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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 4, 761--766

Persistent URL: `http://dml.cz/dmlcz/105524`

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14,4 (1973)

SOME HIGHER ORDER OPERATIONS WITH CONNECTIONS

(Preliminary communication)

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Abstract: Some relations between the higher order connections on a Lie groupoid and the first order connections on the higher order prolongations of this groupoid are studied.

Key words: Connection, jet, Lie groupoid, absolute differentiation.

AMS, Primary: 53C05 Ref. Ž. 3.933.13

We present an abstract of the main results of a paper under the same title which will be published in Czechoslovak Mathematical Journal.

1. Let Φ be a Lie groupoid over B. The partial composition law in Φ as well as the prolongations of this law will be denoted by a dot. If Φ is a groupoid of operators on a fibred manifold (E, p, B), then the r-th non-holonomic prolongation $\tilde{\Phi}^r$ of Φ is a groupoid of operators on the r-th non-holonomic prolongation $\tilde{J}^r E$ of E, [1]. In the semi-holonomic case, the same holds for $\bar{\Phi}^r$ and $\bar{J}^r E$. Let $\tilde{Q}^r(\Phi)$ or $\bar{Q}^r(\Phi)$ be the fibred manifold of all non-holonomic or semi-holonomic elements

of connection of order r on Φ respectively. If $C :$ $: B \longrightarrow Q^1(\Phi)$ is a first order connection, then its r-th prolongation in the sense of Ehresmann is a cross section $C^{(r)} : B \longrightarrow \overline{Q}^{r+1}(\Phi)$, [2]. As usual, $\widetilde{\Pi}^r(B)$ or $\overline{\Pi}^r(B)$ will mean the groupoid of all invertible non-holonomic or semi-holonomic r-jets of B into B. Further, j_r^s will denote the canonical projection of r-jets onto s-jets, $s < r$.

2. Let $X \in \widetilde{Q}_x^{r+1}(\Phi)$, $X = j_x^1 \nu$ and let $Y \in$ $\in Q^1(\widetilde{\Pi}^r(B))$, $Y = j_x^1 \lambda$. We define a mapping $\varkappa_{r+1}$: $: \widetilde{Q}^{r+1}(\Phi) \boxtimes Q^1(\widetilde{\Pi}^r(B)) \rightarrow Q^1(\widetilde{\Phi}^r)$ by $\varkappa_{r+1}(X, Y) =$ $= j_x^1 (\nu(y)\lambda(y).\nu^{-1}(x))$, provided $\boxtimes$ denotes the fibre product over B.

<u>Proposition 1</u>. The mapping $(\varkappa_{r+1}, j_{r+1}^r)$:

$: \widetilde{Q}^{r+1}(\Phi) \boxtimes Q^1(\widetilde{\Pi}^r(B)) \longrightarrow Q^1(\widetilde{\Phi}^r) \boxtimes \widetilde{Q}^r(\Phi)$, $(\varkappa_{r+1}, j_{r+1}^r)(X,Y) =$ $= (\varkappa_{r+1}(X,Y), j_{r+1}^r X)$ is a B-isomorphism.

In the special case $\Phi = \Pi^1(B)$, we introduce a mapping

$\varphi_r : \overline{Q}^r(\Pi^1(B)) \longrightarrow Q^1(\overline{\Pi}^r(B))$ by the following induction:

a) φ_1 is the identity of $Q^1(\Pi^1(B))$,

b) $\varphi_r(X) = \varkappa_r(X, \varphi_{r-1}(j_r^{r-1} X))$, $X \in \overline{Q}^r(\Pi^1(B))$.

<u>Proposition 2</u>. The mapping φ_r is a B-isomorphism.

If a connection $C: B \longrightarrow Q^1(\Phi)$ and a linear connection on the base manifold $L: B \longrightarrow Q^1(\Pi^1(B))$ are given, then we define the prolongation $p(C,L)$ of C with respect to L by $p(C,L) = \varkappa_2(C', L): B \longrightarrow Q^1(\Phi^1)$, where $C' = C^{(1)}$. The r-th prolongation $p^r(C,L)$ of C with respect to L is defined by the iteration $p^r(C,L) = p(p^{r-1}(C,L), L)$, $p^0(C,L) = C$. This is a cross section of $Q^1(\overline{\Phi}^r)$. The relation of $p^r(C,L)$ to the prolongations of C and L in the sense of Ehresmann is described by

<u>Proposition 3</u>. The connections $C^{(r)}: B \rightarrow \overline{Q}^{r+1}(\Phi)$, $L^{(r-1)}: B \longrightarrow \overline{Q}^r(\Pi^1(B))$ and $p^r(C,L): B \longrightarrow Q^1(\overline{\Phi}^r)$ satisfy

$$\varkappa_{r+1}(C^{(r)}, \varphi_r(L^{(r-1)})) = p^r(C,L) .$$

3. We shall give a comparison of the absolute differentiation with respect to the connections of Proposition 3. According to [2], every $X \in \widetilde{Q}^r(\Phi)$ determines a mapping $X^{-1}: \widetilde{J}^r_x E \longrightarrow \widetilde{J}^r_x(B, E_x)$, $W \mapsto X^{-1}.W$, which is said to be the absolute differential with respect to X. Since $\widetilde{\Phi}^r$ is a groupoid of operators on $\widetilde{J}^r E$, the absolute differential with respect to an element $Z \in Q^1(\widetilde{\Phi}^r)$ is a mapping $Z^{-1}: \widetilde{J}^{r+1}_x E \longrightarrow J^1_x(B, \widetilde{J}^r_x E)$. Let $Z_1 \in Q^1_x(\widetilde{\Phi}^{r-1})$ be the element of connection derived from Z by means of the functor $j^{r-1}_r: \widetilde{\Phi}^r \longrightarrow \widetilde{\Phi}^{r-1}$. The map-

ping $Z_1^{-1}: \tilde{J}_x^r E \longrightarrow J_x^1(B, \tilde{J}_x^{r-1} E)$ is extended to a mapping $Z_{1*}^{-1}: J_x^1(B, \tilde{J}_x^r E) \longrightarrow J_x^1(B, J_x^1(B, \tilde{J}_x^{r-1} E))$, $j_x^1 \varphi(y) \mapsto j_x^1(Z_1^{-1}(\varphi(y)))$.

Then

$$Z_{1*}^{-1} \circ Z^{-1}: \tilde{J}_x^{r+1} E \longrightarrow J_x^1(B, J_x^1(B, \tilde{J}_x^{r-1} E)) . \tag{1}$$

Define by induction $N_x^1(B, E_x) = J_x^1(B, E_x)$, $N_x^{r+1}(B, E_x) = J_x^1(B, N_x^r(B, E_x))$. By iterating (1), we obtain a mapping $t(Z^{-1}): \tilde{J}_x^{r+1} E \longrightarrow N_x^{r+1}(B, E_x)$, which will be called the full absolute differential with respect to Z . Analogously to the concept of a semi-holonomic jet, we define a subspace $S_x^r(B, E_x) \subset N_x^r(B, E_x)$ by the following induction:

a) $S_x^1(B, E_x) = J_x^1(B, E_x)$,

b) an element $j_x^1 \sigma \in N_x^r(B, E_x)$ belongs to $S_x^r(B, E_x)$ if σ is a local mapping of B into $S_x^{r-1}(B, E_x)$ satisfying $\sigma(x) = j_x^1[\beta \sigma(y)]$.

<u>Proposition 4</u>. If $Z \in Q_x^1(\tilde{\Phi}^r)$ and $W \in \bar{J}_x^{r+1} E$, then $t(Z^{-1})(W) \in S_x^{r+1}(B, E_x)$.

In particular, let $\bar{C}: B \longrightarrow Q^1(\bar{\Phi}^r)$ be a connection on $\bar{\Phi}^r$ and let $\sigma: B \longrightarrow E$ be a cross section. Then the cross section

$$t(\bar{C}^{-1})(\sigma) : B \longrightarrow \bigcup_{x \in B} S_x^{k+1}(B, E_x) := S^{k+1}(E) ,$$
$$x \mapsto t(\bar{C}^{-1}(x))(j_x^{k+1}\sigma)$$

will be said to be the full absolute differential of σ with respect to $\bar{C}$.

On the other hand, every $Y \in Q_x^1(\tilde{\Pi}^k(B))$, $Y = j_x^1 \lambda$, determines a mapping $\mu(Y) : \tilde{J}_x^{k+1}(B, E_x) \longrightarrow J_x^1(B, \tilde{J}_x^k(B, E_x))$, $j_x^1 \varphi(y) \mapsto j_x^1(\varphi(y)\lambda(y))$. Consider the element $Y_1 \in Q_x^1(\tilde{\Pi}^{k-1}(B))$ derived from Y by means of the functor $j_k^{k-1} : \tilde{\Pi}^k(B) \longrightarrow \tilde{\Pi}^{k-1}(B)$. The mapping $\mu(Y_1)$ is extended to a mapping $\mu(Y_1)_* : J_x^1(B, \tilde{J}_x^k(B, E_x)) \longrightarrow$ $\longrightarrow J_x^1(B, J_x^1(B, \tilde{J}_x^{k-1}(B, E_x)))$, $j_x^1 \varphi(y) \mapsto j_x^1 \mu(Y_1)(\varphi(y))$ and one can construct $\mu(Y_1)_* \circ \mu(Y) : \tilde{J}_x^{k+1}(B, E_x) \longrightarrow$ $\longrightarrow J_x^1(B, J_x^1(B, \tilde{J}_x^{k-1}(B, E_x)))$. By iteration, we obtain a mapping $\tau(Y) : \tilde{J}_x^{k+1}(B, E_x) \longrightarrow N_x^{k+1}(B, E_x)$. In particular, if $Y \in Q_x^1(\bar{\Pi}^k(B))$ and $W \in \bar{J}_x^{k+1}(B, E_x)$, then $\tau(Y)(W) \in S_x^{k+1}(B, E_x)$.

Proposition 5. In the situation of Proposition 3, $t([p^k(C, L)(x)]^{-1})$ is the composition of $[C^{(k)}(x)]^{-1}$ and $\tau(\varphi_k(L^{(k-1)}(x)))$, $x \in B$.

The groupoid $\Phi \times \Pi^1(B)$ operates on $S^{k+1}(E)$ in a natural way. This action can be used for a simple step by step construction of the full absolute differential of a cross section $\sigma : B \to E$ with respect to $p^{r}(C,L)$ by means of C and L only.

Proposition 6. The full absolute differential of σ with respect to $p^{r}(C,L)$ coincides with the absolute differential with respect to the connection $C \times L$ on $\Phi \times \Pi^{1}(B)$ of the full absolute differential of σ with respect to $p^{r-1}(C,L)$.

Proposition 6 gives an interesting consequence for the special case of connections on vector bundles. In the vector bundle case, our prolongation of a connection with respect to a linear connection on the base manifold coincides with the operation treated by Pohl, [3] .

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(Oblatum 8.11.1973)