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Aleš Pultr; Věra Trnková

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ON REALIZATION AND BOUNDABILITY OF CONCRETE CATEGORIES IN WHICH THE MORPHISMS ARE CHOICED BY LOCAL CONDITIONS

A. PULTR, V. TRNKOVÁ, Praha

The present paper is a contribution to the question of full embeddability of concrete categories into categories of algebras (boundability) asked by J.R. Isbell in [1]. In [2] and [3] a method of embedding of categories described by functors is elaborated. Using that method, we prove that also categories the object of which are topological spaces and morphisms selected among continuous mappings by means of local conditions given by some functor, are again boundable. E.g., let us ask whether the category of topological spaces with locally one-to-one continuous mappings is boundable. We obtain a positive answer using local selection by means of binary relations "to be distinct". Similarly we obtain the boundability of, e.g., the following categories:
topological spaces with local homeomorphisms,
topological spaces with quasi-local homeomorphisms (here is a different "localisation" than in previous examples),
topological spaces with open local homeomorphisms,
metric spaces with locally Lipschitzian mappings (here we must, of course, show first how the global Lipschitz condition can be realized by means of a functor),
etc.

We use the notation from [3] and [4]. Without particular mentioning we use the assumption of non-existence of measurable cardinals (or, the weaker assumption (M) from [3]). The reader acquainted with the quoted papers will see easily in which statements we need this assumption and in which not. For brevity, we use the expression F-structure on X for any subset of $F(X)$ where F is a set functor (i.e., a functor from the category of sets into itself); if r, s resp. are F-structures on X, Y resp., we say that a mapping $f: X \to Y$ is r s-compactible if $F(f)(r) \subset s$ in the covariant case, $F(f)(s) \subset r$ in the contravariant one.

The paper is divided into three parts. The first one contains definitions and some theorems. The main results are proved in the second part. Finally, the third part deals with applications of these results to concrete cases.

I.

__Definition 1__: Let F be a covariant set functor, r, s F-structures on topological spaces X, Y resp. We say that a mapping $f: X \to Y$ is locally r s-compactible, if:
for every $x \in X$ there is an open $U \ni x$ with $F(f)\,([F(j_U)\,(F(U))] \cap r) \subset s$ (where j_U is the embedding of U into X.

If F is contravariant, we say that f is locally r s-compactible, if
for every $x \in X$ there is an open $U \ni x$ with $F(j_U)\,(F(f)(s)) \subset F(j_U)(r)$.

__Remark__: The definition is motivated as follows: in covariant (contravariant) case we request that for every $x \in X$ there be an open U containing x such that

$$f|U : (U, (F(j_U))^{-1}(r)) \longrightarrow (Y, s)$$

$$(f|U : (U, F(j_U)(r)) \longrightarrow (Y, s))$$

is compactible.

<u>Definition 2</u>. If r is an F-structure on X we define a $P^- F$ structure r^* on X as follows:

$a \in r^*$ if and only if $a \cap r = \emptyset$.

<u>Lemma 1</u>. Let F be a contravariant set functor. If r, s are F-structures on X, Y respectively, then $f: X \to Y$ is $r\,s$-compactible if and only if it is $r^* s^*$-compactible.

<u>Proof</u>: Let $F(f)(s) \subset r$, $a \in r^*$. Then evidently $P^- F(f)(a) = (F(f))^{-1}(a) \in s^*$. Conversely let $P^- F(f)(r^*) \subset s^*$, $\xi \in s$. Let $F(f)(\xi) \notin r$, then $(F(f)(\xi)) \in r^*$ and, consequently, $P^- F(f)(F(f)(\xi)) = [F(f)]^{-1}(F(f)) \in s^*$; however $\xi \in \in (F(f))^{-1}\ (F(f)(\xi)) \cap s$, which is a contradiction.

<u>Lemma 2</u>: Let F be a contravariant set functor. Let $f: X \longrightarrow Y$ be a mapping, r, s F-structures on X, Y respectively, $U \subset X$. Then

$$F(j_U)(F(f)(s)) \subset F(j_U)(r) \qquad (*)$$

if and only if

$$P^- F(f)([P^- F(j_U)(P^- F(U))] \cap r^*) \subset s^* . \qquad (**)$$

<u>Proof:</u> Let $(**)$ do not hold. Then there exists

$\mathfrak{a} \in P^{-}F(f)([P^{-}F(j_U)(P^{-}F(U))] \cap \mathfrak{r}^{*})$ such that $\mathfrak{a} \cap \mathfrak{s} \neq \emptyset$. Let $P^{-}F(f)(\mathfrak{b}) = (F(f))^{-1}(\mathfrak{b})$, $\mathfrak{b} \in [P^{-}F(j_U)(P^{-}F(U))] \cap \mathfrak{r}^{*}$, i.e., $\mathfrak{b} \cap \mathfrak{r} = \emptyset$ and $\mathfrak{b} = (F(j_U))^{-1}(\mathfrak{c})$ where $\mathfrak{c} \subset F(U)$. Then $\mathfrak{c} \cap$ $\cap F(j_U)(\mathfrak{r}) = \emptyset$. If $\xi \in \mathfrak{a} \cap \mathfrak{s}$ then $F(f)(\xi) \in \mathfrak{b}$, $F(j_U)(F(f)(\xi)) \in \mathfrak{c}$. Then $F(j_U)(F(f)(\xi)) \notin F(j_U)(\mathfrak{r})$, i.e. $(*)$ does not hold.

Let $(*)$ do not hold. Then there exists $\varkappa \in \mathfrak{s}$ such that $\xi = F(j_U)(F(f)(\varkappa)) \notin F(j_U)(\mathfrak{r})$. Consequently $(F(j_U))^{-1}(\xi) \cap \mathfrak{r} = \emptyset$, i.e. $(F(j_U))^{-1}(\xi) \in \mathfrak{r}^{*}$. One-point set (ξ) is an element of $P^{-}F(U)$ and therefore $\mathfrak{a} \in [P^{-}F(j_U)(P^{-}F(U))] \cap \mathfrak{r}^{*}$. Using $(**)$ there is $P^{-}F(f)(\mathfrak{a}) \in \mathfrak{s}^{*}$, i.e. $(F(f))^{-1}(\mathfrak{a}) \cap \mathfrak{s} = \emptyset$. But $\varkappa \in (F(f))^{-1}(\mathfrak{a}) \cap \mathfrak{s}$.

Definition 3. Let F be a covariant (contravariant) set functor, r,s F-structures on topological spaces X,Y respectively. We say that a mapping f: X $\longrightarrow$ Y is quasilocally r s-compactible if:
for every non-void open $U \subset X$ there is a non-void open $V \subset U$ such that

$$F(f)([F(j_V)(F(V))] \cap \mathfrak{r}) \subset \mathfrak{s}$$

$$(F(j_V)(F(f)(\mathfrak{s})) \subset F(j_V)(\mathfrak{r}))\,.$$

Theorem 1. Let F be a contravariant set functor. Let $f: X \longrightarrow Y$ be a mapping, r, s F-structures on topological spaces X, Y respectively. f is locally (quasi-locally) $r\,s$-compactible if and only if it is locally (quasilocally) $r^*\,s^*$-compactible.

Proof: This is an easy consequence of Lemma 2.

Definition 4. Let X, Y be topological spaces. A mapping $f: X \longrightarrow Y$ is termed semi-open if $f(A)$ has a non-void interior whenever A has a non-void interior.

Remark: Every mapping f such that for every open non-void U there is a non-void $V \subset U$ with $f(V)$ open, is, of course, semi-open.

Definition 5. The category $T(F_1, \dots, F_n \mid G_1, \dots, G_m)$ is defined as follows: The objects are systems $(X, \tau, r_1, \dots, r_n, r_1', \dots, r_m')$ where τ is a topology on X, r_i F_i-structures and r_i' G_i-structures, the morphisms from $(X, \tau, r_1, \dots, r_n, r_1', \dots, r_m')$ into $(Y, \sigma, s_1, \dots, s_n, s_1', \dots, s_m')$ are the continuous mappings from (X, τ) into (Y, σ) which are $r_i\,s_i$-compactible for $i = 1, \dots, n$ and locally $r_i'\,s_i'$-compactible for $i = 1, \dots, m$.

The category $T_q(F_1, \dots, F_n \mid G_1, \dots, G_m \mid H_1, \dots, H_p)$ is defined as follows: The objects are systems $(X, \tau, r_1, \dots, r_n, r_1', \dots, r_m', r_1'', \dots, r_p'')$ where τ is a topology on X, r_i F_i-structures, r_i' G_i-structures and r_i'' H_i-structures on X. The morphisms from $(X, \tau, r_1, \dots, r_n, r_1', \dots, r_m', r_1'', \dots, r_p'')$ into $(Y, \sigma, s_1, \dots, s_n, s_1', \dots, s_m', s_1'', \dots, s_p'')$ are the continuous semi-open mappings from (X,τ) into (Y,σ) which are $r_i\,s_i$ -compactible for $i = 1,\dots,n$, locally $r_i'\,s_i'$ -compactible for $i = 1,\dots,m$ and quasilocally $r_i''\,s_i''$ -compactible for $i = 1,\dots,p$.

II.

<u>Definition 6</u>. Let F be a covariant set functor, r an F-structure on a topological space X. Define a $P^- F$-structure $\bar{r}$ and $\tilde{r}$ on X as follows:

$a \in \bar{r}$ if and only if for every $x \in X$ there exists an open $U \ni x$ with $a \cap r \cap [F(j_U)(F(U))] = \emptyset$

$a \in \tilde{r}$ if and only if for every non-void open U there exists a non-void open $V \subset U$ with $a \cap r \cap$ $\cap\,[F(j_U)(F(U))] = \emptyset$.

<u>Lemma 3</u>: A continuous mapping $f\colon X \longrightarrow Y$ is locally $r\,s$-compactible if and only if it is $\bar{r}\,\bar{s}$-compactible.

<u>Proof</u>: We must prove that for a continuous f the following statements are equivalent:

I. $\forall x \in X \;\exists\, U \ni x$ open with
$F(f)([F(j_U)(F(U))] \cap r) \subset s$

II. $P^- F(f)(\bar{s}) \subset \bar{r}$

Let I. hold. Let $a \in \bar{s}$. We must prove $P^- F(f)(a) = (F(f))^{-1}(a) \in \bar{r}$. Let $x \in X$; then there exists $U_1 \ni x$ open such that $F(f)([F(j_{U_1})(F(U_1))] \cap r) \subset s$. Since $a \in \bar{s}$ there exists $V \ni f(x)$ open with $[F(j_V)(F(V))] \cap s \cap a = \emptyset$. Put $U = U_1 \cap f^{-1}(V)$. Then $U \ni x$ is open and it is sufficient to prove

(1) $\quad (F(f))^{-1}(a) \cap [F(j_U)(F(U))] \cap \kappa = \emptyset .$

Since $\quad j_U = j_{U_1} \circ j_{U,U_1}$ there is $F(j_U)(F(U)) \subset$

$\subset F(j_{U_1})(F(j_{U,U_1})(F(U)) \subset F(j_{U_1})(F(U_1))$, consequently $F(f)([F(j_U)(F(U))] \cap \kappa) \subset F(f)([F(j_{U_1})(F(U_1))] \cap \kappa) \subset \mathscr{s}$.

If (1) does not hold then there exists $\eta \in F(U)$ with $\xi = F(j_U)(\eta) \in \kappa \cap (F(f))^{-1}(a)$, i.e.

$F(f)(\xi) \in a$, $F(f)(\xi) \in \mathscr{s}$. There is $f \circ j_U = j_V \circ g$ for a convenient g , consequently $F(f) \circ F(j_U) = F(j_V) \circ F(g)$.

Then $F(f)(\xi) = F(f)(F(j_U)(\eta)) = F(j_V)(F(g)(\eta)) \in F(j_V)(F(V))$. Consequently, $F(f)(\xi) \in [F(j_V)(F(V))] \cap \mathscr{s} \cap a$ which is a contradiction.

Let II. hold. Choose $x \in X$. Since $F(Y) \setminus \mathscr{s} \in \bar{\mathscr{s}}$ there is $(F(f))^{-1}(F(Y) \setminus \mathscr{s}) \in \bar{\kappa}$. Consequently there exists $U \ni x$ open with $[(F(f))^{-1}(F(Y) \setminus \mathscr{s})] \cap$

$\cap \kappa \cap [F(j_U)(F(U))] = \emptyset$. If $\xi \in [F(j_U)(F(U))] \cap$ $\cap \kappa$ then $\xi \notin (F(f))^{-1}(F(Y) \setminus \mathscr{s})$, i.e.

$F(f)(\xi) \notin F(Y) \setminus \mathscr{s}$ i.e. $F(f)(\xi) \in \mathscr{s}$.

<u>Lemma 4</u>. A continuous semi-open mapping $f: X \to Y$ is quasilocally r s-compactible if and only if it is $\tilde{r}$ $\tilde{s}$-compactible.

<u>Proof</u>: We must prove that the following statements are equivalent:

I. For every non-void open $U \subset X$ there exists a non-void open $V \subset U$ with $F(f)([F(j_V)(F(V))] \cap \kappa) \subset \mathscr{s}$.

II. $P^{-}F(f)(\tilde{\mathfrak{s}}) \subset \tilde{\mathfrak{r}}$

Let I. hold. Let $a \in \tilde{\mathfrak{s}}$. We must prove $P^{-}F(f)(a) = (F(f))^{-1}(a) \in \tilde{\mathfrak{r}}$ Let U be open non-void in X. There exists non-void open $U_1 \subset U$ with $F(f)([F(j_{U_1})(F(U_1))] \cap \mathfrak{r}) \subset \mathfrak{s}$. f is semi-open, therefore there exists non-void open $W \subset f(U_1)$. Since $a \in \tilde{\mathfrak{s}}$, there exists non-void open $W_1 \subset W$ with $[F(j_{W_1})(F(W_1))] \cap a \cap \mathfrak{s} = \emptyset$. f is continuous, $V = U_1 \cap f^{-1}(W_1)$ is non-void open, $V \subset U$. Now it is sufficient to prove:

(1) $$[(F(f))^{-1}(a)] \cap [F(j_V)(F(V))] \cap \mathfrak{r} = \emptyset$$

Since $j_V = j_{U_1} \circ j_{V,U_1}$ there is $F(j_V)(F(V)) = F(j_{U_1})(F(j_{V,U_1})(F(V))) \subset F(j_{U_1})(F(U_1))$, consequently,

(2) $$F(f)([F(j_V)(F(V))] \cap \mathfrak{r}) \subset \mathfrak{s}.$$

Let (1) do not hold. There exists $\xi \in [F(j_V)(F(V))] \cap \mathfrak{r}$ such that $F(f)(\xi) \in a$. Then, using (2), there is $F(f)(\xi) \in a \cap \mathfrak{s}$. Since $f(V) \subset W_1$ there is $j_{W_1} \circ g = f \circ j_V$ for convenient g. Put $\xi = F(j_V)(\eta)$. Then $F(f)(\xi) = F(f)(F(j_V)(\eta)) = F(j_{W_1})(F(g)) \in F(j_{W_1})(F(W_1))$ which is a contradiction.

Let II. hold. Choose U non-void open in X; $F(Y) \setminus \mathfrak{s} \in \tilde{\mathfrak{s}}$, consequently $(F(f))^{-1}(F(Y) \setminus \mathfrak{s}) \in \tilde{\mathfrak{r}}$. Then there exists non-void open $V \subset U$ with

$[(F(f))^{-1}(F(Y)\setminus s)] \cap r \cap [F(j_V)(F(V))] = \emptyset$.

If $\xi \in [F(j_V)(F(V))] \cap r$ then $\xi \notin$ $\notin (F(f))^{-1}(F(Y)\setminus s)$ consequently $F(f)(\xi) \notin F(Y)\setminus$ $\setminus s$, i.e. $F(f)(\xi) \in s$.

As an easy consequence of Lemmas 1,3 and 4 we obtain the following two theorems:

Theorem 2. $T(F_1,\dots,F_n \mid G_1,\dots,G_m) \rightrightarrows S(P^-, F_1,\dots,F_n, P^-G_1',\dots,P^-G_m')$ where $G_i' = G_i$ for covariant G_i, $G_i' = P^-G_i$ otherwise.

Theorem 3. $T_q(F_1,\dots,F_n \mid G_1,\dots,G_m \mid H_1,\dots,H_p) \rightrightarrows$

$\rightrightarrows S(P^-, P^+, F_1,\dots,F_n, P^-G_1',\dots,P^-G_m', P^-H_1,\dots,P^-H_p)$, where G_i' and H_i' are defined as in the previous theorem.

Remark: Here and in the following, the topology is always represented by the set of all open subsets.

Corollary: If $F_1,\dots,F_n, G_1,\dots,G_m, H_1,\dots,H_p$ are constructive functors (see [4]), then there exists an integer k and a set A such that

$$T(F_1,\dots,F_n \mid G_1,\dots,G_m) \rightrightarrows S((P^-)^k \circ V_A)$$

$$T_q(F_1,\dots,F_n \mid G_1,\dots,G_m \mid H_1,\dots,H_p) \rightrightarrows S((P^-)^k \circ V_A)$$

Consequently the categories $T(F_1,\dots,F_n \mid G_1,\dots,G_m)$ and $T_q(F_1,\dots,F_n \mid G_1,\dots,G_m \mid H_1,\dots,H_p)$ are in such a case boundable.

III.

Theorem 4. The following categories $\mathcal{K}_1, \ldots, \mathcal{K}_5$ are boundable: objects are always topological spaces, the morphisms are

in $\mathcal{K}_1$: locally one-to-one continuous mappings,

in $\mathcal{K}_2$: open local homeomorphisms,

in $\mathcal{K}_3$: local homeomorphisms,

in $\mathcal{K}_4$: quasiopen continuous mappings (i.e. continuous which are open with the exception of a nowhere dense set),

in $\mathcal{K}_5$: quasilocal homeomorphisms (i.e. continuous mappings which are local homeomorphisms with the exception of a nowhere dense set).

Remark: The reader sees certainly other variations.

Proof: It suffices to represent quoted categories as full subcategories of the categories from Theorems 2 and 3.

$\mathcal{K}_1 \rightrightarrows T(- \mid Q)$ (endow the object by the Q-structure $\{(x, y) \mid x \neq y\}$)

$\mathcal{K}_2 \rightrightarrows T(P^+ \mid Q)$ (the P^+-structures are repeated topology, Q-structures as above)

$\mathcal{K}_3 \rightrightarrows T(- \mid Q, Q \circ P^+)$ (the Q-structure as above; if (X, τ) is a topological space, define a $Q \circ P^+$-structure r on X as follows:
$(A, (a)) \in r \Longleftrightarrow a \notin \bar{A}$)

$\mathcal{K}_4 \rightrightarrows T_{\mathfrak{q}}(- \mid - \mid P^+)$ (the P^+-structures are repeated topologies)

$\mathcal{K}_5 \rightrightarrows T_{\mathfrak{q}}(- \mid - \mid Q, Q \circ P^+)$ analogically as with $\mathcal{K}_3$.

Lemma 5. Let (X, ρ) be a metric space. Define a $P_R \circ Q$ -structure (R is the set of real numbers) $\bar{\rho}$ as follows:

$$\varphi \in \bar{\rho} \iff \exists k \in R \quad \text{with } \varphi(x,y) \leqq k\rho(x,y)$$

Let $(X, \rho), (X', \rho')$ be metric spaces. A mapping f: : $X \to X'$ is Lipschitzian if and only if it is $\bar{\rho}\bar{\rho}'$ -compactible.

Proof: I. Let $f : (X, \rho) \to (X', \rho')$ be Lipschitzian. Then $\rho'(f(x); f(y)) \leqq K \cdot \rho(x, y)$ for every $x, y \in X$. Let $\varphi \in \bar{\rho}'$; then for every $u, v \in X'$ is $\varphi(u, v) \leqq k_1 \cdot \sigma(u, v)$ for convenient k_1. Consequently $[((P_R \circ Q)(f))(\varphi)](x, y) =$

$$= [\varphi \circ (f \times f)](x,y) = \varphi(f(x), f(y) \leqq k_1 \cdot \sigma(f(x), f(y) \leqq$$

$$\leqq K \cdot k_1 \cdot \rho(x, y),$$

i.e. $P_R \circ Q(f)(\varphi) \in \bar{\rho}$

II. Let $f : (X, \rho) \to (X', \rho')$ be $\bar{\rho}\bar{\rho}'$ -compactible. Since $\sigma \in \bar{\sigma}$ there is $\sigma \circ (f \times f) \in \bar{\rho}$.

Theorem 5. The categories of metric spaces with Lipschitzian and locally Lipschitzian mappings are boundable.

Proof: First of them is by lemma 5 realizable in $S(P_R \circ Q)$. The second one, considering of course the induced topologies, is realizable in $T(- \mid P_R \circ Q)$. It is only necessary to verify that the localization from Definition 1 (for F contravariant) really corresponds to the usual localization of the Lipschitz condition. If (X, ρ) is a metric space, (U, ρ_1) its subspace, then $\bar{\rho}_1 = P_R \circ Q(j_U)(\bar{\rho})$. For, if $\varphi \in \bar{\rho}$ then $P_R \circ Q(j_U)(\varphi) =$

$= \varphi \circ (j_U \times j_U) \in \bar{\rho}_1$; if $\psi \in \bar{\rho}_1$ define $\varphi(x, y) = \psi(x, y)$ for $x, y \in U$, $\varphi(x, y) = 0$ otherwise; then $\varphi \in \bar{\rho}$ and $P_R \circ Q(j_U)(\varphi) = \psi$. A mapping $f : (X, \rho) \to (X', \rho')$ is, by Definition 1, locally $\bar{\rho}\,\bar{\rho}'$-contractible if and only if for every $x \in X$ there exists $U \ni x$ open with $((P_R \circ Q(j_U)) \circ (P_R \circ Q(f))(\bar{\rho}') \subset P_R \circ Q(j_U)(\bar{\rho}) = \bar{\rho}_1$, i.e. $\varphi \circ (f\, j_U \times f\, j_U) \in \bar{\rho}_1$ whenever $\varphi \in \bar{\rho}'$. But it holds if and only if $f\, j_U$ is Lipschitzian.

R e f e r e n c e s :

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