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SOME REMARKS ON TENSOR PRODUCTS

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§ 1. A locally convex topology in $E \otimes F$.

Let E and F be two topological vector spaces over the field of real numbers. We shall define in the tensor product $E \otimes F$ a topology, which may be identified with the projective tensor topology (see [2]) in case, when E and F are locally convex spaces. We denote for a subset A of E, B subset of F, by $A \otimes B$ the set of all $x \otimes y \in E \otimes F$, where x is in A, y in B.

For any neighborhood $\mathcal{U}$ of zero element in E, $\mathcal{V}$ neighborhood of zero element in F and for any positive integer n we set

(1) $K^n(\mathcal{U}, \mathcal{V}) = 2^{-n}(\mathcal{U} \otimes \mathcal{V} + \ldots + \mathcal{U} \otimes \mathcal{V})$ (2^n summands on the right side)

(2) $\Omega_{\mathcal{U},\mathcal{V}} = \bigcup_{n=1}^{\infty} K^n(\mathcal{U}, \mathcal{V})$.

The system of all $\Omega_{\mathcal{U},\mathcal{V}}$, where $\mathcal{U}$ varies in the neighborhood system U of zero element in E, $\mathcal{V}$ in neighborhood system V in F, defines a topology on the tensor product $E \otimes F$. This topology is called in following discussion $\mathcal{G}$-topology. It suffices to prove the relation $\Omega_{\mathcal{U},\mathcal{V}} + \Omega_{\mathcal{U},\mathcal{V}} \subseteq \Omega_{\mathcal{W},\mathcal{V}}$, where $\mathcal{W} \in U$, $\mathcal{U} \in U$, $\mathcal{V} \in V$, $\mathcal{U} + \mathcal{U} \subseteq \mathcal{W}$. The proof of the last statement is obvious. $\mathcal{G}$-topology in $E \otimes F$ is locally convex. In order to prove this fact it suffices to show the equality (see [3]) $\frac{1}{2}(\Omega_{\mathcal{U},\mathcal{V}} + \Omega_{\mathcal{U},\mathcal{V}}) = \Omega_{\mathcal{U},\mathcal{V}}$. This follows immediately from the definition (2).

The fundamental system of locally convex neighborhoods in $E \otimes F$ is formed by the collection of all interiors $\Omega^{\circ}_{\mathcal{U},\mathcal{V}}$ of $\Omega_{\mathcal{U},\mathcal{V}}$. The geometric significance of the neighborhoods $\Omega_{\mathcal{U},\mathcal{V}}$ is clear: if we denote by $co(\mathcal{U} \otimes \mathcal{V})$ the convex hull of $\mathcal{U} \otimes \mathcal{V}$ in $E \otimes F$, then $\Omega_{\mathcal{U},\mathcal{V}}$ containing the interior of $co(\mathcal{U} \otimes \mathcal{V})$ is contained in $co(\mathcal{U} \otimes \mathcal{V})$. It follows at once that the closure of $\Omega_{\mathcal{U},\mathcal{V}}$ is equal to the closed convex hull of $\mathcal{U} \otimes \mathcal{V}$ in $E \otimes F$. If E and F are locally convex spaces, then the equivalence of the $\mathcal{G}$-topology with the projective tensor topology (see [2]) follows from the inclusions:

$$co(\mathcal{U} \otimes \mathcal{V}) \subseteq \Omega^{\circ}_{\mathcal{W},\mathcal{V}} \subseteq \Omega_{\mathcal{W},\mathcal{V}} \subseteq co(\mathcal{W} \otimes \mathcal{V}),$$

where $\mathcal{U} + \mathcal{U} \subseteq \mathcal{W}$.

We may define the $\mathcal{G}$-topology in $E \otimes F$ in the following manner, too; we set for any neighborhood $\mathcal{U}$ of 0 in E, $\mathcal{V}$ in F and any positive integer n:

(1′) $K'^{n}(\mathcal{U},\mathcal{V}) = \frac{1}{n}(\mathcal{U} \otimes \mathcal{V} + \dots + \mathcal{U} \otimes \mathcal{V})$ (n summands on the right side)

(2′) $\Omega'_{\mathcal{U},\mathcal{V}} = \bigcup_{n=1}^{\infty} K'^{n}(\mathcal{U},\mathcal{V})$.

It is clear that for any neighborhood $\mathcal{U}$ in E, $\mathcal{V}$ in holds

$$\Omega_{\mathcal{U},\mathcal{V}} \subseteq \Omega'_{\mathcal{U},\mathcal{V}} \subseteq co(\mathcal{U} \otimes \mathcal{V}).$$

The last definition of the $\mathcal{G}$-topology can be acceptably generalized for tensor products of Abelian groups (see [6]).

From (2) it follows at once that the natural bilinear mapping $(x,y) \to x \otimes y$ of $E \times F$ in $E \otimes F$ is continuous.

Theorem 1. Let E and F be two topological vector spaces. There exists a unique locally convex topology on the tensor product $E \otimes F$ having the following properties:

(a) the natural mapping $(x,y) \to x \otimes y$ of $E \times F$ in $E \otimes F$

is continuous on $E \times F$.

(b) If G is a locally convex vector space, $f(x, y)$ a bilinear continuous mapping of $E \times F$ in G, the associatdd linear mapping f^*, defined by the algebraic isomorphism of the space $\mathcal{L}(E, F; G)$ of all bilinear mappings $E \times F \rightarrow G$ onto the space $\mathcal{L}(E \otimes F; G)$ of all linear mappings $E \otimes F \rightarrow G$, is continuous on $E \otimes F$.

Proof. The $\mathcal{G}$-topology has properties (a) and (b). Indeed, (a) was established above, (b) follows from the fact that $f(\mathcal{U}, \mathcal{V}) \subseteq \mathcal{W}$, where $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are neighborhoods of 0 in E, F, G, implies $f^*(\Omega_{\mathcal{U},\mathcal{V}}) \subseteq \mathcal{W}$. The uniqueness of a topology having properties (a) and (b) is clear.

If E is a topological vector space, τ a topology in E, then there exists a locally convex topology in E coarser than τ. For example the topology having as neighborhood of 0 in E the set E only (the coarsest topology in E). The supremum of all locally convex topologies in E coarser than τ is a locally convex topology in E coarser than τ. We denote this topology by τ^*. It is well-known that a linear function f is continuous on (E, τ) if and only if f is continuous on (E, τ^*). The topological dual of E with the topology τ will be denoted by E', $\sigma(E, E')$ means the weak topology in E defined by E', τ_0 a topology in E having as neighborhood the set E only.

The following statements are equivalent:

(a) E' contains an element different from zero-element,

(b) τ^* is larger than τ_0,

(c) $\sigma(E, E')$ is larger than τ_0.

Proof. If $0 \neq f \in E'$, then $\sigma(E, E')$ is larger than τ_0, hence τ^* is larger than τ_0. It suffices to prove (b) implies (a). If (b) is satisfied, the factor space $E/\bar{0}$, where $\bar{0}$ is closure of zero-element in (E, τ^*) is separated locally convex space containing an element different from zero. Clearly E' has the property (a).

The topology τ^* associated to τ in E has as fundamental system of neighborhoods the collection of all $co\,\mathcal{U}$, where $co\,\mathcal{U}$ is the convex hull of $\mathcal{U}$ in E, $\mathcal{U}$ in U.

Proposition 1. Let E and F be two topological vector spaces, τ_1, τ_2 topologies on E and F. Then the $\mathcal{G}$-topology in $E \otimes F$ defined by τ_1 and τ_2 is identical with the $\mathcal{G}$-topology in $E \otimes F$ defined by τ_1^* and τ_2^*.

Proof. In order to prove this proposition, it suffices to prove the inclusions $\Omega_{\mathcal{U},\mathcal{V}} \subseteq \Omega_{co\,\mathcal{U},co\,\mathcal{V}} \subseteq co\,(\mathcal{U} \otimes \mathcal{V})$, where $co\,A$ denotes the convex hull of A. It is evident that $\Omega_{\mathcal{U},\mathcal{V}} \subseteq \Omega_{co\,\mathcal{U},co\,\mathcal{V}}$. Let $z \in \Omega'_{co\,\mathcal{U},co\,\mathcal{V}}$; there exist x_i in $co\,\mathcal{U}$ $(1 \leq i \leq n)$ and y_i in $co\,\mathcal{V}$ $(1 \leq i \leq n)$ such that

$$z = \frac{1}{n}(x_1 \otimes y_1 + \dots + x_n \otimes y_n).$$

As x_i is in $co\,\mathcal{U}$ $(1 \leq i \leq n)$, we may assume that x_i is of the form $x_i = \lambda_1^i x_1^i + \dots + \lambda_{n_i}^i x_{n_i}^i$ where $\sum_{k=1}^{n_i} \lambda_k^i = 1$, $\lambda_k^i \geq 0$ $(1 \leq k \leq n_i,\ 1 \leq i \leq n)$ and $x_r^i \in \mathcal{U}$ $(1 \leq r \leq n_i,\ 1 \leq i \leq n)$. Similarly $y_i = \mu_1^i y_1^i + \dots + \mu_{m_i}^i y_{m_i}^i$, where $\sum_{j=1}^{m_i} \mu_j^i = 1$, $\mu_j^i \geq 0$ $(1 \leq j \leq m_i,\ 1 \leq i \leq n)$ and $y_s^i \in \mathcal{V}$ $(1 \leq s \leq m_i,\ 1 \leq i \leq n)$. The rest of the proof follows from the fact that

$$z = \frac{1}{n}\left(\sum_{k=1}^{n_1}\sum_{j=1}^{m_1} \lambda_k^1 \mu_j^1 (x_k^1 \otimes y_j^1) + \dots\right) \text{ is in } co\,(\mathcal{U} \otimes \mathcal{V}).$$

From the discussion it follows that $\mathcal{G}$-topology in tensor product $E \otimes F$ of two topological vector spaces (E, τ_1), (F, τ_2) is larger than the coarsest topology in $E \otimes F$ if and only if τ_1^* is larger than the coarsest topology in

E and τ_2^* is larger than the coarsest topology in F.

Similarly $\mathcal{G}$-topology in $E \otimes F$ is separated if and only if τ_1^* and τ_2^* are separated topologies in E and F respectively.

§ 2. W-topology in $E \otimes F$.

We assume as in § 1 E and F to be topological vector spaces, U the system of all neighborhoods of 0 in E, V the system of all neighborhoods of 0 in F. For any sequence $(\mathcal{U}_i,\ i = 1, 2, \dots)$, $\mathcal{U}_i \in U$, and for any sequence $(\mathcal{V}_i,\ i = 1, 2, \dots)$, $\mathcal{V}_i \in V$, we define

$$(3)\quad \Omega_{(\mathcal{U}_i),(\mathcal{V}_i)} = \{x \in E \otimes F;\ x \in \sum_i{}^* \mathcal{U}_i \otimes \mathcal{V}_i\},$$

where $\sum_i{}^* \mathcal{U}_i \otimes \mathcal{V}_i$ means the set of all $x_1 \otimes y_1 + \dots + x_n \otimes y_n$, $x_i \in \mathcal{U}_i\ (1 \le i \le n)$, $y_i \in \mathcal{V}_i\ (1 \le i \le n)$, n arbitrary integer. If we choose $\mathcal{U}'_k \in U$, $\mathcal{V}'_k \in V$ $(k = 1, 2, \dots)$ satisfying

$$\mathcal{U}'_k \subseteq \mathcal{U}_{2k-1} \cap \mathcal{U}_{2k},\ \mathcal{V}'_k \subseteq \mathcal{V}_{2k-1} \cap \mathcal{V}_{2k}\ (k = 1, 2, \dots), \text{then}$$

$$\Omega_{(\mathcal{U}'_k),(\mathcal{V}'_k)} + \Omega_{(\mathcal{U}'_k),(\mathcal{V}'_k)} \subseteq \Omega_{(\mathcal{U}_k),(\mathcal{V}_k)}.$$

That proves that the collection of all sets of the form (3) defines a topology in $E \otimes F$ compatible with the structure of a vector space (see [7]). This topology is called in further discussion a W-topology.

The natural bilinear mapping $(x, y) \to x \otimes y$ of $E \times F$ in $E \otimes F$ is continuous on $E \times F$.

<u>Lemma</u>. Let G be a topological vector space, $\mathcal{W}$ a neighborhood of zero element in G. We choose a neighborhood $\mathcal{W}_1$ satisfying $\mathcal{W}_1 + \mathcal{W}_1 \subseteq \mathcal{W}$. If $\mathcal{W}_i$ $(1 \le i \le k-1)$ are defined, we choose a neighborhood $\mathcal{W}_k$ of 0 such that $\mathcal{W}_k + \mathcal{W}_k \subseteq \mathcal{W}_{k-1}$. Then for any k holds

$$\mathcal{W}_1 + \mathcal{W}_2 + \dots + \mathcal{W}_k + \mathcal{W}_k \subseteq \mathcal{W} \quad ((k+1) \text{ summands on the right side}).$$

The proof can be carried out easily by induction.

Theorem 2. If f is a continuous bilinear mapping of $E \times F$ in a topological vector space G, then the associated linear mapping f^* of $E \otimes F$ in G defined by

$$(4) \qquad f^*(x \otimes y) = f(x, y)$$

is continuous on $E \otimes F$. The correspondence $f \longleftrightarrow f^*$ defines an isomorphism of the space $\mathcal{B}(E, F; G)$ of all bilinear continuous mappings $E \times F \to G$ onto the space $\mathcal{L}(E \otimes F; G)$ of all linear continuous mappings $E \otimes F \to G$.

Proof. If $f \in \mathcal{B}(E, F; G)$, $\mathcal{W}$ any neighborhood of zero element in G, we choose $\mathcal{W}_i$ $(i = 1, 2, \dots)$ as in precedent lemma. For suitable neighborhoods $\mathcal{U}_i$, $\mathcal{V}_i$ $(i = 1, 2, \dots)$ of zero element in E and F respectively we have $f(\mathcal{U}_i, \mathcal{V}_i) \subseteq \subseteq \mathcal{W}_i$ $(i = 1, 2, \dots)$. The continuity of f^* follows from $f^*(\Omega_{(\mathcal{U}_i),(\mathcal{V}_i)}) \subseteq \mathcal{W}$. For any $x \in \Omega_{(\mathcal{U}_i),(\mathcal{V}_i)}$ there exist x_i in $\mathcal{U}_i$ $(1 \leq i \leq n)$, y_i in $\mathcal{V}_i$ $(1 \leq i \leq n)$ such that $x = x_1 \otimes y_1 + \dots + x_n \otimes y_n$. From $f^*(x) = \sum_{i=1}^{n} f(x_i, y_i) \in \mathcal{W}_1 + \mathcal{W}_2 + \dots + \mathcal{W}_n \subseteq \mathcal{W}$ we derive $f^*(\Omega_{(\mathcal{U}_i),(\mathcal{V}_i)}) \subseteq \mathcal{W}$. This concludes the proof.

Consequence. On the tensor product $E \otimes F$ W-topology is the unique topology compatible with the structure of a vector space satisfying following conditions:

(a) The natural bilinear mapping $(x, y) \to x \otimes y$ of $E \times F$ in $E \otimes F$ is continuous.

(b) If $f(x, y)$ is a bilinear continuous mapping of $E \times F$ in a topological vector space G, then the linear mapping f^* associated to f is continuous on $E \otimes F$.

The proof is evident.

If E and F are two finite dimensional separated topological spaces, then $\mathcal{G}$-topology on $E \otimes F$ is identical with the W-topology on $E \otimes F$. This follows from ([7], theorem 26). Making use of theorem 2 we may conclude that W-topology on $E \otimes F$ is larger or equal to $\mathcal{G}$-topology on $E \otimes F$. In general these topologies are different.

Example. Let $(X, \mathfrak{M}, \mu)$ be a measure space, $\mathfrak{M}$ a σ-algebra of subsets in X, μ a finite, atomic-free measure on $\mathfrak{M}$. By $\mathcal{S} = \mathcal{S}(X, \mathfrak{M}, \mu)$ we denote the space of all almost everywhere finite measurable functions on X with respect to $\mathfrak{M}$. We may define in $\mathcal{S}$ a topology τ by a metric ρ:

$$\rho(f, g) = \int \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, d\mu \, .$$

In $\mathcal{S}$ there exists a unique open convex set, hence the associated locally convex topology τ^* is the coarsest topology in $\mathcal{S}$. For every k there exists a decomposition $(X_i)_{i \in J}$, J finite, $X_i \cap X_j = \emptyset$ for $i \in J, j \in J, i \neq j$, $X_i \in \mathfrak{M} \, (i \in J)$, $X = \bigcup_{i \in J} X_i$ and $\mu(X_i) \leq \frac{1}{k}$ (see [5]). For any $f \in \mathcal{S}$ we set

$$f_i(x) = \begin{cases} f(x), & x \in X_i, \\ 0, & x \notin X_i, \quad i \in J. \end{cases}$$

Hence $f_i \in \mathcal{S} \, (i \in J)$, $f(x) = \frac{1}{n} \sum_{i \in J} n f_i(x)$, where n is the cardinal of J.

If we define a bilinear continuous mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ by $(f, g) \to f \cdot g$, we may conclude that $f \otimes g \to f \cdot g$ is a continuous mapping of $\mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ with respect to the W-topology. From the proposition 1 of § 1 it follows that linear mapping $f \otimes g \to f \cdot g$ is not continuous in $\mathcal{G}$-topology. This proves W-topology is larger than $\mathcal{G}$-topology in $\mathcal{S} \otimes \mathcal{S}$.

If E and F are E'- and F'-separated, W-topology in $E \otimes F$ is of course separated.

Proposition 2. Let E_i $(i = 1, 2)$, F_i $(i = 1, 2)$ be topological vector spaces, u_i $(i = 1, 2)$ a linear continuous mapping of E_i in F_i $(i = 1, 2)$. Then the linear mapping $u_1 \otimes u_2$ of $E_1 \otimes F_1$ in $E_2 \otimes F_2$ defined by

$$(5) \quad (u_1 \otimes u_2)(x \otimes y) = u_1(x) \otimes u_2(y),$$

where $x \in E_1$, $y \in F_1$, is continuous on $E_1 \otimes F_1$.
The proof is evident.

Proposition 3. If we denote E_i, F_i, u_i $(i = 1, 2)$ as in proposition 2, then $u_1 \otimes u_2$ is an open mapping whenever u_i $(i = 1, 2)$ are open.
Proof. It suffices to prove

$$(6) \quad (u_1 \otimes u_2)\,\Omega_{(\mathcal{U}_i),(\mathcal{V}_i)} \supseteq \Omega_{(u_1(\mathcal{U}_i)),(u_2(\mathcal{V}_i))},$$

where $\mathcal{U}_i \in U$, $\mathcal{V}_i \in V$ $(i = 1, 2, \dots)$.
For a given z' in $E_2 \otimes F_2$ and z' in $\Omega_{(u_1(\mathcal{U}_i)),(u_2(\mathcal{V}_i))}$ we may choose suitable $x_i' \in u_1(\mathcal{U}_i)$ $(1 \leq i \leq n)$, $y_i' \in u_2(\mathcal{V}_i)$ $(1 \leq i \leq n)$ satisfying $z' = x_1' \otimes y_1' + \dots + x_n' \otimes y_n'$.
There exist $x_i \in \mathcal{U}_i$ $(1 \leq i \leq n)$, $y_i \in \mathcal{V}_i$ $(1 \leq i \leq n)$, $u_1(x_i) = x_i'$ $(1 \leq i \leq n)$, $u_2(y_i) = y_i'$ $(1 \leq i \leq n)$. If we define $z = x_1 \otimes y_1 + \dots + x_n \otimes y_n$ we have $(u_1 \otimes u_2)(z) = z'$, $z \in \Omega_{(\mathcal{U}_i),(\mathcal{V}_i)}$.

Proposition 4. If E_i $(1 \leq i \leq n)$, F are topological vector spaces, then the tensor product $(\prod_{i=1}^{n} E_i) \otimes F$ is algebraic and topological isomorphic to $\prod_{i=1}^{n}(E_i \otimes F)$.
Proof. The isomorphism φ of $(\prod_{i=1}^{n} E_i) \otimes F$ on $\prod_{i=1}^{n}(E_i \otimes F)$ is given by (see [7])

$$\varphi((x_1 \otimes y), \dots, (x_n \otimes y)) = (x_1, \dots, x_n) \otimes y.$$

The continuity of φ^{-1} follows from the continuity of

$((x_1, \dots x_n), y) \to x_i \otimes y \ (1 \leq i \leq n)$. Similarly one proves the continuity of φ.

Proposition 5. Let E and F be two topological vector spaces, M subspace in E, N subspace in F. Then the factor space $E \otimes F / _{\Gamma(M,N)}$ is algebraic and topological isomorphic to $(E/_M) \otimes (F/_N)$, where $\Gamma(M, N)$ is a subspace in $E \otimes F$ generated by the set of all $x \otimes y$, x is in M or y is in N.

Proof. We denote by ω, φ, ψ the natural mapping of $E \otimes F$ in $E \otimes F / \Gamma(M,N)$, E in $E/_M$, F in $F/_N$. The natural isomorphism Φ of $E \otimes F / _{\Gamma(M,N)}$ on $(E/_M) \otimes (F/_N)$ is defined (see [1]) by

$$\Phi(\omega(x \otimes y)) = \varphi(x) \otimes \psi(y).$$

The continuity of Φ follows from the continuity of $x \otimes y \to \to \varphi(x) \otimes \psi(y)$ and $\Phi(\omega(\Gamma(M, N))) = 0$. From (6) we may conclude that $\Phi(\omega(\Omega_{(U_i),(V_i)})) \supseteq \Omega_{(\varphi(U_i)),(\psi(V_i))}$. This proves Φ is open.

§ 3. W-topology on the tensor product $G \otimes K$ of Abelian groups.

Let G and K be two Abelian topological groups written in the additive form, $G \otimes K$ their tensor product. Every Abelian group may be regarded as a module over the ring Z of all integers. By a Z-linear (Z-bilinear) mapping we mean a linear (bilinear) mapping of the module with respect to the ring Z.

Every element z of $G \otimes K$ is of the form

$$z = x_1 \otimes y_1 + \dots + x_n \otimes y_n,$$

where $x_i \in G \ (1 \leq i \leq n)$, $y_i \in K \ (1 \leq i \leq n)$ and n is a positive integer.

For any sequence $(\mathcal{U}_i\,,\ i=1,2,\dots)$ of neighborhoods of zero in G and for any sequence $(\mathcal{V}_i\,,\ i=1,2,\dots)$ of neighborhoods of zero in K we define (as in § 2)

$$(3') \quad \Omega_{(\mathcal{U}_i),(\mathcal{V}_i)} = \{x \in G \otimes K,\ x \in \textstyle\sum_i^* \mathcal{U}_i \otimes \mathcal{V}_i\},$$

where $\sum_i^* \mathcal{U}_i \otimes \mathcal{V}_i$ means the set of all $x_1 \otimes y_1 + \dots + x_n \otimes y_n$, $x_i \in \mathcal{U}_i\ (1 \leq i \leq n)$, $y_i \in \mathcal{V}_i\ (1 \leq i \leq n)$ and n is any integer.

The collection of all $\Omega_{(\mathcal{U}_i),(\mathcal{V}_i)}$ defines in $G \otimes K$ a topology compatible with the structure of a group $G \otimes K$. This topology is called W-topology in $G \otimes K$.
The proof of this statement is similar as in § 2. The natural Z-bilinear mapping $(x,y) \to x \otimes y$ of $G \times K \to G \otimes K$ is continuous in $(0,0)$. In general this mapping is not separated continuous on $G \times K$. For example we may consider G additive group of the real numbers with the usual topology, K a discrete group with basis $(e_i)_{i \in J}$. The mappings $x \to x \otimes e_i\ (i \in J)$ are not continuous on G.

<u>Theorem 3</u>. If f is a Z-bilinear mapping of $G \times K$ in a topological group H continuous in $(0,0)$, then the linear mapping f^* of $G \otimes K$ in H associated to f and defined by

$$(4') \qquad f^*(x \otimes y) = f(x,y)$$

is continuous on $G \otimes K$.
The proof is similar as in § 2.

<u>Consequence</u>. W-topology is the unique topology on $G \otimes K$ compatible with the structure of a group and having the following properties:
(a) The natural Z-bilinear mapping $(x,y) \to x \otimes y$ is continuous in $(0,0)$.
(b) If f is a Z-bilinear mapping of $G \times K$ in a topological group H continuous in $(0,0)$ then the associated

mapping f^* defined by $(4')$ is continuous on $G \otimes K$.

If G or K is a discrete group, then $G \otimes K$ is discrete. Hence, W-topology is equal to π-topology on $G \otimes K$ (see [6]), when $G \otimes K$ is torsion-free. From theorem 3 it follows that in general W-topology is larger or equal to π-topology.

<u>Example</u>. We denote by D the ring of all p-adic numbers. Any element x in D is of the form $x = (k_1, k_2, \dots k_n, \dots)$, where k_n is a positive integer, $k_{n+1} = k_n \pmod{p^n}$, $0 \leq k_n < p^n$. A neighborhood $U_n(0)$ of zero element in D is defined by (see [4])

$$U_n(0) = \{x \in D;\ k_1 = \dots = k_n = 0\},\ n = 1, 2, \dots .$$

For any $x \in D$, $y \in D$ we set $f(x, y) = x \cdot y$. It is clear that f is continuous in $\{0, 0\}$. The associated mapping f^* is continuous on $D \otimes D$ with the W-topology. Making use of the fact that π-topology on $D \otimes D$ has as a neighborhood the set $D \otimes D$ only (see [6]), we may conclude that f^* is not continuous on $D \otimes D$ with the π-topology. Hence, W-topology and π-topology are not identical.

If G and K are (b)-groups, (see [6]) (i.e. for any x in G, y in K and any neighborhood U of 0 in G, V of 0 in K, there exist positive integrs m, n satisfying $x \in mU$, $y \in nV$), then $(x, y) \to x \otimes y$ is continuous on $G \times K$ (see [6]) .

Similarly as in § 2 for W-topology in the tensor product $G \otimes K$ of two Abelian groups hold propositions 2,3,4 and 5 .

If G and K are two topological groups, then every character (see [4]) of $G \otimes K$ may be regarded as a Z-bilinear mapping of $G \times K$ in $R/_Z$, where $R/_Z$ is the additive group of real numbers modulo 1 continuous in $(0, 0)$ and conversely. Especially the natural Z-bilinear mapping $(x, \chi) \rightarrow \langle x, \chi \rangle = \chi(x)$ is a character of $G \otimes G^*$ where G is a locally compact group, G^* the group of all characters of G.

References:

[1] N. BOURBAKI, Algèbre multilinéaire, Paris, Hermann 1948.

[2] A. GROTHENDIECK, Produits tensoriels topologiques et espaces nucléaires, Memoirs, Amer.Math.Soc., 1955.

[3] J. von NEUMANN, On complete topological spaces, Trans. Amer.Math.Soc.37(1935), pp.1-20.

[4] L.S. PONTRJAGIN, Continuous groups, Moscow 1954 (Russian).

[5] S. SAKS, Addition to the note on some functionals, Trans. Amer.Math.Soc.53(1933), pp.965-970.

[6] S. TOMÁŠEK, On tensor products of Abelian groups, this Journal, pp.73-83.

[7] H. WHITNEY, Tensor products of Abelian groups, Duke Math. Journal, 4(1938), pp.495-528.