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FUNDAMENTAL CENTRAL DISPERSION IN A SIMPLE SYSTEM

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1. INTRODUCTION

The classical Borůvka's theory of dispersions is concerning especially the sets of all linear differential equations of the 2nd order

$$y'' = q(t)y \tag{q}$$

where $t \in \mathbf{R}$, q : $\mathbf{R} \to \mathbf{R}$ is a continuous function and the equation (q) is oscillatoric on both sides. In this introductory part we shall denote by the symbol (q) partly the equation $y'' = q(t)\, y$ itself, and partly the set of all its solutions. An arbitrary ordered couple $\langle u, v \rangle$ of its linearly independent solutions will be called a basis of the equation (q). Every function q that is a coefficient of the differential equation (q) of the mentioned properties is called a carrier.

It is the matter of transformation of solutions Y of the equation (Q) to solutions $\tilde{Y}$ of the equation $(\tilde{Q})$ of the form

$$\tilde{Y} = \frac{Y(\alpha)}{\sqrt{|\alpha'|}}, \tag{1}$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ are suitable functions. It appears that α has always a continuous derivative of the 3rd order, the lst derivative $\neq 0$, there holds $\lim\limits_{t \to \pm\infty} \alpha(t) = \pm\infty \operatorname{sgn} \alpha'$ and moreover

$$-\{\alpha, t\} + Q(\alpha)\, \alpha'^2 \alpha^{12} = \tilde{Q}(t), \tag{$Q, \tilde{Q}$}$$

where $\{\alpha, t\} = \dfrac{1}{2}\dfrac{\alpha'''}{\alpha'} - \dfrac{3}{4}\dfrac{\alpha''^2}{\alpha'^2} = -\sqrt{|\alpha'|}\left(\dfrac{1}{\sqrt{|\alpha'|}}\right)''$ is the Schwarz derivative. By a symbol $(Q, \tilde{Q})$ we shall denote partly the mentioned differential equation itself, partly the set of all its solutions.

The set $\mathfrak{G}$ of all functions $\alpha : \mathbf{R} \to \mathbf{R}$ having the continuous 3rd derivative, the lst derivative $\neq 0$ and such that $\lim\limits_{t \to \pm\infty} \alpha(t) = \pm\infty \operatorname{sgn} \alpha'$ form a group with regard to composition of functions. Each of such functions is called a phase and $\mathfrak{G}$ is the group of all phases. It appears that $\mathfrak{G} = \bigcup (q, Q)$ where q, Q range over, independently on each other, the set of all carriers. Subsets (q, Q) in $\mathfrak{G}$ are called complexes and are of the following properties

$$(q, Q)^{-1} = (Q, q), \tag{2}$$

$$(q, \tilde{q})(\tilde{q}, Q) = (q, Q). \tag{3}$$

From the relations (2), (3) it follows that (q, q) are subgroups in $\mathfrak{G}$ and that for arbitrary $\boldsymbol{A} \in \mathfrak{G}$ there holds

$$(q, \tilde{q})\, \boldsymbol{A} = (q, Q) \tag{4}$$

iff $\boldsymbol{A} \in (\tilde{q}, Q)$,

since we have $(q, \tilde{q})\, \boldsymbol{A} \subseteq (q, Q)$, $(q, \tilde{q}) \subseteq (q, Q)\, \boldsymbol{A}^{-1} \subseteq (q, \tilde{q})$ for $\boldsymbol{A} \in (\tilde{q}, Q)$. On the contrary if (4) holds for some $\boldsymbol{A} \in \mathfrak{G}$, then necessarily $\boldsymbol{A} \in (\tilde{q}, Q)$ since there exist $\alpha \in (q, \tilde{q})$ and $\beta \in (q, Q)$ such that $\alpha\boldsymbol{A} = \beta$ and thus $\boldsymbol{A} = \alpha^{-1}\beta \in (\tilde{q}, q)(q, Q) = (\tilde{q}, Q)$.

The phases $\alpha \in (q, Q)$ are also called dispersions (q, Q), for $q = Q$ the dispersions of the carrier q as well, or the dispersions of the differential equation (q, q).

Choose a carrier q fixed. Then all other carriers Q can be determined by the formula

$$Q(t) = -\{\alpha, t\} + q(\alpha)\, \alpha'^2, \tag{5}$$

where α is let to range over $\mathfrak{G}$. For $Q \neq \dot{Q}$ the complexes (q, Q) and $(q, \dot{Q})$ are disjoint and there holds $\bigcup_Q (q, Q) \supseteq \mathfrak{G}$ so that there exists a decomposition of the group $\mathfrak{G} = \bigcup_Q\ (q, Q)$ to the complexes (q, Q). Call $\alpha \in (q, Q)$ by q-phase of the carrier Q. Since, according to (4), there holds $(q, q)\ \boldsymbol{A} = (q, Q)$ iff $\boldsymbol{A} \in (q, Q)$, the mentioned decomposition of the group $\mathfrak{G}$ to the classes of q-phases of carriers Q equals the decomposition $\mathfrak{G}_r(q, q)$. Of course, (q, q) is the element of the decomposition which contains the unit ι of the group $\mathfrak{G}$.

For an arbitrary carrier Q the subgroup $(Q, Q) = \boldsymbol{A}^{-1}(q, q)\ \boldsymbol{A}$ where $\boldsymbol{A} \in (q, Q)$ is an arbitrary q-phase of the carrier Q. Similarly it holds $(Q, \tilde{Q}) = \boldsymbol{A}^{-1}(q, q)\ \widetilde{\boldsymbol{A}}$ for arbitrary $\boldsymbol{A} \in (q, Q)$, $\widetilde{\boldsymbol{A}} \in (q, \tilde{Q})$.

The formula (1) represents a certain multiplication between elements Y of the space (Q) and dispersions α of the equation $(Q, \tilde{Q})$ with values in the space $(\tilde{Q})$. It appears that for $\alpha \in \mathfrak{G}$ there holds $(Q)\ \alpha \subseteq (\tilde{Q})$ iff $\alpha \in (Q, \tilde{Q})$. Since for $\beta = \alpha^{-1}$ there holds $Y = \dfrac{\tilde{Y}(\beta)}{\sqrt{|\beta'|}}$, we have an inclusion $(\tilde{Q})\ \beta \subseteq (Q)$ and thus $(\tilde{Q}) \subseteq (Q)\ \alpha \subseteq (\tilde{Q})$. So we get

$$(Q)\ \alpha = (\tilde{Q}) \text{ iff } \alpha \in (Q, \tilde{Q}). \tag{6}$$

Hence there follows also the formula

$$(Q)(Q, \tilde{Q}) = (\tilde{Q}). \tag{6*}$$

In the classical theory of dispersions there exists a significant carrier $q = -1$. The subgroup $\mathfrak{F} = (-1, -1)$ is called fundamental, the elements $\alpha \in (-1, Q)$ are called phases of the equation (Q), or phases of the carrier Q. The corresponding decomposition of the group $\mathfrak{G}$ into the phases according to the carriers is then $\mathfrak{G}/_r\mathfrak{F}$.

The space (-1) of all solutions of the differential equation (-1) has a significant basis $\langle \sin t, \cos t \rangle$. For arbitrary $\alpha \in (-1, Q)$ put $U = \dfrac{\sin \alpha}{\sqrt{|\alpha'|}}$, $V = \dfrac{\cos \alpha}{\sqrt{|\alpha'|}}$. Then $\langle U, V \rangle$ is a basis of the space (Q) of all solutions of the differential equation (Q). We can see that for any $\alpha \in (-1, Q)$ there exist bases $\langle U, V \rangle$ of the equation (Q) such that there holds $\operatorname{tg} \alpha = \dfrac{U}{V}$.

It appears that, on the contrary as well, for any basis $\langle U, V\rangle$ of the space (Q) there exist phases $\alpha \in (-1, Q)$ fulfilling the equation $\operatorname{tg} \alpha = \frac{U}{V}$. All these phases are then called phases of the equation (Q), corresponding to the basis $\langle U, V\rangle$. If α is one of them, then they are all $\alpha + \nu\pi$ where $\nu \in \mathbf{Z}$ ($\mathbf{Z}$ is the set of all integers).

All bases $\langle U_2, V_2\rangle$ of the space (Q) consist of all linearly independent linear combinations of arbitrarily fixed chosen linearly independent solutions $\langle U_1, V_1\rangle$ of the equation (Q).

Hence we have that all phases $\mathbf{A}_2 \in (-1, Q)$ are characterized by the relation $\operatorname{tg} \mathbf{A}_2 = h \operatorname{tg} \mathbf{A}_1$ where h ranges over all real homographies and $\mathbf{A}_1$ is arbitrarily fixed chosen phase corresponding to the basis $\langle U_1, V_1\rangle$.

The relation $\mathbf{A}_2 \sim \mathbf{A}_1$ defined by $\operatorname{tg} \mathbf{A}_2 = h \operatorname{tg} \mathbf{A}_1$ where h is a real homography, is an equivalence on the group $\mathfrak{G}$ and the corresponding decomposition is just $\mathfrak{G}/_r\mathfrak{F}$.

The set of all phases $\alpha \in \mathfrak{G}$, for which $\operatorname{tg} \alpha = \operatorname{tg} t$ holds, is $\alpha(t) = t + \nu\pi$ where $\nu \in \mathbf{Z}$, and thus forms an infinite cyclic subgroup $\mathfrak{Z}$ with the generator $\varepsilon(t) = t + \pi$ which is called the fundamental central dispersion of the carrier -1. The second generator is $\varepsilon^{-1}(t) = t - \pi$.

Note that the phases of the equation (-1) are identical with the dispersions of the equation $(-1, -1)$. Further, the group $\mathfrak{G}$ is decomposed into the class $\mathfrak{P}$ of increasing phases and the class ${}^c\mathfrak{P}$ of decreasing phases. The subgroup $\mathfrak{P}$ has the index 2 in $\mathfrak{G}$.

1.1. Theorem. *The infinite cyclic group $\mathfrak{Z}$ is the centre of the subgroup $\mathfrak{P} \cap \mathfrak{F}$ whereas the centre of the fundamental subgroup is trivial.*

Proof. I. We are going to prove that $\alpha\varepsilon = \varepsilon\alpha$ for arbitrary $\alpha \in \mathfrak{P} \cap \mathfrak{F}$. By this will be ascertained that $\mathfrak{Z}$ is a part of the centre of the group $\mathfrak{P} \cap \mathfrak{F}$.

For arbitrary constants $k, \mathrm{l} \in \mathbf{R}$ there exist constants $\tilde{k}, \tilde{\mathrm{l}} \in \mathbf{R}$ — and vice versa — such that the general solution of the equation (-1), i.e. $Y(x) = k \sin (x + \mathrm{l})$ according to (1) is mapped on $\tilde{Y}$ where

$$\tilde{Y}(x) = \frac{k}{\sqrt{|\alpha'(x)|}} \sin\big((\alpha(x) + \mathrm{l})\big) = \tilde{k} \sin (x + \tilde{\mathrm{l}}). \tag{7}$$

Take $t \in \mathrm{R}$ fixed. The function α is increasing and maps the interval $[t, \varepsilon(t)[$ on the interval $[\alpha(t), \alpha\varepsilon(t)[$. Choose the constants $k, \mathrm{l} \in \mathbf{R}$ such that $\tilde{Y}(t) = 0$. It is sufficient to choose $\mathrm{l} \in (\mathbf{Z}\pi - \alpha(t))$. According to (7) the function $\tilde{Y}$ in the interval $[t, t + \pi[$ has the unique root in number t. According to (7), the first root of the function $\tilde{Y}$ on the right after t is $\alpha(t) + \pi = \varepsilon\alpha(t)$, but this must not be in the interval $[\alpha(t), \alpha\varepsilon(t)[$ so that it fulfils $\alpha\varepsilon(t) \leqq \alpha(t) + \pi$. Thus we have $\alpha\varepsilon(t) \leqq \varepsilon\alpha(t)$ for all $\in \mathbf{R}$.

The function α maps the basis $\langle \sin t, \cos t\rangle$ on the basis which, according to (7), has for suitable constants $a, b, c, d \in \mathbf{R}$ the expression $\frac{\sin \alpha}{\sqrt{|\alpha'|}} = a \sin (t + b)$, $\frac{\cos \alpha}{\sqrt{|\alpha'|}} = c \sin (t + d)$. Hence we get $\frac{\sin \alpha\varepsilon}{\sqrt{|\alpha'(\varepsilon)|}} = \frac{\sin \varepsilon\alpha}{\sqrt{|\alpha'|}}$, $\frac{\cos \alpha\varepsilon}{\sqrt{|\alpha'(\varepsilon)|}} = \frac{\cos \varepsilon\alpha}{\sqrt{|\alpha'|}}$ so that $\operatorname{tg} \alpha\varepsilon = \operatorname{tg} \varepsilon\alpha$ and thus $\alpha\varepsilon = \varepsilon\alpha + \tilde{\mu}\pi$ where $\tilde{\mu} \in \mathbf{Z}$. By differentiation we get $\alpha'(\varepsilon) = \alpha'$ so that there even hold the formulas $\sin \alpha\varepsilon = \sin \varepsilon\alpha$, $\cos \alpha\varepsilon = \cos \varepsilon\alpha$ and thus $\alpha\varepsilon = \varepsilon\alpha + 2\nu\pi$ where $\nu \in \mathbf{Z}$. Nevertheless, for further considerations the relation $\alpha\varepsilon = \alpha + \mu\pi$ where $\mu \in \mathbf{Z}$ will be sufficient enough. Since α is increasing and $\varepsilon(t) = t + \pi > t$, we have $\alpha\varepsilon - \alpha > 0$ and thus $\mu > 0$.

Altogether we have $\alpha\varepsilon = \alpha + \mu\pi$ where $0 < \mu \leqq 1$ and thus there holds $\alpha\varepsilon = = \alpha + \pi = \varepsilon\alpha$.

II. We shall prove on the contrary that an arbitrary element γ of the centre of $\mathfrak{P} \cap \mathfrak{F}$, resp. $\mathfrak{F}$, belongs to $\mathfrak{Z}$. With regard to I, this will prove $\mathfrak{Z}$ to be the centre of $\mathfrak{P} \cap \mathfrak{F}$ and the centre of $\mathfrak{F}$ to be a subgroup of $\mathfrak{Z}$.

For any $\alpha \in \mathfrak{P} \cap \mathfrak{F}$, resp. $\alpha \in \mathfrak{F}$, there holds $\alpha\gamma = \gamma\alpha$. Since γ, $\alpha \in \mathfrak{F}$, real homographies H, h exist such that $\operatorname{tg} \gamma = H \operatorname{tg} t$, $\operatorname{tg} \alpha = h \operatorname{tg} t$. Then there holds $hH = = Hh$ for all real homographies h, and thus H is the identity in $\mathbf{R}$. Then we have $\operatorname{tg} \gamma = \operatorname{tg} t$ and consequently $\gamma \in \mathfrak{Z}$.

III. We shall prove that for an arbitrary $\alpha \in {}^{c}\mathfrak{P} \cap \mathfrak{F}$ there holds $\alpha\varepsilon = \varepsilon^{-1}\alpha$. Consequently for arbitrary $\gamma \in \mathfrak{Z}$ and arbitrary $\alpha \in {}^{c}\mathfrak{P} \cap \mathfrak{F}$ there holds $\alpha\gamma = \gamma^{-1}\varepsilon$.

Choose $t \in \mathbf{R}$ fixed. The function α is decreasing and maps the interval $[t, \varepsilon(t)[$ on the interval $]\alpha\varepsilon(t), \alpha(t)]$. Choose constants k, $l \in \mathbf{R}$ such that $\tilde{Y}(t) = 0$, see I. According to (7), $\tilde{Y}$ in the interval $[t, t + \pi[$ has the unique root t. According to (7) the immediately preceding root of $\tilde{Y}$ before t is $\alpha(t) - \pi = \varepsilon^{-1}\alpha(t)$ and because it cannot be in the interval $]\alpha\varepsilon(t), \alpha(t)]$ it fulfils the inequality $\alpha(t) - \pi \leqq \alpha\varepsilon(t)$. Thus we have $\varepsilon^{-1}\alpha(t) \leqq \alpha\varepsilon(t)$ for all $t \in \mathbf{R}$.

In the same way as in I, we get $\alpha\varepsilon = \alpha + \mu\pi$ where $\mu \in \mathbf{Z}$. Since α is decreasing and $\varepsilon(t) > t$, we have $\alpha\varepsilon - \alpha < 0$ and consequently $-\pi \leq \alpha\varepsilon(t) - \alpha(t) = \mu\pi < 0$ so that $\mu = -1$. Thus we have $\alpha\varepsilon(t) = \alpha(t) - \pi = \varepsilon^{-1}\alpha(t)$ for all $t \in \mathbf{R}$.

IV. We shall prove the centre of the subgroup $\mathfrak{F}$ to be trivial. Let γ be an arbitrary element of the centre of $\mathfrak{F}$. According to II. there is $\gamma \in \mathfrak{Z}$. For all $\beta \in {}^{c}\mathfrak{P} \cap \mathfrak{F}$ there hold at the same time $\beta\gamma = \gamma\beta$ and $\beta\gamma = \gamma^{-1}\beta$ and thus $\gamma = \gamma^{-1}$ or $\gamma^2 = \iota$. But $\gamma = \varepsilon^{\nu}$ for suitable $\nu \in \mathbf{Z}$. Because of $\mathfrak{Z}$ being an infinite cyclic group, $\varepsilon^{2\nu} = \iota$ is possible only for $\nu = 0$, and consequently $\gamma = \iota$.

Note that for arbitrary $\nu \in \mathbf{Z}$ and arbitrary $y \in (-1)$ the dispersion $\varepsilon^{\nu} \in \mathfrak{Z}$ transforms, according to (1), the solution y to $(-1)^{\nu}\, y$. Other dispersions $\gamma \in (-1, -1)$ are not of this property because if $\dfrac{\sin \gamma}{\sqrt{|\gamma'|}} = \pm \sin t$, $\dfrac{\cos \gamma}{\sqrt{|\gamma'|}} = \pm \cos t$ with the same sign, then $\operatorname{tg} \gamma = \operatorname{tg} t$ and therefore $\gamma \in \mathfrak{Z}$.

For an arbitrary carrier q the group (q, q) consists of dispersions of the equation (q, q). For an arbitrary phase $\alpha \in (-1, q)$ the conjugate subgroup $\alpha^{-1}\mathfrak{Z}\alpha \overset{\text{def}}{=} \mathfrak{Z}_q$ is the centre of $\mathfrak{P} \cap (q, q) = \alpha^{-1}(\mathfrak{P} \cap \mathfrak{F})\, \alpha$ and consists just of all central dispersions of the equation (q, q). The centre $\mathfrak{Z}_q$ is then an infinite cyclic group. Its generator $\varphi > \iota$ is called the fundamental central dispersion of the carrier q. We can write $\mathfrak{Z}_q = \{\varphi^{\nu}\}_{\nu \in \mathbf{Z}}$.

Central dispersions $\gamma \in \mathfrak{Z}_q$ are characterized by the fact that, in the sense of formula (1), they transform solutions $y \in (q)$ on $\pm y$, where the sign does not depend on y and equals to $(-1)^{\nu}$ for $\gamma = \varphi^{\nu}$. Even if some $\gamma \in (q, q)$ transforms all $y \in (q)$ to $\pm y$ without any supposition about the sign, then the sign is necessarily independent on y and thus $\gamma \in \mathfrak{Z}_q$. The solutions $y \in (q)$ are namely infinitely many, whereas the sings being only two. Therefore two linearly independent solutions u, $v \in (q)$ necessarily exist transforming with the same sign. Then for phases α of the basis $\langle u, v\rangle$ there holds $\operatorname{tg} \alpha\gamma = \operatorname{tg} \alpha$ and accordingly $\alpha\gamma = \varepsilon^{\nu}\alpha$ for a suitable $\nu \in \mathbf{Z}$ or $\gamma = \alpha^{-1}\varepsilon^{\nu}\alpha = \varphi^{\nu} \in \mathfrak{Z}_q$.

An important role in the theory of dispersions is played by the so-called Abelian relations. For the fundamental central dispesion ε of the carrier -1, for the fundamental central dispersion φ of the carrier q and for any phase $\alpha \in (-1, q)$ there holds

$$\alpha\varphi = \varepsilon^{\pm 1}\alpha \tag{8}$$

with the sign $\pm$ according to the fact if $\alpha \in \mathfrak{P}$ or $\alpha \in {}^c\mathfrak{P}$. In the consequence of (8) there holds $\alpha\varphi^\nu = \varepsilon^{\pm\nu}\alpha$ for any $\nu \in \mathbf{Z}$.

From the Abelian relations the mentioned relation $\alpha\varepsilon^\nu = \varepsilon^{\pm\nu}\alpha$ already follows for $\alpha \in (-1, -1)$ with the sign $\pm$ according to the fact if $\alpha \in \mathfrak{P}$ or $\alpha \in {}^c\mathfrak{P}$. By means of the automorphism $x \to \beta^{-1}\, x\, \beta$ of the group $\mathfrak{G}$ for $\beta \in (-1, Q)$ the mentioned relation will be transferred to (Q, Q). For arbitrary $\boldsymbol{A} \in (Q, Q)$ and the fundamental central dispersion $\Phi \in \mathfrak{Z}_Q$ there holds $\boldsymbol{A}\Phi^\nu = \Phi^{\pm\nu}\boldsymbol{A}$ with the sign $\pm$ according to that if $\boldsymbol{A} \in \mathfrak{P}$ or $\boldsymbol{A} \in {}^c\mathfrak{P}$.

The differential equation $(Q, \tilde{Q})$ is equivalent with two differential equations

$$(Q, \tilde{Q})^+ \qquad \sqrt{\alpha'}\left(\frac{1}{\sqrt{\alpha'}}\right)'' + Q(\alpha)\,\alpha'^2 = \tilde{Q}(t),$$

$$(Q, \tilde{Q})^- \qquad \sqrt{-\alpha'}\left(\frac{1}{\sqrt{-\alpha'}}\right)'' = Q(\alpha)\,\alpha'^2 + \tilde{Q}(t).$$

All solutions of $(Q, \tilde{Q})^+$, in case if Q, $\tilde{Q}$ range over all carriers, form the subgroup $\mathfrak{P}$ of increasing phases, whereas the solutions of $(Q, \tilde{Q})^-$ form the coset ${}^c\mathfrak{P}$ of decreasing phases.

If we consider only the increasing phases, we get a simplified modification of the theory of dispersions which, however, preserves on principle the main transformation properties of the original theory and is used in favour with many authors.

The basis of the abstract theory of dispersions will be an arbitrary abstract group $\mathfrak{G}$ (with the unit ι) the elements of which will be called phases. In this group a subgroup $\mathfrak{F}$, called the fundamental subgroup, will be given. Similarly as in the classical theory it will be a matter of the decomposition $\mathfrak{G}/_r\mathfrak{F}$. To each class $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$ a symbol q, the so-called carrier, is assigned in such a way that the correspondence between classes $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$ and carriers q may be one-to-one. The carrier that is assigned to the subgroup $\mathfrak{F}$ will be denoted by e. We introduce the denotation: $\mathfrak{F}\alpha = (e, q)$ iff $\alpha \in (e, q)$. The elements $\alpha \in (e, q)$ will be called the phases of the carrier q. For every ordered couple $\langle q, Q\rangle$ of the carriers we denote by the symbol (q, Q) the complex $\alpha^{-1}\mathfrak{F}\boldsymbol{A}$ where $\alpha \in (e, q)$, $\boldsymbol{A} \in (e, Q)$. There evidently holds $(q, Q) = \alpha^{-1}\mathfrak{F}\boldsymbol{A}$ independently on the choice of α in (e, q) and $\boldsymbol{A}$ in (e, Q). Specially, $\mathfrak{F} = (e, e)$ and the subgroup $(q, q) = \alpha^{-1}\mathfrak{F}\alpha$ is conjugate with $\mathfrak{F}$ by means of elements $\alpha \in (\varepsilon, g)$. The elements $\alpha \in (q, Q)$ are called dispersions (q, Q), when $q = Q$ also dispersions of the carrier q. Specially $\alpha \in \mathfrak{F}$ are both phases and dispersions of the carrier e at the same time. Evidently every complex $\alpha^{-1}\mathfrak{F}\boldsymbol{A}$, α, $\boldsymbol{A} \in \mathfrak{G}$ is of the form (q, Q) for suitale carriers q, Q, where $\alpha \in (e, q)$, $\boldsymbol{A} \in (e, Q)$.

2. System $\langle\mathfrak{G}\rangle$. Following the classical theory we investigate an abstract modification which corresponds to the classical modification with only increasing phases.

2.1. Definition. *Let $\mathfrak{G}$ be an arbitrary group. Let $A \subseteq \mathfrak{G}$, $A \neq \emptyset$ be an arbitrary subset. By the symbol ${}^nA$ we shall denote the normalizator, by ${}^zA$ the centralizator, by ${}^iA$ the invertor, and by ${}^{\mathfrak{z}}A$ the centre of subset A in the group $\mathfrak{G}$.*

2.2. Definition. *By the system* $\langle\mathfrak{G}\rangle$ *we shall call an arbitrary group* $\mathfrak{G}$ (*with the unit* ι), *where a so-called fundamental subgroup* $\mathfrak{F}$ *is given, the centre* ${}^{\mathfrak{z}}\mathfrak{F}$ *of which is an infinite cyclic group* (*with a generator* ε).

2.3. Lemma. *For an arbitrary infinite cyclic group* $\mathfrak{C} \subseteq \mathfrak{G}$ *there holds* ${}^{n}\mathfrak{C} = {}^{z}\mathfrak{C} \cup {}^{i}\mathfrak{C}$.

Proof. The Theorem was introduced by O. BORŮVKA, see [4], and follows from the fact that the infinite cyclic group has with one generator γ still just one generator γ^{-1} more, and that for $x \in \mathfrak{G}$ fixed the restriction of the automorphism $\alpha \to x^{-1}\alpha x$ of the group $\mathfrak{G}$ to the infinite cyclic subgroup $\mathfrak{C}$ transforms $\mathfrak{C}$ to an infinite cyclic subgroup $\mathfrak{C}$ iff it transforms some genrator of $\mathfrak{C}$ to some generator of $\mathfrak{C}$. The disjunctivity of summands follows from the absence of involutory elements, i.e. $\gamma \neq \iota$ such that $\gamma^2 = \iota$, in an infinite cyclic group.

2.4. Lemma. *There hold inclusions* $\mathfrak{F} \subseteq {}^{z\mathfrak{z}}\mathfrak{F} \subseteq {}^{n\mathfrak{z}}\mathfrak{F}$.

2.5. Lemma. *For* $\alpha, \beta \in \mathfrak{G}$ *there holds* $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$ *iff* $\beta\alpha^{-1} \in {}^{z\mathfrak{z}}\mathfrak{F}$.

2.6. Lemma. *For* $\alpha, \beta \in \mathfrak{G}$ *there holds* $\alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha = \beta^{-1\mathfrak{z}}\mathfrak{F}\beta$ *iff* $\beta\alpha^{-1} \in {}^{n\mathfrak{z}}\mathfrak{F}$.

2.7. Theorem. *For an arbitrary carrier* q *the centre* $\mathfrak{z}(q, q)$ *of the subgroup* (q, q) *is an infinite cyclic group. For all* $\alpha \in (e, q)$ *there holds* $\mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha$. *One of its generators is* $\alpha^{-1}\varepsilon\alpha$ *independently on the choice of* α *in* (e, q).

Proof. Let $\alpha \in (e, q)$. Put $y = \alpha^{-1}x\alpha$, $\gamma = \alpha^{-1}\beta\alpha$. Then $\gamma \in \mathfrak{z}(q, q) \equiv y\gamma = \gamma y$ for any $y \in (q, q) \equiv \alpha^{-1}x\beta\alpha = \alpha^{-1}\beta x\alpha$ for any $x \in \mathfrak{F} \equiv \beta \in {}^{\mathfrak{z}}\mathfrak{F}$. By that it is proved that $\mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha$. Evidently $\mathfrak{z}(q, q)$ is an infinite cyclic group with a generator $\alpha^{-1}\varepsilon\alpha$ which is the same for any $\alpha \in (e, q)$ according to 2.5. and 2.4.

2.8. Definition. *For every carrier* q *put* $\varphi_q = \beta^{-1}\varepsilon\beta$ *where* $\beta \in (e, q)$. *Iff for any* $\alpha \in \mathfrak{G}$ *there holds* $\{q, \mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha\} = \{q, \varphi_q = \alpha^{-1}\varepsilon\alpha\}$, *we shall call* φ_q *the fundamental central dispersion of the carrier* q, *and* $\langle\mathfrak{G}\rangle$ *will be called a system with fundamental central dispersions.*

2.9. Remark. *In a system* $\langle\mathfrak{G}\rangle$ *with fundamental central dispersions the same centres have the same fundamental central dispersion without regard to which carriers they belong.*

2.10. Theorem. *If* φ *is a fundamental central dispersion, then* φ^{-1} *is not a fundamental central dispersion for any carrier.*

Proof. If $\varphi^{-1} = \alpha^{-1}\varepsilon\alpha$ for some $\alpha \in (e, q)$, then $\varphi^{-1} \in \mathfrak{z}(q, q)$ and also $\varphi \in \mathfrak{z}(q, q)$ where $\varphi = \alpha^{-1}\varepsilon\alpha$. Hence $\varphi^{-1} = \alpha^{-1}\varepsilon^{-1}\alpha$ and thus $\varepsilon^{-1} = \varepsilon$ which is a contradiction.

2.11. Theorem. *If for one* $\alpha \in \mathfrak{G}$ *there is* $\{q, \mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha\} = \{q, \varphi_q = \alpha^{-1}\varepsilon\alpha\}$, *then* ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$.

Proof. Put $N = \{q, \mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha\}$ and suppose that also $N = \{q, \varphi_q = \alpha^{-1}\varepsilon\alpha\}$. First of all it holds that $\bigcup_{Q \in N} (e, q) = ({}^{n\mathfrak{z}}\mathfrak{F})\,\alpha$ because $\beta \in ({}^{n\mathfrak{z}}\mathfrak{F})\,\alpha \equiv \beta\alpha^{-1} \in {}^{n\mathfrak{z}}\mathfrak{F} \equiv \beta^{-1\mathfrak{z}}\mathfrak{F}\beta = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha \equiv \beta \in (e, q)$ for $q \in N$. Let $\gamma \in {}^{n\mathfrak{z}}\mathfrak{F}$. Put $\beta = \gamma\alpha$ so that $\beta \in ({}^{n\mathfrak{z}}\mathfrak{F})\,\alpha = \bigcup_{q \in N} (e, q)$ and thus $\beta^{-1}\varepsilon\beta = \alpha^{-1}\varepsilon\alpha$, or $\gamma = \beta\alpha^{-1} \in {}^{z\mathfrak{z}}\mathfrak{F}$. Then ${}^{n\mathfrak{z}}\mathfrak{F} \subseteq {}^{z\mathfrak{z}}\mathfrak{F}$ and consequently ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$.

2.12. Theorem. *If* ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$, *then* $\langle\mathfrak{G}\rangle$ *is asystem with fundamental central dispersions.*

Proof. If ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$, then ${}^{n\mathfrak{z}}\mathfrak{F} = {}^{z\mathfrak{z}}\mathfrak{F}$ and for any $\alpha \in \mathfrak{G}$ there holds $\{q, \mathfrak{z}(q, q) = \alpha^{-1\mathfrak{z}}\mathfrak{F}\alpha\} = \{q, \varphi_q = \alpha^{-1}\varepsilon\alpha\}$.

2.13. Corollary. *A necessary and sufficient condition for* $\langle\mathfrak{G}\rangle$ *to be a system with fundamental central dispersions is that* ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$.

2.14. Theorem. *Let* ${}^{i\mathfrak{z}}\mathfrak{F} = \emptyset$. *Then for any carrier* q, *an arbitrary phase* $\beta \in (e, q)$ *and both fundamental central dispersions* $\varphi \in \mathfrak{z}(q, q)$, $\varepsilon \in {}^{\mathfrak{z}}\mathfrak{F}$ *fulfil the Abelian relation* $\beta\varphi = \varepsilon\beta$.

Proof. The fundamental central dispersion is, by means of arbitrary $\beta \in (e, q)$, defined by the formula $\varphi = \beta^{-1}\varepsilon\beta$.

2.15. Definition. *Let $\langle\mathfrak{G}\rangle$ be an arbitrary system in the sense of definition 2.2. A binary relation $<$ on the group $\mathfrak{G}$ be called a pseudo-order of the system $\langle\mathfrak{G}\rangle$, if it holds*

$$\text{a) } \alpha < \beta \Rightarrow \alpha \neq \beta,\ \beta \not< \alpha$$
$$\text{b) } \alpha < \beta \Rightarrow x\alpha < x\beta \text{ for any } x \in \mathfrak{G}$$
$$\text{c) } \alpha < \beta \Rightarrow \alpha x < \beta x \text{ for any } x \in \mathfrak{G}$$
$$\text{d) the generator } \varepsilon \text{ of the entre } \mathfrak{Z} \text{ fulfils } \iota < \varepsilon.$$

2.16. Theorem. *Let ${}^{i\mathfrak{z}}\mathfrak{Z} = \emptyset$ hold in a system $\langle\mathfrak{G}\rangle$. Then the relation $<$ between elements $\alpha, \beta \in \mathfrak{G}$, defined by the relation*

$$(9) \qquad \alpha < \beta \equiv \beta\alpha^{-1} = \gamma^{-1}\varepsilon\gamma \text{ for some } \gamma \in \mathfrak{G},$$

is a pseudo-order of the system $\langle\mathfrak{G}\rangle$.

Proof. Let $\alpha < \beta$. Then there exists $\gamma \in \mathfrak{G}$ such that $\beta\alpha^{-1} = \gamma^{-1}\varepsilon\gamma$.

a) If $\alpha = \beta$, then we have $\iota = \gamma^{-1}\varepsilon\gamma$ and therefore $\gamma\iota = \varepsilon\gamma$ or $\iota = \varepsilon$, which is a contradiction.

If $\beta < \alpha$ held, then we should have $\alpha\beta^{-1} = x^{-1}\varepsilon x$ for some $x \in \mathfrak{G}$, or $\beta\alpha^{-1} = x^{-1}\varepsilon^{-1}x$ and accordingly $x^{-1}\varepsilon^{-1}x = \gamma^{-1}\varepsilon\gamma$, or $x^{-1}\gamma \in {}^{i\mathfrak{z}}\mathfrak{Z}$ which is impossible.

b) Let $x \in \mathfrak{G}$ be arbitrary. By multiplication from the left side by x and from the right side by x^{-1} we get $(x\beta)(x\alpha)^{-1} = x\beta\alpha^{-1}x^{-1} = x\gamma^{-1}\varepsilon\gamma x^{-1} = (\gamma x^{-1})^{-1}\varepsilon(\gamma x^{-1})$ and consequently it holds $x\alpha < x\beta$.

c) Let $x \in \mathfrak{G}$ be arbitrary. Then $(\beta x)(\alpha x)^{-1} = \beta\alpha^{-1} = \gamma^{-1}\varepsilon\gamma$ and thus it holds $\alpha x < \beta x$.

d) For $\iota \in \mathfrak{G}$ there holds $\iota^{-1}\varepsilon\iota = \varepsilon = \varepsilon\iota^{-1}$ and therefore $\iota < \varepsilon$.

2.17. Corollary. *Let ${}^{i\mathfrak{z}}\mathfrak{Z} = \emptyset$ hold in a system $\langle\mathfrak{G}\rangle$. Then in the pseudo-order* (9) *for any fundamental central dispersion φ there holds $\iota < \varphi$.*

Proof. To the fundamental central dispersion φ there exists $\alpha \in \mathfrak{G}$ such that $\alpha^{-1}\varepsilon\alpha = \varphi = \varphi\iota^{-1}$ and thus $\iota < \varphi$.

2.18. Theorem. *Have a system $\langle\mathfrak{G}\rangle$ with an arbitrary pseudoorder relation $<$. Then it holds ${}^{i\mathfrak{z}}\mathfrak{Z} = \emptyset$.*

Proof. Let $x \in {}^{n\mathfrak{z}}\mathfrak{Z}$. Then $x\varepsilon = \varepsilon^{\pm 1}x$. Since $\iota < \varepsilon$, we have $\varepsilon^{-1} < \iota$, $x < x\varepsilon$, $\varepsilon^{-1}x < x$. Therefore $x\varepsilon = \varepsilon^{-1}x$ cannot hold, because $x\varepsilon < x$ would be and at the same time $x < x\varepsilon$. Therefore it holds necessarily $x\varepsilon = \varepsilon x$ and thus $x \in {}^{z\mathfrak{z}}\mathfrak{Z}$. We have then ${}^{n\mathfrak{z}}\mathfrak{Z} \subseteq {}^{z\mathfrak{z}}\mathfrak{Z}$ and consequently ${}^{i\mathfrak{z}}\mathfrak{Z} = \emptyset$.

2.19. Corollary. *For a system $\langle\mathfrak{G}\rangle$ the following statements are equivalent:*

a) ${}^{i\mathfrak{z}}\mathfrak{Z} = \emptyset$,

b) in $\langle\mathfrak{G}\rangle$ *it is possible to define a pseudo-order relation,*

c) in $\langle\mathfrak{G}\rangle$ *fundamental central dispersions can be defined by the relation $\varphi = \alpha^{-1}\varepsilon\alpha$ for $\alpha \in \mathfrak{G}$.*

2.20. Corollary. *Let a system $\langle\mathfrak{G}\rangle$ be pseudo-ordered. Then for arbitrary $\mu < \nu \in \mathbf{Z}$ and for any fundamental central dispersion φ there holds $\varphi^{\mu} < \varphi^{\nu}$ and accordingly every centre $\{\varphi^{\nu}\}_{\nu \in \mathbf{Z}}$ is completely ordered by the relation $<$.*

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