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THE TOPOLOGICAL NATURE OF ALGEBRAIC CONTRACTIONS

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Abstract: One shows that if $c : W \longrightarrow W/V$ is the coequalizer of a constant map and a closed immersion in the category of affine schemes of a countable type over a field k, then c is also a topological coequalizer with respect to the Zariski topologies. If $k = \mathbb{R}$ or $\mathbb{C}$ and W/V has the induced product topology, then c is on compact balls a topological coequalizer with respect to the strong topology on W. Finally, if W_m is a closed orbit under the action of G on W, the group quotient of W by G exists if and only if the group quotient of W/W_m by G exists.

Key words: Affine scheme of a countable type over k closed immersion, algebraic contraction, topological cokernel, strong open subset, Zariski topology, submersive, invariant ideals.

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§ 0. Introduction. Let k/G be the category of based affine schemes of a countable type over a field k. The main references here are [2] and [3]. If (W, Q) is an element of k/G, $W = \mathrm{Spec}\, A$ for some countably generated k algebra A, $Q \in W$ and $Q = \mathrm{Spec}\, k$. Suppose that (V, Q) is another element of k/G and

$$i : (V, Q) \longrightarrow (W, Q)$$

is a closed immersion in k/G. This implies that V is the zeroes in W of an ideal in A.

Then, from [2], we know that the cokernel of i in k/G is the map

$$c : (W, Q) \to (\mathrm{Spec}(k + I), V) .$$

Definition 1. The map c is called the algebraic contraction of V in W. We write $\mathrm{Spec}(k+I) = W/V$.

In this paper, we will demonstrate the following propositions.

§ 1) Algebraic contractions are surjective.

§ 2) If $c . (W, Q) \to (W/V, V)$ is an algebraic contraction, then, as a scheme of countable type over k, $W - V$ is isomorphic to $W/V - V$.

§ 3) If $c . (W, Q) \to (W/V, V)$ is an algebraic contraction and U is an affine open of W/V containing V, then, the restriction

$$c' : (c^{-1}(U), Q) \to (U, V)$$

is an algebraic contraction.

§ 4) c is the topological cokernel of i if V, W and W/V are endowed with the Zariski topologies.

Consider the situation when k is $\mathbb{C}$ (or $\mathbb{R}$), the field of complex numbers (or the field of real numbers). Suppose that k has the usual topology. We endow k^{N}, the set theoretic product of k indexed by the natural numbers N, with the product topology. Let (W, Q) be an element in k/G. $W = \mathrm{Spec}\, A$ and A has the form $k[X_1, \dots, X_n, \dots]/J$ where J is an ideal in the polynomial ring $k[X_1, \dots, X_n, \dots]$ in a countable number of variables. Then, W can be identified with a closed affine subscheme of the affine space k^N , and the topology on W induced by the product topology on k^N is called the product topology on W .

Now, suppose that C_i , $i = 1, 2, \dots$, are compact subsets of W and that C_i^o denotes the interior of C_i in the product topology. Furthermore, we require that

1) $C_i \subset C_{i+1}$.

2) $C_{i+1}^o - C_i \cup \{R\}$, for some $R \in W$, is connected.

3) $W = \bigcup_{i=1}^{\infty} C_i$.

In this situation, we make the following definition.

<u>Definition 2</u>. $\mathfrak{U}$ <u>is a strong open subset of</u> W (<u>with respect to the</u> C_i) if and only if $\mathfrak{U} \cap C_i$ is open in C_i with respect to the product topology, $i = 1, 2, \dots$. The collection of strong open subsets of W form a <u>strong topology</u> (<u>with respect to the</u> C_i).

§ 5) Suppose that k is $\mathbb{C}$ (or $\mathbb{R}$) and that W is an affine scheme of finite type over k . If W has the product topology and V (more precisely, V reduced) has smooth components, then there is a strong topology on W/V such that

$$c : W \to W/V$$

is a topological cokernel.

We point out the following theorem to be found in Kelly [6], p. 145, which shows that there are substantial difficulties in extending this result to all elements (W, Q)

of $\mathcal{k}/G$.

16 Theorem. If an infinite number of coordinate spaces are non-compact, then each compact subset of the product is nowhere dense.

Let G be an algebraic group acting on an affine scheme $W = \operatorname{Spec} A$ of finite type over $\mathcal{k}$. Suppose that the action of G on W is closed and that $\mathcal{k}$ is algebraically closed. The reader is referred to Mumford [8], for the notions that we now introduce. Our notation is the following:

i) A^G is the collection of elements in A invariant under G .

ii) R is the collection of (closed) orbits of W under the action of G .

iii) I_κ is the (reduced) defining ideal of κ, κ an element of R .

Note that every closed subset of W (Zariski topology) contains a maximal ideal M of A and if $\mathcal{k}$ is algebraically closed, an element of A must take on a value in $\mathcal{k}$ at M . Therefore, as one can easily show,

$$A^G = \bigcap_{\kappa \in R} (\mathcal{k} + I_\kappa) .$$

We consider $\kappa_1, \kappa_2, \dots, \kappa_m$, a finite number of orbits of G and their union

$$W_m = \bigcup_{i=1}^{m} \kappa_i .$$

The action of G on W induces an action of G on the algebraic contraction W/W_m . Furthermore, let $W^G = \operatorname{Spec} A^G$ and $c^G: W \longrightarrow W^G$ be the map of affine schemes induced by the inclusion $A^G \longrightarrow A$. We state now informally the results to be demonstrated in § 6. More precision will be found in § 6.

§ 6) A categorical quotient (W^G, c^G) of W by G exists, c^G is submersive and W^G is an affine scheme of countable type over k if and only if the corresponding assertion for W/W_m (instead of W) is true.

The section in which the result i) above is proven, is § i , $i = 1,2,3,4,5,6$.

§ 1 The surjectivity of algebraic contractions

We use the notation of § 0. Let $c^*: k+I \longrightarrow A$ be the inclusion map of k algebras corresponding to an algebraic contraction c . Suppose that J is a prime ideal of $k + I$ which generates A . Then,

$$1 = \sum_{i=1}^{m} a_i j_i$$

where $j_i \in J$ and $a_i \in A$, $i = 1, 2, \dots, m$. If $t \in I$,

$$t = \sum_{i=1}^{m} (t a_i) j_i \in J ,$$

and, thus, $J \supset I$. But, I is maximal in $k + I$. This implies that $I = J$. As I is an ideal in A , it is impossible that it generates A . Therefore, JA is a (proper) ideal of A , and, as

$$J' \cap (k + I) = J$$

for a minimal prime ideal J' of JA, it follows that c is surjective.

§ 2. Algebraic contractions outside points of contraction

Again, we use the notation of § 0. Let

$$f_1, \ldots, f_m, \ldots$$

be a generating set of I as an A module. Then,

$$W - V = \bigcup_{m=1}^{\infty} \operatorname{Spec}(A_{f_m})$$

where A_{f_m} is the localization of A at f_m. Also,

$$W/V - \{V\} = \bigcup_{m=1}^{\infty} \operatorname{Spec}((k + I)_{f_m})$$

where $(k + I)_{f_m}$ is the localization of $k + I$ at f_m.

We must show that, under the induced map

$$c_m^* : (k + I)_{f_m} \longrightarrow A_{f_m} ,$$

$(k + I)_{f_m}$ is isomorphic to A_{f_m} as a k algebra.

Clearly, c_m^* is an injection. Suppose that

$$x = \frac{a}{(f_m)^n} \in A_{f_m}$$

It follows that

$$x = \frac{a f_m}{(f_m)^{n+1}}$$

belongs to $(k + I)_{f_m}$.

§ 3 . Affine localizations of algebraic contractions are algebraic contractions

The result promised in § 0 is an immediate consequence of the next proposition.

Proposition 1. In the category of countably generated k algebras, localization preserves equalizers.

Proof. Let $i : E \to A$ be the equalizer of $f, g : A \to B$ in the category of countably generated k algebras. Suppose, furthermore, that S is a multiplicative system in E . We need to show that E_S is the equalizer of $f_S, g_S : A_S \to B_{f \circ i(S)}$. Note that $f \circ i(S) = g \circ i(S)$.

i) $i_S : E_S \to A_S$, the map i localized at S , is injective. $i_S(a/s) = 0$ implies that $i(a)/i(s) = 0$. There is an $s' \in S$ so that $i(s')i(a) = 0$. Then, $i(s'a) = 0$ and $s'a = 0$ in E_S . Hence, $a/s = 0$.

ii) $f_S \circ i_S(a/s) = g_S \circ i_S(a/s)$. This is clear.

iii) Suppose that $f_S(a/s) = g_S(a/s)$. Then there is an $s' \in S$ so that

$$f \circ i(s')(f(a) - g(a)) = 0 .$$

As $f \circ i(s') = g \circ i(s')$,

$$f(i(s')a) - g(i(s')a) = 0 .$$

Therefore, $i(s')a \in E$, and

$$a/i(s) = i(s')a/i(ss')$$

belongs to E_S .

i), ii), and iii) imply Proposition 1.

§ 4. Algebraic contractions are topological quotients with respect to the Zariski topology

Let $\mathfrak{U}$ be an open neighborhood of V . We must show that $c(\mathfrak{U})$ is open in W/V . As c is an isomorphism outside V , this will be done if we show that $c(\mathfrak{U}')$ is open in W/V for an open neighborhood $\mathfrak{U}'$ of V contained in $\mathfrak{U}$.

$\mathfrak{U}$ is covered by affine opens $W_{f_m} = \operatorname{Spec}(A_{f_m})$. As $W - \mathfrak{U}$ and V have no points in common,

$$\sum_m (f_m) + I = A \ .$$

Here, $\sum_m (f_m)$ is the ideal generated by the f_m . Hence,

$$1 = f + t$$

where $f \in \sum_m (f_m)$ and $t \in I$. But,

$$c(W_f) = (W/V)_f$$

is open in W/V. As $W_f \subset \mathfrak{U}$ and W_f is a neighborhood of V , we may take $\mathfrak{U}' = W_f$.

§ 5. Algebraic contractions can be topological quotients for appropriate strong topologies

We reduce to the case when $W = k^n$, $n < \infty$, $k = \mathbb{C}$ (or $\mathbb{R}$).

Consider the diagram

$$\begin{tikzcd} W \arrow[r, "c"] \arrow[d, "i"'] & W/V \arrow[d, "i'"] \\ k^n \arrow[r, "c'"] & k^n/V \end{tikzcd}$$

where i' is induced because of the functoriality of co-equalizers. Here, k^n and W have the product topologies. Then, if J is the ideal of W, i' corresponds to the natural map

$$k + I + J \longrightarrow (k + I)/J$$

of k algebras and is thus a closed immersion. Suppose that there are compact subsets C'_i of k^n/V defining a strong topology on k^n/V such that the algebraic contraction c' is a topological quotient. A diagram chase shows that

$$C_i = C'_i \cap W/V$$

are compact subsets of W/V defining a strong topology on W/V such that c is a topological quotient.

Hence, we need only show:

Suppose that k is $\mathbb{C}$ (or $\mathbb{R}$) and that k^n is affine n space. If k^n has the product topology and V is a closed affine subscheme of k^n with smooth components, then there is a strong topology on k^n/V such that $c : k^n \longrightarrow k^n/V$ is a topological cokernel.

This result, however, is an easy consequence of the next proposition, setting $C_i = c(\bar{B}_i)$.

<u>Proposition 2</u>. Let V be a closed affine subscheme

of k^n, $n < \infty$, $k = \mathbb{C}$ (or $\mathbb{R}$). Suppose that V has smooth components and that $\overline{B}_r$ is the closed ball of radius r in k^n about 0. If k^n and k^n/V have the product topology

$$c : k^n \longrightarrow k^n/V$$

restricts to a topological quotient

$$c : \overline{B}_r \longrightarrow c(\overline{B}_r) .$$

Proof. Note that

$$c : \overline{B}_r - (V \cap \overline{B}_r) \longrightarrow c(\overline{B}_r) - (c(V))$$

is a homoomorphism. Hence, we are finished if we show that every open neighborhood $\mathcal{U}$ of $V \cap \overline{B}_r$ is mapped to an open neighborhood $c(\mathcal{U})$ of $c(V)$.

$V_1, \ldots, V_j$ will be the components of V and V_i will have some defining equations

$$F_i^1 = \ldots\ldots = F_i^{mi} = 0$$

where mi is an integer bigger than zero and $1 \le i \le j$. We assume, furthermore, that

$$m1 \ge m2 \ge \ldots\ldots \ge mj .$$

Set

$$S = \{(s1, s2, \ldots, sj) \mid 1 \le si \le mi\} .$$

Then, if, for $s = (s1, s2, \ldots, sj) \in S$,

$$F_s = F_1^{s1} F_2^{s2} \ldots\ldots F_j^{sj}$$

and the elements of S are enumerated

$$s_1, s_2, \ldots, s_m ,$$

$m = m1 \cdot m2 \cdot \ldots \cdot mj$. c can be writte

$$c = ((F_{s_1})^{\gamma 1}, \dots, (F_{s_m})^{\gamma m}, G_{m+1}, \dots)$$

where $\gamma 1, \dots, \gamma m$ are integers bigger than zero, and where $(F_{s_1})^{\gamma 1}, \dots, (F_{s_m})^{\gamma m}, G_{m+1}, \dots$ belong to I, the ideal of V, and generate the k algebra $k + I$. Notice that we need to take powers of the F_{s_i}, $1 \leq t \leq m$, as I need not be reduced; and that the product topology on k^n/V is independent of the generators chosen for $k + I$.

Let $D_\xi = \{x \in k \mid |x| < \xi\}$, F_ξ be the product of D_ξ m times and

$$G_\xi = (F_\xi \times (\textstyle\times_{i=m+1}^{\infty} k_i))$$

where $k_i = k$, $i = m+1, m+2, \dots$.

Claim: For each $P \in \overline{B}_\pi$, there is an open set U' containg P with the property:

If $\sigma > 0$ (thus, $\sigma \in R$), there is a $\xi > 0$ such that every point of

$$c^{-1}(G_\xi \cap c(U'))$$

lies within a (Euclidean) distance σ of $V \cap U'$.

Assume that the claim is known. As $\overline{B}_\pi$ is compact, for each $\sigma > 0$, there is a $\xi > 0$ such that every point of $c^{-1}(G_\xi \cap c(\overline{B}_\pi))$ lies within a distance σ of $V \cap \overline{B}_\pi$. If U is an open neighborhood of $V \cap \overline{B}_\pi$ and $B(U)$ is its boundary, let

$$\sigma = \min\{d(R, R') \mid R \in \overline{B}_\pi \cap V, R' \in B(U)\}.$$

Here, d denotes the Euclidean distance and $0 < \delta < \infty$ as both $\overline{B}_n \cap V$ and $B(\mathcal{U})$ are compact. Clearly,

$$c^{-1}(G_\xi \cap c(\overline{B}_n)) \subset \mathcal{U}$$

and

$$G_\xi \cap c(\overline{B}_n) = c(c^{-1}(G_\xi \cap c(\overline{B}_n)))$$

is open. Hence, the proof of Proposition 2 will be complete as soon as the claim is demonstrated.

<u>Proof of Claim</u>: Consider $P \in \overline{B}_n \cap V$. The V_i, $i = 1, \dots, j$, can be arranged so that

$P \in \bigcap_{i=1}^{q} V_i$ and $P \notin \bigcup_{i=q+1}^{j} V_i$ for some integer q satisfying $1 < q \leq j$. By choosing appropriate linear combinations of

$$F_i^1, \dots, F_i^{mi},$$

for $i \geq q+1$, one can guarantee that

$$F_i^p(P) \neq 0$$

for $i \geq q+1$ and $1 \leq p \leq mi$. Hence, there is a closed (compact) ball $\overline{B}_\rho(P)$ of radius ρ around P such that

$$F_i^p(Q) \neq 0$$

for $Q \in \overline{B}_\rho(P)$, $i \geq q+1$ and $1 \leq p \leq mi$. Also, if

$$\theta = \min\{|F_i^p(Q)| \mid Q \in \overline{B}_\rho(P),\ i \geq q+1,\ 1 \leq p \leq mi\},$$

$\theta > 0$ as $\overline{B}_\rho(P)$ is compact. Thus, if for all $s \in S$ and $Q \in \overline{B}_\rho(P)$,

$$|F_s(Q)| < \xi,$$

then

$$|(F_1^{s1}(Q)\cdot F_2^{s2}(Q)\cdot \ldots \cdot F_j^{sj}(Q))^{\gamma s}| < \xi$$

and

$$|F_1^{s1}(Q)\cdot F_2^{s2}(Q)\cdot \ldots \cdot F_q^{sq}(Q)| < \xi^{1/\gamma s}/\theta^{j-q} \quad .$$

Hence, we can assume that

$$P \in \bigcap_{i=1}^{j} V_i$$

and that I is reduced.

Next, we show that if ξ is small enough and $Q \in$ $\in \bar{B}_\rho(P)$,

$*$) For some i, $1 \leq i \leq j$,

$$F_i^1(Q), F_i^2(Q), \ldots, F_i^{mi}(Q)$$

must be small.

Suppose, for instance, that F_1^μ, $1 \leq \mu \leq m1$, is not small. As

$$|F_1^\mu(Q)\cdot F_2^{s2}(Q)\cdot \ldots \cdot F_j^{sj}(Q)| < \xi \quad ,$$

$$|F_2^{s2}(Q)\cdot \ldots \cdot F_j^{sj}(Q)|$$

must be small for $(\mu, s2, \ldots, sj) \in S$. Induction, then, implies that one has small values

$$F_i^1(Q), \ldots, F_i^{mi}$$

for some i such that $2 \leq i \leq j$. Otherwise, F_1^μ is small for $1 \leq \mu \leq m1$, in which case $*$) is true.

The proof is reduced to the case where V has one reduced smooth component.

We select defining equations $F_1, F_2, \ldots, F_m$ for V. Let $Q \in \overline{B}_\rho(P)$. For every i, $1 \leq i \leq m$, F_i can be written

$$F_i(X) = \frac{\partial F_i}{\partial X_1}(Q)(X_1 - Q_1) + \ldots + \frac{\partial F_i}{\partial X_n}(Q)(X_n - Q_n) +$$

higher degree terms

where $X = (X_1, X_2, \ldots, X_n)$ and $Q = (Q_1, Q_2, \ldots, Q_n)$.

If ρ is small enough, the higher degree terms can be disregarded. Let

$$A(Q) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1}(Q) & \cdots\cdots & \frac{\partial F_1}{\partial X_n}(Q) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial X_1}(Q) & \cdots\cdots & \frac{\partial F_m}{\partial X_n}(Q) \end{bmatrix}$$

$A(Q) \cdot (X - Q) = \xi$ is the equation of the subspace of k^n parallel to the tangent space $T(Q)$ to V at Q whose distance from $T(Q)$ and, hence, Q is determined by ξ. The coefficients of $A(Q)$ are bounded as $Q \in \overline{B}_\rho(P)$, a compact set. Therefore, for a given $\delta > 0$, there is an $\xi > 0$ such that a point X in $\overline{B}_\rho(P)$ satisfying $|F_i(X)| < \xi$ for $i = 1, 2, \ldots, m$ must lie within a distance δ of $V \cap \overline{B}_\rho(P)$. Taking $U = B_\rho(P)$, the interior of $\overline{B}_\rho(P)$, the proof is complete.

<u>Example 1.</u> If, in Proposition 2, one takes the product topology on k^n/V, $c : k^n \to k^n/V$ is not necessarily the topological quotient. Let $k = \mathbb{R}$ and

$n = 2$. Suppose V is the Y axis. Then, the image of the open subset

$$U = \{ (x, y) \mid x < e^{-y} \}$$

of k^2 under c is not open. For instance, the sequence

$$X_m = (e^{1/m}, 1/m)$$

lies outside U but converges to the image of the Y axis under c .

§ 6. <u>Some geometric invariant theory</u>

Our notation is that of § 0. First, we collect some results which will be useful.

Let $W^G = \mathit{Spec}\, A^G$ and $c^G : W \longrightarrow W^G$ be the map defined by the inclusion $A^G \rightarrow A$. Then, according to Mumford [8], p. 8, a categorical quotient (W^G, c^G) of W by G exists and c^G is submersive when the following conditions hold.

<u>i</u>) If $\sigma : G \times W \longrightarrow W$ defines the operation of G on W and $P_2 : G \times W \longrightarrow W$ is the second projection, then

$$c^G \circ \sigma = c^G \circ P_2 \ .$$

<u>ii</u>) $\underline{o}_W{}^G$ is the subsheaf of invariants of $c^G{}_*(\underline{o}_W)$.

<u>iii</u>) If X is an invariant closed subset of W , $c^G(X)$ is closed in W^G ; if X_i , $i \in I$, form a set of invariant closed subsets of W , then

$$c^G(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} c^G(X_i) \quad .$$

As in Mumford [8], p.28, one can deduce that iii) is implied by the relation:

$$\underline{\text{iii}'}) \qquad \sqrt{(\sum_{i\in I} A_i)} \cap A^G = \sqrt{\sum_{i\in I} (A_i \cap A^G)}$$

where the A_i are G invariant ideals in A correspon - ding to the X_i . Note that the radical operation $\surd$ commutes with $\cap$.

Next, we restate the first result promised in § 0.

<u>Proposition 3</u>. Suppose that W_m is the finite union of orbits $\kappa_1, \kappa_2, \ldots, \kappa_m$ of G . A categorical quotient (W^G, c^G) of W by G exists, c^G is submersive and W^G is an affine scheme of countable type over k if and only if, for the induced action of G on W/W_m , a categorical quotient (W_m^G, c_m^G) of W_m by G exists, c_m^G is submersive and W_m^G is an affine scheme of countable type over k . Moreover, if W^G exists, $W^G = W_m^G$.

<u>Proof</u>. There is a countable subset R'' of R such that $\bigcup_{\kappa\in R''} \kappa$ is dense in W and $R' = \{\kappa_1, \kappa_2, \ldots, \kappa_m\} \subset R''$. If we write $R'' = \{\kappa_1, \kappa_2, \ldots, \kappa_i, \kappa_{i+1}, \ldots\}$, it follows that

$$A^G = \bigcap_{i=1}^{\infty} (k + I_{\kappa_i}) \quad .$$

Define now

$$E_1 = k + I_{\kappa_1} \quad ,$$

$$E_2 = k + (I_{r_2} \cap (k + I_{r_1}))$$

and , inductively, for each positive integer $j > 0$,

I) $$E_{j+1} = k + (I_{r_{j+1}} \cap E_j) .$$

Then, for each integer $j > 0$, by means of induction, one can prove without difficulty that

II) $$E_j = \bigcap_{i=1}^{j} (k + I_{r_i}) .$$

Let $B = \bigcap_{r \in R'} (k + I_r)$. Relations I and II imply

$$k + (I_{r_j} \cap E_m) = \bigcap_{i=1}^{m} (k + I_{r_i}) \cap (k + I_{r_j})$$

for each integer $j > m$, the order in R'' being immaterial. Hence, on taking countable intersections, $A^G = B^G = \bigcap_{r \in R''} (k + I_r)$. Let $c_m^G : W/W_m \to W^G$ be the affine map of schemes defined by the inclusion $A^G \longrightarrow B$.

For both W and W/W_m , Condition) above is obviously true. One can derive Condition ii) for both W and W/W_m from Proposition 1. Hence, in order to complete the proof of Proposition 3, it is necessary to show that iii′) is valid for W if and only if it is valid for W/W_m .

As $\operatorname{Spec} E_j \longrightarrow \operatorname{Spec} E_{j+1}$ has been shown in § 4 to be a topological quotient, for each integer $j > 0$, so is the composite

$$\operatorname{Spec} E_1 \longrightarrow \operatorname{Spec} E_2 \longrightarrow \dots\dots \longrightarrow \operatorname{Spec} E_m .$$

Therefore, the map

$$\operatorname{Spec} A \to \operatorname{Spec} \bigcap_{i=1}^{m} (k + I_{x_i})$$

is the topological quotient shrinking each x_i, $i = 1, 2, \ldots, m$, to a point. Hence

$$\text{III)} \quad \sqrt{\left(\sum_{i \in I} A_i\right)} \cap \left(\bigcap_{i=1}^{m} (k + I_{x_i})\right) = \sqrt{\sum_{i \in I} \left(A_i \cap \left(\bigcap_{i=1}^{m} (k + I_{x_i})\right)\right)} .$$

Intersecting this last equality with $B^G = A^G = \bigcap_{x \in R''} (k + I_x)$, we discover that $\sqrt{\left(\sum_{i \in I} A_i\right)} \cap A^G = \sqrt{\sum_{i \in I} (A_i \cap A^G)}$ when iii′) is valid for W / W_m. Since every G invariant ideal B' in B is of the form $A' \cap B$ for some G invariant ideal A' in A, the validity of iii′) for W implies the validity of iii′) for W / W_m. q.e.d.

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