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A SURFACE IN A SPACE WITH PROJECTIVE CONNEXION

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Let us consider a 2-dimensional domain of parameters Ω. To every point $u \in \Omega$ let there correspond a 3-dimensional centre-projective space $P_3(u)$ with centre $A_0(u)$. Let γ be an arc connecting two points u^1, u^2 of the domain of parameters $\Omega (\gamma \in \Omega)$ and let there be given the homology $H(u^1 u^2 \gamma)$ between the local spaces $P_3(u^1)$, $P_3(u^2)$. Let a connexion be given by the equations $dA_i = \omega_i^j A_j$, $\omega_i^i = 0$, $\omega_i^j = \pi_{i\alpha}^j du^\alpha$ $(i, j = 0, 1, 2, 3;\ \alpha = 1, 2$. The König's variety P_{02}^3 defined in this way is called a surface π with projective connexion. It is possible to choose the coordinate system (reper) on the surface π in such a way that the connexion is given by the equation

$$dA_0 = \omega_0^0 A_0 + du A_1 + dv A_2 ; \quad dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \beta du A_2 + (1-h) dv A_3 ;$$

$$dA_2 = \omega_2^0 A_0 + \gamma dv A_1 + \omega_2^2 A_2 + (1+h) du A_3 ; \quad dA_3 = \omega_3^0 A_0 + \omega_3^1 A_1 + \omega_3^2 A_2 + \omega_3^3 A_3 ;$$

$$\omega_k^i = a_k^i du + b_k^i dv , \quad \omega_i^i = 0 , \quad [du\, dv] \neq 0 , \quad (i, k = 0, 1, 3) ;$$

$$\text{let} \quad a = a_0^0 - a_1^1 - a_2^2 + a_3^3 , \quad b = b_0^0 - b_1^1 - b_2^2 + b_3^3 .$$

Consider upon the surface π a curve c which has contact of the first order with the asymptotic $u = const$ at a point A_0. Consider the tangents to the asymptotics $v = const$ from the points on the curve c. They form a ruled surface. Then there exists such a quadric such that one system of its lines has a line-contact of second order with our ruled surface. This quadric we shall denote by $Qu(\nu)$ ($\nu + 1$ is the Smith-Meshke's invariant of the contact of the curve c with the asymptotic $u = const$). If we interchange the asymptotics we get a quadric $Qv(\nu)$. In the two parametric bundle of quadrics $Q(\nu, \lambda) = Q_v(\nu) + \lambda Q_u(\nu)$

there exists for $\lambda = \frac{h+1}{h-1}$ a singular quadric consisting of a tangent plane of the surface π at the point A_0 and a plane τ which does not contain the point A_0 (if $h \neq 0$, then the surface is not without torsion; if $h = 0$ then both planes pass through the point A_0). If we consider the characteristics of one-parametric ($u = const$ or $const$) systems of such planes τ of singular quadrics (which are not tangent planes) we have for fixed an invariant point $P(\nu)$ in τ, which is their point of intersection. The points $P(\nu)$ lie on a line n which passes through the point A_0 of the surface π - the first pseudo-normal of the surface with projective connexion. If we consider the dualisation $\tilde{P}^3_{0\tau}$ of the surface we have a dual quadric $\tilde{Q}$ to the quadric $Q(\nu, \frac{h+1}{h-1})$. The quadric $\tilde{Q}$ consists of two planes, the line of intersection of which is the first normal n_0. The normal n_0 and a pseudonormal n are generally different lines. A canonical plane is determined by these lines. The second pseudonormal is dual to the first one. The summit of the quadric $Q(\nu, \frac{h+1}{h-1}$ is a second normal of the surface π.

There are ∞^3 osculating quadrics Q_3 of the surface at the point A_0. Then $h = 0$, there exist exactly three curves passing through the point A_0 on the surface and having contact of the third order with a certain Q_3. But when $h \neq 0$, there is a differential equation of the second order $v'' + a_1 v' + a_2 (v')^2 + a_3 (v')^3 + a_4 = 0$, the solution of which are the curves of the surface π having contact of the third order with Q_3 at a point A_0. The osculating planes of these curves are the tangent planes of a cone of third degree. This cone has three singular tangent planes passing through one straight line. We call this line a normal $n_1(\xi, \eta)$ with respect to Q_3 (n_1 depends on two parameters only). If

$\xi = -\frac{1}{3} \frac{hu}{1+h}, \eta = \frac{1}{3} \frac{hv}{1-h}$ we obtain the normal n.

Among the ∞^3 quadrics Q_3 it is possible to find

∞^1 quadrics Q_1 in the following way. Consider a straight line p passing through the point A_0 of the surface π, which is not a line of tangent plane at the point A_0. Let R_1 be the surface formed by the tangents to the asymptotic curves $v = const$ from the points on the asymptotic curve $u = const$. And let R_2 be the corresponding surface, if we interchange the asymptotics. Let r_1 be a line on the surface R_1 containing the point A_0 and analogously r_2 on the surface R_2. The surfaces R_1, R_2 are clearly non-developable surfaces. Two straight lines p, r_1 (or p, r_2) determine a plane τ_1 (or τ_2). On the line r_1 (or r_2) there is a point P_1 (or P_2) which is a tangent point of the plane τ_1 (or τ_2) with the surface R_1 (or R_2). The straight line q which is determined by the two points P_1, P_2 is called the reciprocal line to p. The quadric Q_1 is that quadric Q_3 with respect to which p, q are polar lines. The equation of Q_1 is in the local coordinate system $x^0x^3 - x^1x^2 = k(x^3)^2$.

Among the normals $n_1(\xi, \eta)$ we have $n_1(0,0) = n_1$ which belongs to the quadric Q_1. The normals n and n_1 determine a plane. The intersection of this plane with the tangent plane is a tangent to the curve $a\,du - b\,dv = 0$ passing through the point A_0. The second normal $\tilde{n}_1$ is the polar line to n_1 with respect to $Q_1(x^3 = 0, hx^0 - bx^1 - ax^2 = 0)$. The normal n_1 is not a line of the plane determined by n and n_0. Let γ be a tangent curve to the curve $a\,du - b\,dv = 0$ at a point A_0 and let γ have a contact of third order with the osculating quadric Q_1 of the surface π. From the points on the curve γ consider tangents to the asymptotic lines; then we obtain two systems of ∞^1 quadrics G_1, G_2, which have contact of the second order with one of the considered line surface. The quadric G_1 has in a convenient coordinate system the equation

$$(1+h)x^1x^2 - x^0x^3 - \frac{b}{a}h(x^1)^2 - 3\frac{a}{b}x^2x^3 +$$
$$+ \left(b + 3\frac{b^2}{a^2} + \frac{1}{2}\cdot\frac{hv}{1+h} + \frac{1}{2}\,\frac{hn}{1+h}\,\frac{b}{a}\right)x^1x^3 = 0.$$

Now let us consider homologies conserving the surface element of the third order.

For a surface with projective connexion, there are two kinds of homologies of local space onto itself. In homologies of the first kind, the asymptotes are identical while in homologies of the second kind one asymptote corresponds to the other and vice versa. It can be shown that the homology of the first kind exists also in the case, when on the surface π the relation $h \neq 0$ holds. For the existence of the homology of the second kind, however, the condition $h = 0$ is necessary and sufficient. There are ∞^3 homologies of the second kind. Among them are ∞^1 perspective homologies, the summits of which necessarily lie on Darboux's tangents at a considered point.

If $h = 0$ on a surface π, then the singular quadric consists of two planes. One is a tangent plane of π at the point A_0 and the other τ_1 is a plane containing the point A_0. There exists an infinity of curves (determined by the equation $adu - bdv = 0$), such that the characteristics of the one-parametric system of planes τ_1 along one curve of the system are invariant lines, the f i r s t n o r m a l s n^0 of the surface with projective connexion without torsion. The s e c o n d n o r m al $\tilde{n}^0$ is a polar line to n^0 with respect to the main Lie's quadric $Q(0,1)$.

We shall call "pseudogeodetics" on a surface π the curves $v = \varphi(u)$ for which $\int \phi(u, v, du, dv)$ has an extremal value, if ϕ is an invariant differential form on the given surface π. If we take $\phi = adu \pm bdv$ it can be shown that the necessary and sufficient condition for $v = \varphi(u)$ to be a pseudogeodetic curve is that the relation $a_v \mp b_u = 0$ should hold. In this case, however, each curve which passes through the given point is a pseudogeodetic curve. If we use the invariant forms $\sqrt{2ab\,du\,dv}$, $\sqrt{k_1 a^2 du^2 - k_2 b^2 dv^2}$, $\dfrac{ab\,du\,dv}{k_1 adu + k_2 bdv}$ (k_1, k_2 are arbitrary parameters), we have given the respective Euler - Lagrange's differential equations of the pseudogeodetics. The corresponding osculating planes of the pseudogeodetics always envelop a cone of the third degree. There exists a straight line which is the intersection of three singular tangent planes to this cone. We thus have three invariant straight lines (with respect to invariant differential forms) which lie in one plane. This invariant plane is formed by lines (the normals) $n^{\circ}(\lambda)$: $x^1 = \frac{1}{2}\left\{b_1^1 - b_2^2 + \frac{\partial \lg ab}{\partial v} - \lambda \frac{\partial \lg a}{\partial v}\right\} x^3$; $x^2 = \frac{1}{2}\left\{a_2^2 - a_1^1 + \frac{\partial \lg ab}{\partial u} - \lambda \frac{\partial \lg b}{\partial u}\right\} x^3$.

For values of the parameter $\lambda = 0, \frac{3}{2}, 3$, we obtain three normals corresponding to our three forms. We have a bundle $n^{\circ}(\xi, \lambda)$ which is formed by $n^{\circ}(\lambda)$ and n_0 . Among the normals $n^{\circ}(\lambda)$ there does not exist a line such that the developable surfaces of the congruence of these lines intersect the conjugate net on π .

By Wilczynski's directrix of a surface with projective connexion we shall understand a generalization of that line from a projective space, where we consider the definition by means of a linear complex. The W.d. has the following characteristic in a projective space: if we take an arbitrary straight line p passing through the point A_0 of a surface π and a straight line q reciprocal to p , we get two line congruences Γ_1 , Γ_2 in a correspondence C . If C is a developable correspondence and if the developable ruled surfaces of these congruences intersect a conjugate net on π , then p is a W.d. Such a straight line on a surface with projective connexion, however, does not exist, not even when the surface is without torsion. If, however, we want C to be only a developable correspondence, or Γ_1 , Γ_2 to cut a conjugate net by their developable ruled surfaces, then we get the W.d. as a solution of a certain system of partial differential equations. This system has only one solution if initial conditions are given, that means if p is chosen at one point. In this case, however, W.d. defined in such a way are not identical with the W.d. studied by A.Švec, by means of the definition by a linear complex, on the assumption that we choose p at the particular point as a W.d. of Švec's system.

Let us consider two surfaces π , $\bar{\pi}$ with projective connexion. Let π and $\bar{\pi}$ be in an asymptotic correspondence C. Let H be a homology between the local spaces at the considered points. It is possible to show that a linearising line (introduces by E.Čech) of a tangent to an asymptotic $v = const$ is H-characteristic only when $\beta = \bar{\beta}$. If π is given, then $\bar{\pi}$ depends on five functions of one variable. In order for a linearising line of a tangent to an asymptotic $dv = 0$ to be a tangent to one curve of a system $a\,du - b\,dv = 0$ passing through a given point it is necessary and sufficient that the equation $a(\bar{a}^0_0 - a^0_0 + a^1_1 - \bar{a}^1_1 - 2k^0_1) + b(\bar{\beta} - \beta) = 0$ should hold. If π is given, then $\bar{\pi}$ depends on five functions of one variable.

The space with projective connexion (3-dimensional) without torsion, where through each point pass three surfaces, on which the system of curves $a\,du - b\,dv = 0$ is undetermined ($a = b = 0$) is characterized by the equations

$$R^3_{112} - R^3_{12} = R^3_{212} - R^1_{12} = R^3_{312} - R^3_{12} = 0,$$

where

$$[d\omega^\alpha] = [\omega^\beta(\omega^\alpha_\beta - \delta^\alpha_\beta \omega^0_0)] - \frac{1}{2} R^\alpha_{\gamma\varepsilon}[\omega^\gamma \omega^\varepsilon],$$

$$[d\omega^i_j] = [\omega^i_k \omega^k_j] - \frac{1}{2} R^i_{j\gamma\varepsilon}[\omega^\gamma \omega^\varepsilon], \quad R^\alpha(\gamma\varepsilon) = R^i_j(\gamma\varepsilon) = 0.$$

(B.CENKL, The normals of a surface in a space with projective connexion, sent to be printed;
A.ŠVEC, Sur la geométrie différentielle d´une surface plongée dans un espace à trois dimensions à connexion projective, sent to be printed).