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ON RECONSTRUCTING OF INFINITE FORESTS

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§ 1. Introduction. It is well known that every finite tree (i.e. an undirected connected graph without cycles) can be reconstructed from the collection of its maximal subgraphs, maximal subtrees or non-isomorphic maximal subtrees (see [2,3,4]). (By a subgraph we mean throughout this paper a proper subgraph.) N.St.A. Nash Williams proposed the analogous problem for infinite trees [5]. We give here a partial answer to this question.

A ray is a one way infinite path, a forest is a graph every component of which is a tree. We prove that every rayless forest can be reconstructed from the collection of all its non-isomorphic maximal subforests. We prove even that the knowledge of almost all graphs from this collection suffices. On the other hand, we show that the general statement "every forest can be reconstructed from the collection of all its subgraphs" is not true. This statement being true for finite forests, we exhibit a counterexample of an infinite forest which may be regarded as the simplest example of a non-reconstructible graph, see also [1].

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§ 2. Infinite rayless trees. Let $T = (V(T), E(T))$ be a fixed rayless tree. Denote by $\mathcal{J}(T)$ the set of all vertices of T of an infinite degree. (The degree $d(x, T)$ of a vertex x is the cardinality of the set $\{y; [x, y] \in \in E(T)\}$.) It is easy to prove:

Lemma 1: $\mathcal{J}(T) \neq \emptyset$ iff T is infinite.

Let T_0 be the minimal subtree of T which contains $\mathcal{J}(T)$. Define T_n as the minimal subtree of T containing $\mathcal{J}(T_{n-1})$. Since every tree T_n is rayless we have $T_n \neq T_{n+1}$ for $n = 0, 1, \dots$. Further, there is an n such that $T_n = \emptyset$, hence, by Lemma 1, there is T_m such that T_m is a finite tree.

Let $A(T)$ be the group of all automorphisms of the tree T. We have $f(T_n) = T_n$ for every $n = 0, 1, \dots$ and for every $f \in A(T)$. Let $c(T)$ be the center of the tree T_m. (The center of a tree is the intersection of all diameters of T, recall that $|C(T)| \leq 2$.) Thus $f(C(T)) = C(T)$ for every $f \in A(T)$.

Hence the permutation group $A(T)$ has analogous properties to the automorphism group of a finite tree, particularly it can be obtained by (infinite) applying of a direct sum and wreath product to a system of symmetric groups.

Let us remark that the following holds:

Let T be an infinite rayless tree, $x \in p(T)$ (denote by $p(T)$ the set of all pendant vertices, i.e. the set of all vertices of degree 1). Then we have $C(T) = C(T-x)$. Here the tree $T - x$ is defined by $V(T-x) = V(T) \setminus \{x\}$, $E(T-x) = E(T) \setminus \{[x, x_{-1}]\}$, where $[x, x_{-1}] \in E(T)$.

Denote by $[T]$ the isomorphism type of the tree T. Put $\mathcal{UK}(T) = \{[T-x]; x \in p(T)\}$.
In the case that $C(T)$ is a single point, we call the tree central. In the case that $C(T)$ are two points (which form an edge), we call the tree bicentral.

Lemma 2: Let T, S be infinite rayless bicentral trees; $\{x, x'\} = C(T)$, $\{y, y'\} = C(S)$. Let $T^{\vee}$ be the tree defined by $V(T^{\vee}) = V(T) \cup \{c\}$ $E(T^{\vee}) = (E(T) \setminus \{[x, x']\}) \cup \{[x, c], [c, x']\}$ where $c \notin V(T)$. Define analogously $S^{\vee}$. Then $\mathcal{UK}(T) = \mathcal{UK}(S)$ iff $\mathcal{UK}(T^{\vee}) = \mathcal{UK}(S^{\vee})$.

Proof is obvious since $C(T) = C(T-a)$ $a \in p(T)$.
In view of the above lemma we can restrict ourselves to central trees. Thus, let T, S be infinite central rayless trees.
A branch of a tree T at a point x is every maximal subtree of T which contains x as a pendant vertex. A limb of T is every branch at $C(T)$.

Lemma 3: Let $\varkappa : p(T) \longrightarrow p(S)$ be a bijection such that $[T-x] = [S - \varkappa(x)]$ for almost all $x \in p(T)$. Then $T \simeq S$.

Proof: Let $V = V(C(T), T)$ $U = V(C(S), S)$ where $V(x, T) = \{y; [x, y] \in E(T)\}$. Let T_x be the limb of T at $C(T)$ containing $x \in V$, analogously S_x. Let the relation $R \subset V \times U$ be defined by
$(x, y) \in R \Longleftrightarrow (T_x, C(T)) \simeq (S_y, C(S))$

(here we mean the root-isomorphism, i.e. an isomorphism of T_x and S_y which maps $C(T)$ onto $C(S)$).

We prove that there exists an $f: V \longrightarrow U$ such that 1) f is one-to-one ,

2) $(T_x, C(T)) \simeq (S_{f(x)}, C(S))$.

According to the Hall theorem it suffices to prove $|R(A)| \geq |A|$ for every finite subset A of V (we put $R(A) = \{y; \exists x \in A, (x,y) \in R\}$). In the way of contradiction let us suppose $|R(A)| < |A|$ for a finite subset $A \subset V$, we can assume that A is chosen in such a way that $B \subsetneqq A$ implies $|R(B)| \geq |B|$. It is $|R(A)| = |A| - 1$.

We distinguish two cases:

I. $|A| > 1$:

We claim $(S_y - a, C(S)) \not\simeq (S_y, C(S))$ for every $y \in R(A)$, $a \in p(S_y)$. Let $a \in p(T_x)$, $x \in A$ then there exists an isomorphism $\varphi: T - a \longrightarrow S - b$ and $\varphi(V(T_x) \setminus a) \not\subset \bigcup\{V(S_a); a \in R(A)\}$ (for otherwise $|R(A)| \geq |A|$), consequently $(T_x - a, C(S)) \not\simeq (T_x, C(S))$ for every $x \in A$ and $a \in p(T_x)$. This proves the claim by the definition of R .

Let $a \in p(S_y)$, $y \in R(A)$, $\varphi: S - a \longrightarrow T - b$ be an isomorphism. Then obviously $b \in V(T_z)$, $z \in A$ and $\varphi(S_y - a) \subset \bigcup\{T_x ; z \neq x \in A\}$. By the assumption on A there is a one-to-one mapping $\psi: A \setminus \{z\} \longrightarrow R(A)$ such that $(T_x, C(T)) \simeq (S_{\psi(x)}, C(S))$.

According to the claim proved above it is $\varphi(y) \neq \psi^{-1}(y)$. Put $\chi = \psi \circ \varphi$. Then χ is a permutation on the set

$R(A)$, hence there exists an n such that $(\chi)^n(y) = y$, then $\psi^{-1}\chi^n = t \in A$ contradicts the claim as $(T_t, C(T)) \simeq (S_y, C(S)) \simeq (S_y - a, C(S))$.

II. $|A| = 1$:

a) Assume that there exist $v \in V$ and $u \in U$ such that $(T_v, C(T)) \not\simeq (S_t, C(S))$ for every $t \in U$ and $(S_u, C(S)) \not\simeq (T_t, C(T))$ for every $t \in V$. First, let $a \in p(T) \cap p(T_v)$ then necessarily $T - a \simeq S - b$ and $b \in p(S_u)$. Hence there exists a bijection $\varphi : V \setminus \{v\} \longrightarrow U \setminus \{u\}$ such that $(T_t, C(T)) \simeq (S_{\varphi(t)}, C(S))$.
Secondly, if $a \in p(T) \cap p(T_t)$, $t \neq v$ and $\varphi : T - a \longrightarrow S - b$ is an isomorphism then $b \in p(S_x)$ $x \neq u$ and

$$\varphi(T_v) = S_x \setminus \{b\}$$
$$\varphi(T_t \setminus \{a\}) = S_u .$$

Since $|U| \geq 2$ and $|V| \geq 2$ we have that

$$(S_u, C(T)) \simeq (T_t - a, C(T)) \simeq (S_{\varphi(t)} - b, C(S)) \simeq (T_v, C(T))$$

for convenient a and b, a contradiction.

b) By I, II a), we may suppose that there exists a monomorphism $\varphi : S \longrightarrow T$ and that there exists $v \in V$ such that $(T_v, C(T)) \not\simeq (S_t, C(S))$ for every $t \in U$.

Let $a \in p(T) \cap p(T_x)$ $x \neq v$, then $T - a \simeq S - b$ and $b \in p(S_y)$ where $(S_y - b, C(S)) \simeq (T_v, C(T))$. But $(T_{\varphi(y)}, C(T)) \simeq (S_y, C(S))$; thus there exists $b' \in p(T) \cap p(T_{\varphi(y)})$ such that $(T_{\varphi(y)} - b', C(T)) \simeq (S_y - b, C(S))$. Then $T - b' \simeq S - b''$ and thus there exists a $x \in \mathcal{U}$ such that $(S_x, C(S)) \simeq (T_v, C(T))$. This is a contradiction.

Hence by I and II we may suppose that there are mappings $f : \mathcal{V} \longrightarrow \mathcal{U}$ and $g : \mathcal{U} \longrightarrow \mathcal{V}$ which satisfy

1) f, g are one to one,

2) $(T_x, C(T)) \simeq (S_{f(x)}, C(S))$ and $(S_y, C(S)) \simeq$
$\simeq (T_{g(y)}, C(T))$ for every $x \in \mathcal{V}$ and $y \in \mathcal{U}$.

Then there is a bijection $h : \mathcal{V} \longrightarrow \mathcal{U}$ such that $(T_x, C(T)) \simeq (S_{h(x)}, C(S))$ for every $x \in V$. (This may be proved as follows: Put $V = \mathcal{V} \setminus g(\mathcal{U})$, then $x \in V$ implies $(T_x, C(T)) \simeq (T_y, C(T))$ for infinitely many different $y \in \mathcal{V}$. Thus we may easily construct a bijection e.g. by $h|_{g(\mathcal{U})} = g^{-1}$, $h|_V =$ identity.) This proves the lemma.

<u>Theorem 1</u>: Let T, S be rayless trees. Then $\mathcal{UK}(S) = \mathcal{UK}(T)$ iff $T \simeq S$.

Proof follows by Lemma 3, the finite case by [4].

<u>Remark</u>: In [6] there is proved a theorem on reconstructing of an asymmetric tree T (i.e. a tree which possesses no no-trivial automorphisms) from the collection of all its asymmetric subtrees. The similar theorem for infinite rayless trees seems to be harder for one can construct an asymmetric rayless tree T such that $T - x$ is not an asymmetric tree for every $x \in p(T)$.

§ 3. <u>Rayless forests</u>.

<u>Theorem 2</u>: Let S, T be rayless forests. Then $\mathcal{UK}(S) = \mathcal{UK}(T)$ iff $S \simeq T$.

Proof: Clearly one direction is needed to prove only.

Let $\mathcal{U}\mathcal{K}(\mathcal{S}) = \mathcal{U}\mathcal{K}(T)$. By [2] we can assume that all the tree components are infinite. Denote by T_ι, $\iota < \alpha$ ($\mathcal{S}_\iota$, $\iota < \beta$, respectively) all the tree components of T ($\mathcal{S}$, respectively). Clearly $\alpha = \beta$ and we can assume (by Lemma 2) that all the trees T_ι, $\iota < \alpha$ ($\mathcal{S}_\iota$, $\iota < \alpha$ respectively) are central.

Let $c \notin \bigcup\{V(T_\iota); \iota < \alpha\}$ $c' \notin \bigcup\{V(\mathcal{S}_\iota); \iota < \alpha\}$. Define the tree $\widetilde{T}$ by

$$V(\widetilde{T}) = V(T) \cup \{c\},\ E(\widetilde{T}) = E(T) \cup \{[c, c(T_\iota)];\ \iota < \alpha\}.$$

Define analogously the tree $\widetilde{\mathcal{S}}$.

Then $\mathcal{U}\mathcal{K}(T) = \mathcal{U}\mathcal{K}(\mathcal{S})$ implies $\mathcal{U}\mathcal{K}(\widetilde{T}) = \mathcal{U}\mathcal{K}(\widetilde{\mathcal{S}})$. Hence $\widetilde{T} \simeq \widetilde{\mathcal{S}}$ by Theorem 1 and thus $T \simeq \mathcal{S}$.

§ 4. An example: Let T be the tree every degree of which is $\alpha \geq 2$. Denote by $X \cup Y$ the disjoint union of the graphs X and Y. Then $\mathcal{U}\mathcal{K}(T_\alpha \cup T_\alpha) = \mathcal{U}\mathcal{K}(T_\alpha)$ for every $\alpha \geq \aleph_0$. This is evident since $[X] \in \mathcal{U}\mathcal{K}(T_\alpha)$ implies that X is a forest with α components which are all isomorphic to T_α.

In this connection we conjecture that every forest with an endpoint is reconstructible from the collection of all its maximal subgraphs.

R e f e r e n c e s

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