: B \longrightarrow D$, $k: C \longrightarrow D$ such that $hf = kg$. Thus, $\underline{K}$ has the <u>amalgamation property</u> if $\underline{K} = \mathrm{Amal}(\underline{K})$.

In general, our category-theoretic terminology will be consistent with that of Herrlich and Strecker [12]. In particular, a morphism $f: A \longrightarrow B$ $\underline{K}$-algebras A and B is an epimorphism if, whenever $h,k: B \longrightarrow C$ are morphisms to a $\underline{K}$-algebra C such that $hf = kf$ then $h = k$. The following result interrelates the above concepts; it has a consequence which is useful throughout this paper.

1.1. <u>Lemma</u>. Let $\underline{K}$ be a category of algebras, $A \in \mathrm{Amal}(\underline{K})$, and $f: A \longrightarrow B$ be an epimorphic embedding of A into $\underline{K}$-algebra B. Then, f is essential.

<u>Proof</u>: Suppose $g: B \longrightarrow C$ is a morphism to $\underline{K}$-algebra C such that gf is an embedding. As A is in the amalgamation class of $\underline{K}$, there is a $\underline{K}$-algebra D and embeddings $h: C \longrightarrow D$ and $k: B \longrightarrow D$ such that $hgf = kf$. Since f is an epimorphism $hg = k$. As k is an embedding, it follows that g is also an embedding.

1.2. <u>Proposition</u>. Let $\underline{H}$ be a full reflective subcategory of a category of algebras $\underline{K}$ with the functor $R: \underline{K} \longrightarrow \underline{H}$ as the reflection. For each $\underline{K}$-algebra A suppose that the morphism $\Phi_A: A \longrightarrow R(A)$ mapping A into its $\underline{H}$-reflection is an embedding. Then, provided that $\underline{H}$ has the amalgamation property, a $\underline{K}$-algebra A is $\mathrm{Amal}(\underline{K})$ if and only if Φ_A is essential.

<u>Proof</u>: Suppose A is $\underline{K}$-algebra such that Φ_A is essential and let $f: A \longrightarrow B$ and $g: A \longrightarrow C$ be embeddings of A into $\underline{K}$-algebras B and C. Since $R(f): R(A) \longrightarrow R(B)$ and $R(g): R(A) \rightarrow$

$\longrightarrow R(C)$ are such that $R(f)\Phi_A = \Phi_B f$ and $R(g)\Phi_A = \Phi_C g$, both $R(f)$ and $R(g)$ are embeddings. Let $h: R(B) \longrightarrow F$ and $k:$ $: R(C) \longrightarrow F$ be embeddings into a $\underline{H}$-algebra F such that $hR(f) = kR(g)$. Then, $(h\Phi_B)f = hR(f)\Phi_A = kR(g)\Phi_A =$ $= (k\Phi_C)g$ and $A \in \mathrm{Amal}(\underline{K})$.

Conversely, suppose $A \in \mathrm{Amal}(\underline{K})$. Because $\underline{H}$ is a full reflective subcategory and each morphism $\Phi_B: B \longrightarrow R(B)$ is an embedding, each Φ_B $(B \in \underline{K})$ is an epimorphism [12; Proposition 36.3, p. 276]. By Lemma 1.1 Φ_A is essential.

The following lemma was noted, without proof, by Jónsson in his fundamental paper [14] concerning the amalgamation property.

1.3. Lemma. The category of fields has the amalgamation property.

Proof: Let $f: A \longrightarrow B$ and $g: A \longrightarrow C$ be embeddings of a field A into fields B and C. Let M be a maximal ideal of the commutative ring $B \otimes_A C$ and D be the associated quotient field. Then, D and the associated embeddings $h: B \longrightarrow D$, $h(b) = \frac{b \otimes 1}{M}$ $(b \in B)$ and $k: C \longrightarrow D$, $k(d) = \frac{1 \otimes d}{M}$ $(d \in D)$ effect the required amalgamation. For more details, see [20; Chapter 3, § 15].

1.4. Corollary. The category of integral domains and unitary homomorphisms has the amalgamation property.

Proof: The familiar embeddings of an integral domain into its field of quotients is essential and maps the domain to its reflection in the full reflective subcategory of fields. Thus, the result follows from 1.2 and 1.3.

Despite 1.4 it is not hard to show that the category Semp Rng of commutative semiprime rings with identity and unitary homomorphisms does not have the amalgamation property. Indeed, let D be any integral domain possessing a prime element p. Consider D to be embedded in its field of quotients Q(D) and also in the semiprime ring D[x]/ (px). This latter ring is semiprime since the ideal (px) is the intersection of the two prime ideals (p) and (x). Let $\bar{x}$ denote the class of x in the quotient ring and suppose f: $Q(D) \longrightarrow R$ and g: : $D[x]/(px) \longrightarrow R$ are embeddings in Semp Rng into a ring R which have equal restrictions to D. As $g(p)g(\bar{x}) = 0$, $f(p) = g(p)$ and $f(p)f(p^{-1}) = 1$, we obtain $g(\bar{x}) = 0$ and so $\bar{x} = 0$, a contradiction. Thus, the amalgamation property does fail in Semp Rng; in Section 2, we will determine Amal(Semp Rng).

A category $\underline{K}$ of algebras has the congruence extension property if for each subalgebra B of a $\underline{K}$-alegbra A and each congruence relation Θ on B, there exists a congruence relation Φ on A such that $\Phi \cap (B \times B) = \Theta$. Bacsich [1; Lemma 1.2] observed that the congruence extension property holds in a variety $\underline{K}$ if and only if whenever f: $A \longrightarrow B$ is an embedding and g: $A \longrightarrow C$ is a surjective homomorphism then there exists a $\underline{K}$-algebra D, a surjective homomorphism h: $B \longrightarrow D$ and an embedding k: $C \longrightarrow D$ such that hf = kg.

A category $\underline{K}$ of algebras is said to have transferable injections if, whenever f: $A \longrightarrow B$ is an embedding and g: $A \longrightarrow C$ is any homomorphism, there exists a $\underline{K}$-algebra D, a homomorphism h: $B \longrightarrow D$ and an embedding k: $C \longrightarrow D$ such that hf = kg.

This notion is due to Bacsich [1]; in [1; Lemma 1.7] he proved that <u>a variety has transferable injections if and only if it has both the amalgamation property and the congruence extension property</u>.

The following lemma is due to Grätzer and Lakser [11; Theorem 3]. It is a very effective weapon when one is trying to show that a variety has the amalgamation property.

1.5. <u>Lemma</u>. Let $\underline{K}$ be a variety satisfying the congruence extension property and assume that each subalgebra of each subdirectly irreducible $\underline{K}$-algebra is itself subdirectly irreducible. Then $\underline{K}$ satisfies the amalgamation property if and only if, whenever A, B, C are subdirectly irreducible $\underline{K}$-algebras and $f: A \longrightarrow B$, $g: A \longrightarrow C$ are embeddings, there exists a $\underline{K}$-algebra D and embeddings $h: B \longrightarrow D$, $k: C \longrightarrow D$ such that $hf = kg$.

Throughout the remainder of the paper all rings are assumed to be commutative and possess an identity element, and all morphisms are unitary. Let R be such a ring. As usual, ann S denotes the annihilator of a nonempty subset S of R. Then, R is a <u>Rickart ring</u> if ann x is a principal ideal generated by an idempotent for each $x \in R$. Alternatively, R is a Rickart ring if and only if, for each $x \in R$, ann ann x is a principal ideal generated by a necessarily unique idempotent which is denoted by $\pi(x)$. It turns out that we can regard a Rickart ring R as an algebra of the form $(R;+,-,\cdot,\pi,0.1)$ of type $\langle 2,2,2,1,0,0\rangle$ such that the reduct $(R;+,-,\cdot,0,1)$ is a commutative ring with identity 1 and π is a unary operation satisfying the following equations: $\pi(0) = 0$, $\pi(xy) = \pi(x)\pi(y)$, $\pi(\pi(x)) = \pi(x)$, $\pi(x).x = x$. Thus, Rickart rings give rise to a variety of type $\langle 2,2,2,1,0,0\rangle$; the accompanying notions of subalgebra, congruence, homomorphism etc. are referred to as π-subalgebra, π-congruence, π-homomorphism etc., respectively, in order to avoid confusion with the similar ring-theoretic notions. This variety has been examined by the author in [3] and [5]. In the variety of Rickart rings, an algebra is subdirectly irreducible if and only if it is an integral domain, [3; Theorem 1]; any integral domain D can be regarded as a Rickart ring if $\pi : D \longrightarrow D$ is defined by $\pi(0) = 0$ and $\pi(d) = 1$ for any $0 \neq d \in D$. Moreover, it follows from [3; Theorem 2] that $\Theta \longrightarrow \{x \in R: x \equiv 0(\Theta)\}$ is a lattice-isomorphism from the lattice of π-congruences of a Rickart ring R onto the lattice of (ring-) ideals of R which are generated by their idempotents. Now, if a Rickart ring R is a π-subring of a Rickart ring R_1 then the Boolean algebra of idempotents of R is a Boolean subalgebra of the Boolean algebra of idempotents of R_1, and if J is an ideal of R which is generated by its idempotents then $J_1 = \{r_1 \in R_1: r_1 = r_1 e \text{ for some } e \in J,\ e^2 = e\}$ is an ideal of R_1, which is generated by its idempotents, and $J = J_1 \cap R$. Thus, <u>the variety of Rickart rings has the congruence extension property</u>.

Recall that a (not necessarily commutative) ring R is called (von Neumann) <u>regular</u> if, for each $x \in R$, there is $y \in R$ such that $x = xyx$. If R is a <u>commutative</u> regular ring and $x,y \in R$ are such that $x = xyx$ then $z = xy^2$ is the unique element such that $xzx = x$ and $zxz = z$. This seems to have been first observed by Gillman, Fine and Lambek [8; *Lemma 10.1*, p. 76]; it was also noted by Olivier [15; *Lemme 2*] and has been recently generalized by Raphael [16; *Lemma 4*]. The point is that a commutative regular ring R can be considered as an algebra $(R;+,-,\cdot,\wp,0,1)$ of type $\langle 2,2,2,1,0,0\rangle$ such that the reduct $(R;+,-,\cdot,0,1)$ is a ring and the unary operation $\wp :$ $: R \longrightarrow R$ satisfies the equations $x^2\wp(x) = x$ and $\wp(x)^2x =$ $= \wp(x)$. Since ring-homomorphisms commute with $\wp$, the category of (commutative) regular rings is <u>isomorphic</u> to the variety of all algebras of type $\langle 2,2,2,1,0,0\rangle$, as defined above. This was certainly known to Raphael [16], is implicit in the work of Olivier [15], and in this paper we make use of the fact that we have a variety. Of course, a regular ring is a Rickart ring; better still a regular ring R, considered as an algebra $(R;+,-,\cdot,\wp,0,1)$, has a derived π-algebra $(R;+,-,\cdot,\pi,0,1)$ obtained by defining $\pi(x)$ to be the term $\pi(x) = x\wp(x)$. Thus, we can consider the category (variety) of regular rings to be a full subcategory of the variety of Rickart tings; the subcategory is actually reflective, see [3; Theorem 4]. Thus, <u>the variety of regular rings has the congruence extension property</u>. Of course, in a regular ring, each ideal is generated by its idempotents and the map $\Theta \longrightarrow \{x: x \equiv 0(\Theta)\}$ is a lattice-isomorphism of the lattice of $\wp$-congruences onto the lattice of all ring-ideals. Because all prime ideals are maximal in a regular ring and regular rings are semiprime, the subdirectly irreducible algebras in the variety of regular rings are precisely all fields (with $\wp$ defined by $\wp(0) = 0$ and $\wp(x) = x^{-1}$ if $x \neq 0$).

1.6. Theorem. The category (variety) of commutative regular rings and the variety of Rickart rings each have transferable injections.

Proof: This follows from the above remarks, 1.3, 1.4 and 1.5, together with the aforementioned characterization, due to Bacsich [1; Lemma 1.7], of varieties with transferable injections. It should be noted that instead of using 1.4 and 1.5 to show that the variety of Rickart rings has the amalgamation property, we could proceed from the result on regular rings, directly via 1.2, since the embedding of a Rickart ring into its regular-reflection, namely its ring of quotients, is π-essential. See [3; Theorem 4] for details.

Taylor [18, Theorem 2.3] has shown that a variety has enough injective algebras, in the sense that each algebra is a subalgebra of an injective algebra, if and only if it has transferable injections and is residually small. A variety is residually small if there is a cardinal m such that each subdirectly irreducible algebra in the variety has power $\leq \underline{m}$. Of course, we can find fields and integral domains of increasingly large power and so neither of the varieties of 1.6 is residually small. In [5] it is shown that the variety of Rickart rings (and as a consequence the variety of regular rings, also) has no non-trivial injective algebras. This should be contrasted with the variety of distributive pseudocomplemented lattices mentioned by Taylor [18; Remarks 2.5] as an example of a variety with transferable injections but not enough injectives; there the complete Boolean algebras form the insufficient class of injective algebras.

In [2], Bacsich showed that each algebra in a variety $\underline{V}$ with transferable injections has an epimorphic hull i.e. for each $\underline{V}$-algebra A, there is a $\underline{V}$-algebra E(A), unique up to isomorphism, which is an (essential) epimorphic extension of A and through which any epimorphic embedding of A into another $\underline{V}$-algebra factors uniquely. This notions was introduced earlier for commutative rings by Storrer [17; Section 3]. In [2; Section 5] Bacsich asks for more examples of epimorphic hulls. Because of [17; Satz 6.1] all epimorphisms in the variety of commutative regular rings are surjective, and it might be interesting to note that recently Gardner [7] has shown that this persists in the category (but not variety, see [16]) of non-commutative regular rings without an identity element. But reflections preserve epimorphisms, and so [3; Theorem 4] allows us to state:

1.7. Proposition. The epimorphic hull of an algebra in the variety Rickart rings is its classical ring of quotients.

2. Semiprime rings. Wiegand [19] showed by means of a sheaf-theoretic approach, that there is a reflection T: : Rng $\longrightarrow$ Reg Rng of the category (variety) Rng commutative rings into the full subcategory Reg Rng of regular rings such that the natural map η_A: A $\longrightarrow$ T(A) mapping A into its reflection T(A) is an epimorphism. See also Olivier [15]. We wish to use the restriction T_s: Semp Rng $\longrightarrow$ Reg of T and the results of Section 1 to determine Amal(Semp Rng). Before doing so, it may be worth noting that the existence and above

property of η_A follow from general considerations. Firstly, any semiprime ring can be embedded in the direct product of fields and so can be considered as a relative subalgebra of a member of the <u>variety</u> <u>Reg</u>. Hence, by Grätzer [9; Theorem 1, Corollary 1, p. 181], each semiprime ring has a free extension in the variety <u>Reg</u>. In other words, we have the existence of the reflector T_s: <u>Semp Rng</u> $\longrightarrow$ <u>Reg</u> and, by [12; Proposition 36.3, p. 276], the embedding $\eta_A^s: A \longrightarrow T_s(A)$ of semiprime A into $T_s(A)$ is an epimorphism. Secondly, H: <u>Rng</u> $\longrightarrow$ <u>Semp Rng</u>, $H(A) = A/\surd A$ ($\surd A = \{x \in A$: x is nilpotent$\}$) is a reflection and the natural map $\lambda_A: A \longrightarrow H(A)$ is a surjection. Thus, T exists for it is the composition T_sH, and $\eta_A: A \longrightarrow T(A)$ is an epimorphism as it is the composition of epimorphisms.

In what follows it is more convenient to use T in place of T_s. The next result was virtually proved by Wiegand [19; Theorem 7]. We have added Condition (i); it is a consequence of the proof of (a)$\Longleftrightarrow$(b) of Wiegand's Theorem 7, replaced Wiegand's conditiom (a) by, for our purposes, a clearer alternative which was mentioned by Wiegand just before his theorem, and given a tidier version of his Condition (d) in our Condition (v). We will omit the simple details. But, perhaps we should observe that if R is a subring of R_1 then R_1 is an essential extension of R in <u>(Semp) Rng</u> if and only if for each ideal $0 \neq J_1$ of R_1, $J_1 \cap R \neq 0$, or equivalently for each, $0 \neq r_1 \in R_1$, there is $s_1 \in R_1$ such that $0 \neq s_1 r_1 \in R$, while ${}_R R_1$ is an essential extension of the module ${}_R R$ in the category of R-modules if and only if, for each submodule $0 \neq M_1$ of ${}_R R_1$, $M_1 \cap R \neq 0$, or equivalently for each $0 \neq r_1 \in R_1$, there is $s \in R$ such that $0 \neq sr_1 \in R$. The point is that we do not have to distinguish between these notions in the following result.

2.1. Lemma. The following conditions on a semiprime ring R are equivalent:

(i) The extension $\eta_R: R \longrightarrow T(R)$ is essential in Semp Rng.

(ii) The extension $\eta_R: {}_R R \longrightarrow {}_T T(R)$ is essential in the category of R-modules.

(iii) Every non-empty subset of Spec(R) that is open in the patch topology contains a non-empty set of the form $\{P \in \mathrm{Spec}\, R: s \notin P\}$ for some $s \in R$.

(iv) Distinct compact open subsets of Spec(R), endowed with the spectral topology, have distinct closures.

(v) For each finitely generated ideal J of R, ann ann $J = \sqrt{J}$.

In the above, Spec(R) is the set of prime ideals of R, the spectral topology is the usual hull-kernel topology with sets of the form $\{P \in \mathrm{Spec}(R): s \notin P\}$, $s \in R$, as an open sub-base, and the patch-topology is the topology which has these sets and their set-complements as an open sub-base. As usual $\sqrt{J} = \{x \in R: x^n \in J \text{ for some } n \geq 1\} = \cap \{P \in \mathrm{Spec}(R): J \subseteq P\}$.

From 2.1, (i)$\Longleftrightarrow$(v), 1.2 and 1.6 we obtain

2.2. Theorem. Amal (Semp Rng) is the class of all $R \in$ Semp Rng such that ann ann $J = \sqrt{J}$ for each finitely generated ideal J of R.

In connection with Section 1, it is interesting to note that a Rickart ring satisfies the conditions of 2.2 (and 2.1) if and only if it is regular. In [19; Example 4], Wiegand gave an example of a ring satisfying the conditions of 2.2 which is not regular. Actually, there exist many rings with this property. We are able to assert this because of work of Hochster [13] and the duality theory of bounded distributive lattices. We can only include the barest of outlines.

The category of spectral spaces, as defined in Hochster [13], is, from the classical work of M.H. Stone, the dual of the category of bounded distributive lattices, see [10; Chapter 2, Section 11] and also [4]. On the other hand, a very deep theorem of Hochster [13; Theorem 6] says that the spectral spaces are precisely those topological spaces which arise as spaces Spec(R) for some (semiprime) ring R when given the spectral (= hull-kernel) topology (note: Spec(R) and Spec($R/\sqrt{R}$) are homeomorphic). Then, using an argument similar to Wiegand's proof of (iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) in 2.1, or alternatively using duality etc. for bounded distributive lattices as given in [4], we can show: 2.1 (iii), 2.1 (iv), Spec(R) in the patch topology satisfies: each open set contains a compact open of the spectral topology, Spec(R) in the spectral topology is the dual of a disjunctive lattice, are equivalent. Here a bounded distributive lattice L is <u>disjunctive</u> (weakly complemented and section semicomplemented are alternative terms) if for any x, y, $x \neq y$ there exists z such that $x \wedge z = 0$ and $y \wedge z \neq 0$, or vice-versa, or equivalently if and only if the lattice is join-dense in its minimal Boolean extension (this is relevant to 2.1 (iii)). In this correspondence the Boolean algebras are those disjunctive lattices which are corresponding to the regular rings. Thus, there are many non-regular rings satisfying 2.1, 2.2 since there are many non-Boolean disjunctive lattices, e.g. the lattice of closed subspaces of a non-discrete T_1-space, the lattice of subsets of power less than an infinite cardinal $\underline{m}$ of a set X of power $\underline{m}$, together with X itself, as the largest element.

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