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Krzysztof Moszyński

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REMARKS ON POLYNOMIAL METHODS FOR SOLVING SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

KRZYSZTOF MOSZYŃSKI

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Summary. For a large system of linear algebraic equations $Ax = b$, the approximate solution x_k is computed as the k-th order Fourier development of the function $1/z$, related to orthogonal polynomials in $L^2(\Omega)$ space. The domain Ω in the complex plane is assumed to be known. This domain contains the spectrum $\sigma(A)$ of the matrix A. Two algorithms for x_k are discussed.

Two possibilities of preconditioning by an application of the so called Richardson iteration process with a constant relaxation coefficient are proposed.

The case when Jordan blocs of higher dimension are present is discussed, with the following conclusion: in such a case application of the Sobolev space $H^s(\Omega)$ may be reasonable, with s equal to the dimension of the maximal Jordan bloc. The paper contains several numerical examples.

Keywords: Fourier Expansion, Orthogonal polynomials on $L^2(\Omega)$ space, approximate solution of linear algebraic equations, Richardson Iteration, preconditioning

AMS classification: 65F10

1. INTRODUCTION

Consider a system of linear algebraic equations

$$Ax = b \tag{1}$$

with an $n \times n$ non-singular matrix A, in general complex and non-hermitian.

Let Ω be a bounded (open) domain of the complex plane $\mathbf{C}$ containing the spectrum $\sigma(A)$ of A, and such that

$$0 \notin \overline{\Omega}.$$

We assume that Ω is known; in other words we assume that we are able to estimate roughly the location of the spectrum $\sigma(A)$ of A. For any open K with a regular Jordan boundary Γ such that

$$\sigma(A) \subset K \subset \overline{K} \subset \Omega$$

and for any complex function f analytic on Ω, the well known Dunford integral formula reads

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - A)^{-1} dz$$

where Γ is taken with positive orientation.

Let $\|\cdot\|$ be any operator norm of an $n \times n$ matrix, and let $\|\cdot\|_{\overline{K}\infty}$ be the supremum norm on $\overline{K}$. The last formula implies that

$$\|f(A)\| \leqslant \frac{|\Gamma|}{2\pi} \sup_{z \in \Gamma} \|(z - A)^{-1}\| \, \|f\|_{\overline{K}\infty} = C\|f\|_{\overline{K}\infty}$$

where the constant C depends on A, $\overline{K}$ and the norm $\|\cdot\|$, but not on f. It is easy to see that C depends strongly on $\mathrm{cond}(A)$.

Put now $f(z) = \varphi(z) - w_N(z)$, where $\varphi(z) = 1/z$, and $w_N(z)$ is a complex polynomial of degree $\leqslant N$; we get

$$\|A^{-1} - w_N(A)\| \leqslant C\|\varphi - w_N\|_{\overline{K}\infty}. \tag{2}$$

This estimate suggests the following

General Algorithm:

1. find a polynomial w_N approximating $\varphi(z) = 1/z$ on Ω in the $\|\cdot\|_{\overline{K}\infty}$ norm;
2. take $x_N = w_N(A)b$ as an approximate solution of (1).

Various realizations of this algorithm are possible; for example:

(i) interpolation over Ω [3],

(ii) optimal $L^2(\Omega)$ approximation,

(iii) optimal $C(\overline{K})$ approximation in the $\|\cdot\|_{\overline{K}\infty}$ norm.

We shall discuss here only the second possibility: optimal $L^2(\Omega)$ polynomial approximation.

The interpolation method ensures the so called optimal convergence when $N \to \infty$ (see Section 2), provided the interpolation knots are chosen in a special way. L. Reichel proposes to construct to this end a certain conformal mapping related to Ω. Once the interpolation knots, are given the interpolatory polynomial is constructed by successive application of the Richardson iteration formula. However, here the problem of the order of interpolation knots arises, [3].

We can avoid these difficulties by applying the optimal $L^2(\Omega)$ polynomial approximation of $\varphi(z) = 1/z$ in order to define w_N. Also in this case w_N converges optimally when $N \to \infty$, however, we have to restrict ourselves to simply connected domains Ω (see [2]). Moreover, the algorithm of construction of the optimal polynomial $w_N(z)$ is a little more complicated than in the case (i).

In order to overcome the drawbacks of this method we may divide the whole process into two stages:

Stage 1. Construction of w_N for a given Ω. This preparatory stage may be accomplished even a long time before actually solving the problem (1), and the results represent a certain information which enables us to construct on the second stage the vector $x_N = w_N(A)b$ for any matrix A such that $\sigma(A) \subset \Omega$. The problem of the computing time on this stage seems to be not very essential

Stage 2 is the solver. It should rapidly calculate the vector $x_N = w_N(A)b$.

Since the quality of approximation depends on N, and not on the dimension n of the system (1), we hope that methods of this kind may furnish a class of rapid solvers for big systems (1), when $N \ll n$.

In practice $N \leqslant 40$; clearly the computing time on the stage 2 also depends on n, but the complexity of this part of the process is of the order of N multiplications of A by a vector only.

It is always possible to apply for (1) the so called Gauss transformation:

$$A^*Ax = A^*b. \tag{3}$$

The matrix of the system (3) is hermitian and positive definite, and, in principle, all known methods as that of Tchebycheff, conjugate gradient and others may be applied here.

Unfortunately, the Gauss transformation squares the condition number $\mathrm{cond}(A)$. The effectivness of polynomial methods depends on the effectivness of the approximation over the domain Ω (which reduces to an interval in the case of (3)). Clearly the global polynomial approximation to $\varphi(z) = 1/z$ on Ω is not effective when Ω is large and/or very close to zero. On the other hand, the measure of the extent and closeness of $\sigma(A)$ to zero is just the condition number $\mathrm{cond}(A)$. Hence it seems to be better rather to apply a certain preconditioning process shrinking the spectrum $\sigma(A)$ and/or moving it farther from zero that to apply the Gauss transformation.

2. Optimal $L^2(\Omega)$ approximation of $\varphi(z) = 1/z$ -implementation of the stage 1

The most natural way to realize the stage 1 of the method seems to be the application of the Gramm-Schmidt orthogonalization process to a certain set of basic polynomials, in order to generate a system of orthogonal polynomials $P_0, P_1, P_2, \ldots, P_N$ and then to calculate the Fourier coefficients c_k of the function $\varphi(z) = 1/z$. We then get

$$w_N(z) = \sum_{k=0}^{N} c_k P_k(z).$$

It is worth noting here that orthogonal polynomials in the complex domain obey in general the quasitriangular recurrence formula

$$zP_k(z) = \sum_{j=0}^{k+1} \alpha_{kj} P_j(z) \tag{4}$$

and not (except a very special cases) the three-term one. Hence in this case the Lanczos type orthogonalization process (4) is no more superior to that of Gramm-Schmidt

$$\Phi_{k+1}(z) = \sum_{j=0}^{k+1} \gamma_{kj} P_j(z), \tag{5}$$

because both produce quasitriangular matrices of coefficients. Here $\{\Phi_k\}$ are certain given basic polynomials, subject to orthogonalization. There is at least one advantage of the Gramm-Schmidt process: it is easy to apply here the Green formula which replaces double integrals by single ones in scalar products defining the coefficients γ_{kj} and the Fourier coefficients c_k of the function $\varphi(z) = 1/z$.

For the coefficients of formulas (4) and (5) we have

$$\alpha_{kj} = \int_\Omega zP_k(z)P_j(z)\,d\Omega \quad k = 0, 1, \ldots, N-1;\ j = 0, 1, \ldots, k, \tag{6}$$

$$\alpha_{kk+1} = \left(\int_\Omega |z|^2 |P_k(z)|^2 d\Omega - \sum_{j=0}^{k} |\alpha_{kj}|^2\right)^{1/2} > 0,$$

$$\gamma_{kj} = \int_\Omega \Phi_{k+1}(z)\overline{P}_j(z)\,d\Omega \quad k = -1, 0, 1, \ldots, N-1;\ j = 0, 1, \ldots, k,$$

$$\gamma_{kk+1} = \left(\int_\Omega |\Phi_{k+1}(z)|^2 d\Omega - \sum_{j=0}^{k} |\gamma_{kj}|^2\right)^{1/2} > 0,$$

$$c_k = \int_\Omega \left(\overline{P}_k(z)/z\right) d\Omega \quad k = 0, 1, \ldots, N.$$

By easy recurrence applied to the last three formulas of (6), we get relations involving only integrals of known functions Φ_j and $\varphi(z) = 1/z$. This is not possible for the coefficients α_{kj}.

$$\text{(7a)} \qquad \gamma_{kj} = \frac{1}{\gamma_{j-1j}}\Big(\int_\Omega \Phi_{k+1}(z)\overline{\Phi}_j(z)\,\mathrm{d}\Omega - \sum_{s=0}^{j-1}\gamma_{j-1s}\gamma_{ks}\Big)$$
$$k = -1, 0, 1, 2, \ldots, N-1;\ j = 0, 1, 2, \ldots, k;$$

$$\text{(7b)} \qquad \gamma_{k,k+1} = \Big(\int_\Omega |\Phi_{k+1}(z)|^2 \mathrm{d}\Omega - \sum_{j=0}^{k}|\gamma_{kj}|^2\Big)^{1/2};$$

$$\text{(7c)} \qquad c_k = \frac{1}{\gamma_{k-1k}}\Big(\int_\Omega (\overline{\Phi}_k(z)/z)\,\mathrm{d}\Omega - \sum_{j=0}^{k-1}\gamma_{k-1j}c_j\Big) \qquad k = 0, 1, 2, \ldots, N.$$

In the recurrence relations (7) the double integrals over Ω may be replaced by single integrals over positively oriented boundary $\partial\Omega$ with help of the so called complex Green formula:

$$\text{(8)} \qquad \int_\Omega f(z)\overline{g}'(z)\,\mathrm{d}\Omega = \frac{1}{2\mathrm{i}}\int_{\partial\Omega} f(z)\overline{g}(z)\,\mathrm{d}z.$$

Clearly this operation may signifi-ıntly simplify the numerical process.

Put, for example,

$$\Phi_k(z) = \Big(\frac{z - z_0}{d}\Big)^k \qquad k = 0, 1, 2, \ldots, N$$

where z_0 is a fixed point chosen somewhere in the central part of Ω and $d \cong$ diameter$(\Omega)/2$. For many regions Ω such a choice ensures a rather stable computational process.

If $0 \notin \overline{\Omega}$, the Fourier expansion w_N of $\varphi(z) = 1/z$

$$\text{(9)} \qquad w_N(z) = \sum_{j=0}^{N} c_j P_j(z)$$

converges to φ in the $L^2(\Omega)$ norm provided the following condition of Markuszewicz and Farrell is satisfied:

Ω is a bounded, simply connected open domain of $\mathbb{C}$ *such that its boundary $\partial\Omega$ is at the same time the boundary of a certain unbounded domain* (see [2]).

Moreover, the polynomial w_N converges optimally to φ in the $\|\cdot\|_{\overline{K}\infty}$ norm, where $\sigma(a) \subset K \subset K \subset \Omega$, provided $\partial\Omega$ is a Jordan curve. The term *optimally* means the following:

There exists a conformal mapping $\psi(w) = cw + c_0 + c_1/w + \ldots$, *with* $c > 0$ *of the external domain of the unit disc* $|w| \leqslant 1$ *onto* $\mathbf{C} \setminus \overline{K}$. *Let* C_r *be the image of the disc* $|w| \leqslant r$, $r > 1$ *by* ψ, *and*

$$\varrho = \sup\{r > 1 \mid 0 \notin C_r\}.$$

We say that w_N *converges optimally to* $\varphi(z) = 1/z$ *if*

$$\limsup_{N\to\infty}(\|w_N - \varphi\|_{\overline{K}\infty})^{1/N} = 1/\varrho \quad \text{(see [2])}.$$

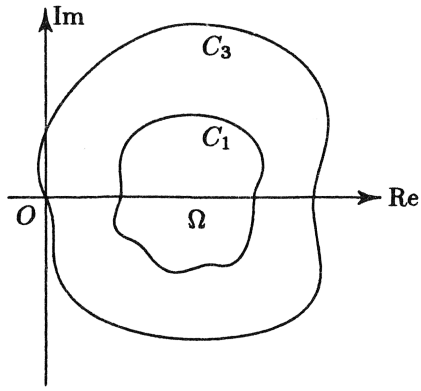

Fig. 1

In connection with the inequality (2) this condition implies the convergence of the method. We will extend this convergence discussion in Section 4.

The stage 1 of the algorithm may be implemented exactly as the formulas (5), (6) and (9) show, with application of the Green formula (8).

As the result we obtain

1. the triangular matrix of coefficients γ_{kj}, $k = -1, 0, 1, \ldots, N$, $j = 0, 1, \ldots, k+1$;
2. the vector of Fourier coefficients $c_0, c_1, \ldots, c_N$.

This is the information which should be stored. If needed, it may be used as part of the data for the stage 2 of the algorithm.

3. Implementation of the stage 2 of the algorithm

At this stage the vector

$$x_N = w_N(A)b$$

has to be computed. Provided that the first stage of the process was executed as described in Section 2, the most natural way to compute x_N seems to be as follows. The data of this stage are:

- the matrix A of dim $n \times n$, $n \gg N$.
- the triangular matrix (γ_{kl}), $k = -1, 0, 1, \ldots, N$, $l = 0, 1, \ldots, k+1$;
- the Fourier coefficients $c_0, c_1, \ldots, c_N$.

For $k = 0, 1, 2, \ldots, N$ put

$$\begin{aligned} v_k &= \Phi_k(A)b, \\ u_k &= P_k(A)b, \\ x_k &= w_k(A)b, \end{aligned} \tag{10}$$

In order to compute x_N, the following simple recurrence may be applied:
initiation:

$$\begin{aligned} v_0 &= b, \\ u_0 &= b/\gamma_{-10}, \\ x_0 &= c_0 u_0; \end{aligned} \tag{11}$$

iteration

$$\begin{aligned} v_{k+1} &= \frac{A - z_0}{d} v_k, \\ u_{k+1} &= \frac{1}{\gamma_{k,k+1}} \Big(v_{k+1} - \sum_{j=0}^{k} \gamma_{kj} u_j \Big), \\ x_{k+1} &= x_k + c_{k+1} u_{k+1}. \end{aligned} \tag{12}$$

Formulas (12) follow directly from (5).

It is worth noting that the process (11) (12) seems to be perfectly stable: at any step the Fourier expansion x_k is computed (orthogonal projection!). Experiments confirm this supposition.

Observe that the process (11) (12) needs the following, rather extensive working memory space:

$u_0, u_1, \ldots, u_N$:	$n(N+1)$	words,
v, x	$2n$	words.

This is the only drawback of the process (11) (12).

It is possible to avoid this inconvenience and to reduce the working memory space, however, the author does not think this to be very recommendable. In order to do it, the first stage of the process should be slightly modified. After computing γ_{kj} and c_k we may present $w_N(z)$ as follows:

$$w_N(z) = a_0 + a_1(z - z_1) + \ldots + a_N(z - z_1)^N$$

with

$$a_j = \frac{w_N^{(j)}(z_1)}{j!} \quad j = 0, 1, \ldots, N,$$

z_1 being a certain chosen point of Ω. To compute a_j the iterative process similar to (5) may be applied:

$$w_N^{(j)}(z_1) = \sum_{k=j}^{N} c_k P_k^{(j)}(z_1)$$

with

$$P_{k+1}^{(j)}(z_1) = \left(\Phi_{k+1}^{(j)}(z_1) - \sum_{l=j}^{k+1} \gamma_{kl} P_l^{(j)}(z_1)\right) \Big/ \gamma_{kk+1},$$
$$k = -1, 0, 1, \ldots, N,$$
$$j = 0, 1, \ldots, N.$$

Here only two vectors, namely $a_0, a_1, \ldots, a_N$ and $c_0, c_1, \ldots, c_N$ should be stored. The second stage data are

- the matrix A;
- the vectors $a_0, \ldots, a_N$ and $c_0, \ldots, c_N$.

The solver has to compute recurrently:

$$x_0 = a_N b,$$
$$x_{k+1} = (A - z_1)(x_k + a_{N-k-1} b) \quad k = 0, 1, \ldots, N - 1.$$

No problem with large working space in this version, but now the nice feature of the stable process, computing at any step of the iteration the orthogonal projection, is lost. Perhaps, we can meet certain serious stability problems. Another possibility is to present $w_N(z)$ as the interpolatory polynomial with certain fixed knots, say z_1, $z_2, \ldots, z_{N+1}$, and its coefficients expressed by successive divided differences. But now returns the old problem of the proper choice of interpolatory knots and their order.

In the end, observe that, if needed, in order to get a better approximation of the solution of (1), the whole process of the stage 2 may be recycled in the following way.

Put

$$q_0 = x_N = w_N(A)b, \tag{13}$$
$$s_0 = r_N = b - Ax_N,$$
$$q_{k+1} = q_k + w_N(A)s_k$$

with

$$s_k = b - Aq_k \quad k = 0, 1, 2, \ldots.$$

We take q_k as the k-th cyclic approximation to $x = A^{-1}b$. Observe that for the residuals s_k we have

$$s_{k+1} = b - Aq_k - Aw_N(A)s_k = \big(I - Aw_N(A)\big)s_k$$

and hence

$$s_k = \big(I - Aw_N(A)\big)^k s_0 = \big(I - Aw_N(A)\big)^k r_N.$$

It is clear that the cyclic process will converge when $k \to \infty$ provided $|1 - zw_N(z)| < 1$ for all $z \in \Omega$; this is a very week requirement concerning the approximation of $\varphi(z) = 1/z$ by $w_N(z)$.

4. Preconditioning by Richardson iteration

Consider the Richardson iteration algorithm

$$y_{k+1} = y_k + h(b - Ay_k), \tag{14}$$

where y_0 is arbitrary and $h \neq 0$. It is easy to find the general solution y_k of the difference equation (14):

$$y_k = A^{-1}[b - (I - hA)^k \varrho_0] \tag{15}$$

where $\varrho_k = b - Ay_k$ $k = 0, 1, 2, \ldots$. The process (14) converges iff the spectrum $\sigma(A)$ is contained in the interior of the disc $|1 - hz| < 1$, or if

$$|1/h - z| < 1/h.$$

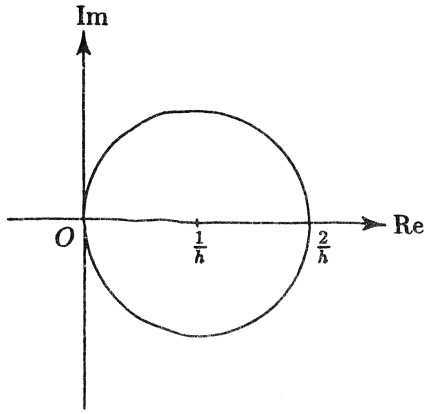

fig. 2

If the process (14) converges, then $y_k \to A^{-1}b$ when $k \to \infty$.
For the residual vectors ϱ_k we have

$$\varrho_k = (I - hA)^k \varrho_0 = V_k(A)\varrho_0$$

where

$$V_k(z) = (1 - hz)^k \tag{16}$$

is the so called *residual polynomial*: $V_k(0) = 1$. Such a polynomial always defines an interpolatory polynomial $q_{k-1}(z)$ of degree $\leqslant k - 1$ for $\varphi(z) = 1/z$, with knots being just the zeros of V_k; we have

$$V_k(z) = 1 - zq_{k-1}(z). \tag{17}$$

This is a general rule (see [1]).

Since $V_k(z)$ has a unique zero of multiplicity k: $1/h$, hence $1/h$ is the unique knot of multiplicity k of $q_{k-1}(z)$ for interpolation of φ. Moreover, the vectors y_k and q_{k-1} are connected by the relation

$$y_k = y_0 + q_{k-1}(A)\varrho_0.$$

Proposition 1.

$$q_{k-1}(z) = \frac{1 - (1 - hz)^k}{z}; \tag{18}$$

moreover, the following recurrence relation holds:

$$q_0(z) = h, \quad q_1(z) = h(2 - hz), \quad q_k(z) = h + q_{k-1}(z)(1 - hz). \tag{19}$$

Assume now that the domain Ω containing the spectrum $\sigma(A)$ of A is contained in the interior of the disc $|z - 1/h| < 1/|h|$. Choosing $h > 0$ small enough, we get the configuration as in Fig. 3.

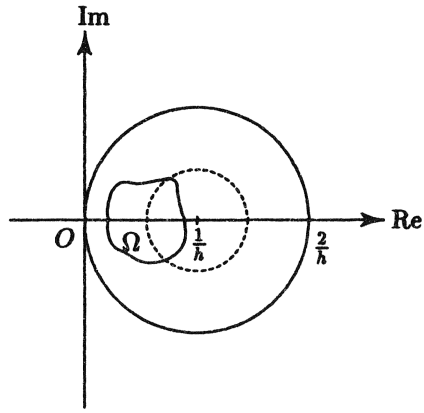

Fig. 3

Clearly, the best approximation is attained near the knot $1/h$, i.e. the Richardson iteration process shrinks the spectrum of A towards zero. Hence we have:

Preconditioning by pre-iteration

Suppose the situation is as in Fig. 3.

– First execute several steps of the Richardson iteration

$$y_{k+1} = y_k + h(b - Ay_k)$$

with h small enough, in order to ensure the arrangement as in Fig. 3. This *preiteration* cuts the part of the spectrum close to $1/h$-the high part of the spectrum of A.

– After, say, m steps of the Richardson process, start the polynomial method with a polynomial w_N ensuring the best approximation near zero, and neglecting the far regions of the spectrum.

We get

$$x_{N+m} = y_m + w_N(A)\varrho_m \tag{20}$$

with

$$\varrho_m = b - Ay_m = (I - hA)^m \varrho_0.$$

For the residual vector we have

$$r_{m+N} = b - Ax_{m+N} = \big(I - Aw_N(A)\big)(I - hA)^m \varrho_0.$$

The first bracket from the left reduces the residuum on the part of Ω close to zero. The second bracket reduces it on the part of Ω far from zero.

Preconditioning by pre-multiplication

Consider as before the configuration as in Fig. 3. Define the matrix B_k $k = 0, 1, 2, \ldots$ as follows:

$$B_k = G_k(A)$$

with $G_k(z) = zq_{k-1}(z)$, where q_{k-1} is the polynomial defined by (18). Since in a certain sense $q_{k-1}(A)$ approximates A^{-1}, hence the spectrum of $B_k = G_k(A) = Aq_{k-1}(A)$ should be closer to 1 than that of A. From (19) we get the following iterative process for B_k:

$$B_1 = hA, \tag{21}$$
$$B_{k+1} = (I - hA)B_k + hA, \quad h = 1, 2, 3 \ldots.$$

At the same time, let us define

$$b_1 = hb, \tag{22}$$
$$b_{k+1} = (I - hA)b_k + b_1.$$

We have

Proposition 2. *If the spectrum $\sigma(A)$ is contained in the open disc*

$$|z - 1/h| < 1/h - p$$

where $0 < h < 1$, $0 < p < 1/h$, then the spectrum $\sigma(B_k)$ is contained in the open disc

$$|z - 1| < (1 - hp)^k = 1 - khp + \frac{k(k-1)}{2!}h^2p^2 + \ldots = 1 - khp + O(h^2p^2).$$

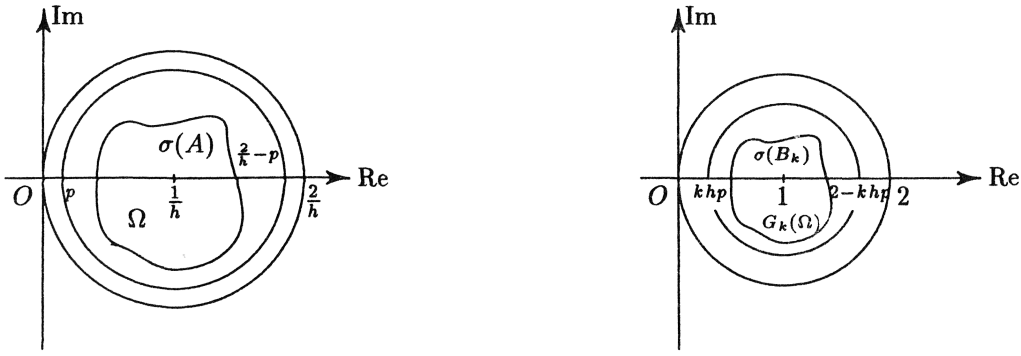

Fig. 4

Proof. Since $B_k = G_k(A)$, let us transform the circle $1/h + (1/h - hz)e^{i\varphi}$ $0 \leqslant \varphi < 2\pi$ by means of the transformation $G_k(z) = 1-(1-hz)^k$. We get $G_k(1/h + (1/h - p)e^{i\varphi}) = 1 - (-1)^k(1-hp)^k e^{ik\varphi}$. □

Proposition 3. *Equations*

$$Ax = b, \tag{1}$$

$$B_k x = b_k, \tag{23}$$

where B_k and b_k are defined by (21) *and* (22), *are equivalent.*

Proof. We have

$$B_1 x = hAx = hb = b_1.$$

Assuming that $B_k x = b_k$, by (21) and (22) we get

$$B_{k+1}x = (I - hA)B_k + hAx = (I - hA)b_k + b_1 = b_{k+1}.$$

□

Let $\mathfrak{A}$ be the class of problems (1) such that $\sigma(A)$ is contained in the disc $|z-1/h| < 1/h - p$, and $\mathfrak{B}_k$ the class of the corresponding problems (23). Put (see Fig. 4)

$$\operatorname{cond}(\mathfrak{A}) = \frac{2/h - p}{p} = \frac{2}{hp} - 1,$$
$$\operatorname{cond}(\mathfrak{B}_k) = \frac{2 - khp}{khp} = \frac{2}{khp} - 1.$$

We have

$$\operatorname{cond}(\mathfrak{B}_k) = \frac{1}{k}\big(\operatorname{cond}(\mathfrak{A}) + 1\big) - 1. \tag{24}$$

5. Jordan form and Sobolev spaces $H^k(\Omega)$

First, let us define the class of Sobolev spaces of analytic functions. For any bounded (open) domain $\Omega \subset \mathbf{C}$ and any natural k put

$$H^k(\Omega) = \Big\{ f \text{ analytic in } \Omega \mid \sum_{j=0}^{k} \int_\Omega |f^{(j)}(z)|^2 \, \mathrm{d}\Omega < \infty \Big\},$$

$$H^0(\Omega) = L^2(\Omega).$$

In $H^k(\Omega)$ the scalar product $(f,g)_{\Omega k} = \sum_{j=0}^{k} \int_\Omega f^{(j)}(z)\overline{g}^{(j)}(z)\,\mathrm{d}\Omega$ defines the corresponding norm $\|\cdot\|_{\Omega k}$.

Lemma (see [2]). *Let $z_0 \in \Omega$, $R = \operatorname{dist}(z_0, \partial\Omega)$, $f(z) = \sum_{l=0}^{\infty} a_l (z - z_0)^l$, $f \in L^2(\Omega)$. Then*

$$\|f\|_{\Omega_0}^2 \geqslant \frac{\pi}{j+1} \sum_{j=0}^{\infty} (|a_j|^2 R^{2(j+1)}).$$

Proof. Put $K = \{z \mid |z - z_0| < R\}$; then

$$\|f\|_{\Omega_0}^2 = \int_\Omega |f(z)|^2 \mathrm{d}\Omega \geqslant \int_K |f(z)|^2 \mathrm{d}K = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} a_j \overline{a}_l \int_K (z - z_0)^j \overline{(z - z_0)^l} \mathrm{d}K.$$

But

$$\int_K (z - z_0)^j \overline{(z - z_0)^l} \mathrm{d}K = \int_0^R r^{j+l+1} \mathrm{d}r \int_0^{2\pi} e^{i(j-l)\varphi} \mathrm{d}\varphi = \frac{R^{2j+2}}{2j+2} \begin{cases} 2\pi & \text{if } j = 1 \\ 0 & \text{if } j \neq 1. \end{cases}$$

Hence

$$\int_\Omega |f(z)|^2 \mathrm{d}\Omega \geqslant \int_K |f(z)|^2 \mathrm{d}K = \frac{2\pi}{2j+2} \sum_{j=0}^{\infty} |a_j|^2 R^{2(j+1)}.$$

□

Proposition 4. *Let $K \subset \Omega$ be compact and $f \in L^2(\Omega)$ analytic on Ω. Then*

$$\|f^{(j)}\|_{K\infty} \leqslant \frac{(j+1)! \, \|f\|_{\Omega_0}}{\pi \big(\inf_{z \in K} \operatorname{dist}(z, \partial\Omega) \big)^{j+1}}, \tag{25}$$

$$\|f^{(j)}\|_{K\infty} \leqslant \frac{\|f^{(j)}\|_{\Omega_0}}{\pi \big(\inf_{z \in K} \operatorname{dist}(z, \partial\Omega) \big)} \leqslant \frac{\|f\|_{\Omega j}}{\pi \big(\inf_{z \in K} \operatorname{dist}(z, \partial\Omega) \big)}. \tag{26}$$

Proof. Since $a_j = f^{(j)}(z_0)/j!$, we get (25) and (26) directly from Lemma. □

Suppose that A has a Jordan decomposition of the form $A = TJT^{-1}$ with $J = \operatorname{diag}[J_1, J_2, \ldots, J_p]$, where $J_i = \lambda_i + E_i$ is the $s_i \times s_i$ diagonal block with the eigenvalue λ_i and nilpotent E_i, $\sum_{i=1}^{p} s_i = n$, $s = \max\{s_i\}$.

If g is analytic on Ω, $0 \notin \overline{\Omega}$ and $\sigma(A) \subset K \subset \Omega$, where K is compact, we get $g(A) = Tg(J)T^{-1}$ and $g(J) = \operatorname{diag}[g(J_1), g(J_2), \ldots, g(J_p)]$. Moreover, $g(J_i) = g(\lambda_i + E_i) = \sum_{j=1}^{s_i-1} \big(g^{(j)}(\lambda_i)/j!\big)(E_i)^j$. For the operator norm $\|\cdot\|$ we obtain

$$\|g(A)\| \leqslant C \sum_{j=0}^{s-1} \|g^{(j)}\|_{K_\infty} \leqslant C_1 \Bigg(\sum_{j=0}^{s-1} \frac{1}{\big(\inf\limits_{z \in K} \operatorname{dist}(z, \partial\Omega)\big)^{j+1}} \Bigg) \|g\|_{\Omega_0}.$$

However, the formula (26) yields

$$\|g(A)\| \leqslant C_2 \frac{\|g\|_{\Omega_{s-1}}}{\inf\limits_{z \in K} \operatorname{dist}(z, \partial\Omega)}.$$

Let w_N be a polynomial of degree N and put $g(z) = 1/z - w_N(z)$. This is an analytic function in Ω. The last inequalities imply

$$\|A^{-1} - w_N(A)\| \leqslant C_1 \Bigg(\sum_{j=0}^{s-1} \frac{1}{\big(\inf\limits_{z \in K} \operatorname{dist}(z, \partial\Omega)\big)^{j+1}} \Bigg) \|\varphi - w_N\|_{\Omega_0}, \tag{27}$$

$$\|A^{-1} - w_N(A)\| \leqslant \frac{C_2}{\inf\limits_{z \in K} \operatorname{dist}(z, \partial\Omega)} \|\varphi - w_N\|_{\Omega_{s-1}} \tag{28}$$

where $\varphi(z) = 1/z$.

Conclusion. *Very often the spectrum $\sigma(A)$ of the matrix A of the system (1) comes close to the point zero in $\mathbf{C}$. In such a situation*

$$\inf_{z \in K} \operatorname{dist}(z, \partial\Omega)$$

is small, and if Jordan blocs of dimension $s > 1$ occur in A, then comparing the inequalities (27) and (28) one should conclude that it is perhaps better to minimize $\|\varphi - w_N\|_{\Omega_{s-1}}$ rather than $\|\varphi - w_N\|_{\Omega_0}$. Hence, if we suppose that Jordan blocs of dimension $s > 1$ are present, we should rather apply the norm $\|\cdot\|_{\Omega_{s-1}}$ to construct the polynomial w_N on the stage 1 of out algorithm.

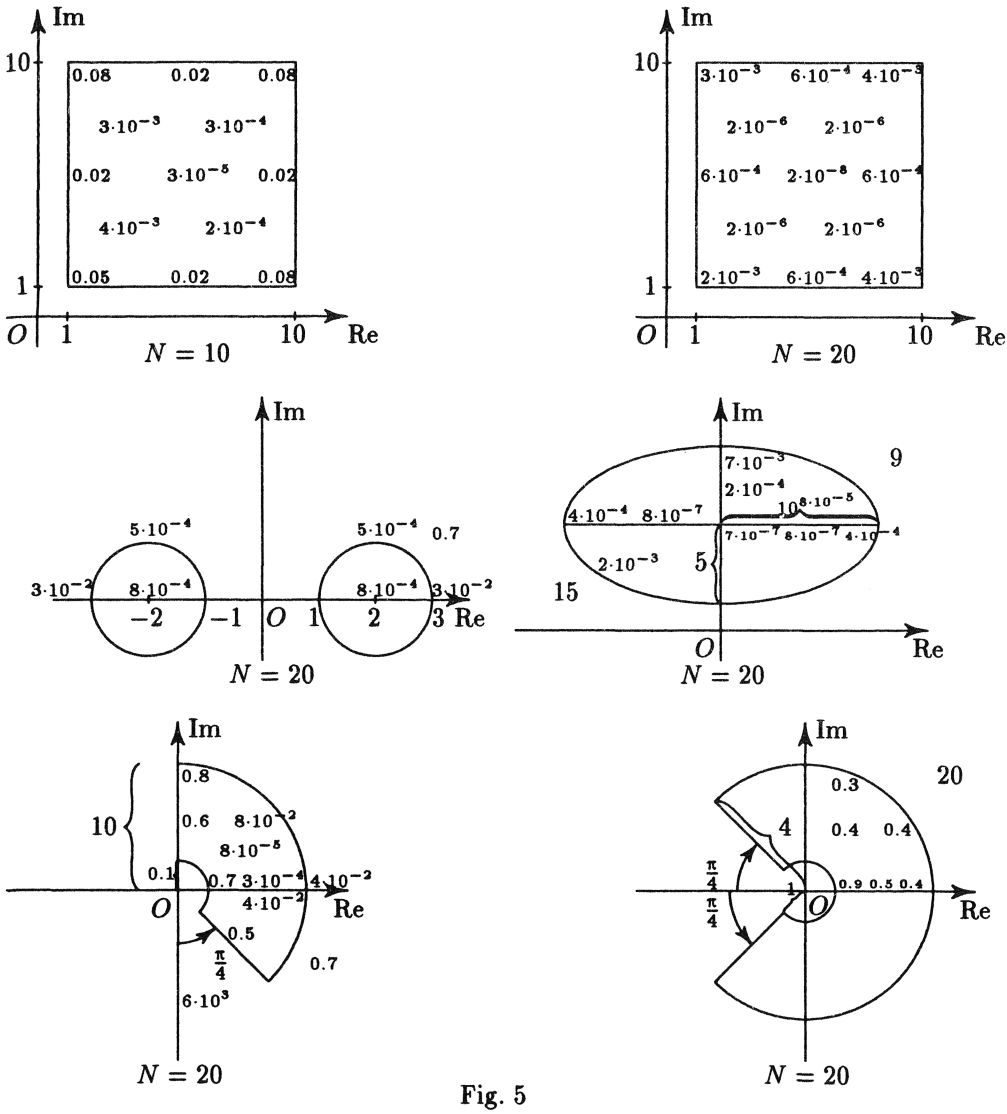

Fig. 5

6. Examples

Fig. 5 shows various domains Ω. Numbers inside the domains represent values of $|1 - zw(z)_N|$ with N equal to 10 or 20.

Relatively poor approximation was obtained for two last domains, especially for that which approaches better the entire ring. This phenomenon is comprehensible because there is no good polynomial approximation for $1/z$ on a ring centred at zero.

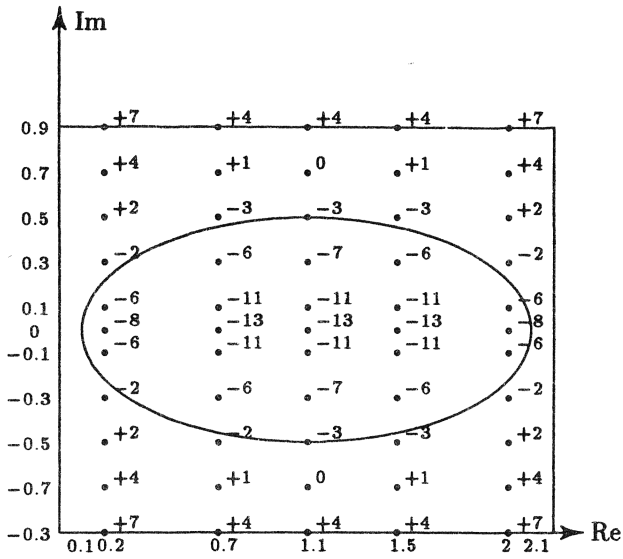

Fig. 6

Fig. 6 Ω shows an ellipse with boundary $\partial\Omega$ given by

$$z = 1.1 + \cos\varphi + 0.5\mathrm{i}\sin\varphi \quad 0 \leqslant \varphi < 2\pi$$

with $N = 40$. The numbers inside the rectangle represent the exponents of 10 of $|1 - zw_N(z)|$ calculated in a neighborhood of Ω.

In the table given below we present the best and worst coordinates r_i of the residual vector $r = b - Aw_s(A)b$ computed for the system of 50 equations. The full complex matrix A was generated in such a way that its spectrum is located in the central part of the ellipse

$$z = 5 + 4\cos\varphi + 2\sin\varphi, \quad 0 \leqslant \varphi < 2\pi$$

and all its Jordan boxes are of dimension 2. The degree s of w_s was taken equal to 30, 25, 20, 15 and 10.

In all experiments, on the *first stage* the algorithm given by formulae (7) and (8) was applied with the use of 16 point composed Gauss-Legendre quadrature. On the *second stage*, algorithm based on formulas (11) (12) was applied.

s	30	25	20	15	10
best	$10^{-12} + \mathrm{i}10^{-15}$	$10^{-10} + \mathrm{i}10^{-15}$	$10^{-9} + \mathrm{i}10^{-16}$	$10^{-7} + \mathrm{i}10^{-15}$	$10^{-4} + \mathrm{i}10^{-16}$
worst	$10^{-7} + \mathrm{i}10^{-7}$	$10^{-6} + \mathrm{i}10^{-6}$	$10^{-4} + \mathrm{i}10^{-3}$	$10^{-3} + \mathrm{i}10^{-3}$	$10^{-2} + \mathrm{i}10^{-2}$

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Author's address: Prof. *Krzysztof Moszyński*, Warsaw University, Institute of Applied Mathematics and Mechanics, Banacha 2, 02-097 Warszawa, Poland.