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Aplikace matematiky, Vol. 34 (1989), No. 6, 475–479

Persistent URL: http://dml.cz/dmlcz/104378

CONVERGENCE OF THE ACCELERATED OVERRELAXATION METHOD

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(Received August 17, 1988)

Summary. The convergence of the Accelerated Overrelaxation (AOR) method is discussed. It is shown that the intervals of convergence for the parameters σ and ω are not always of the following form: $0 \leqq \omega \leqq \omega_1$, $-\sigma_1 \leqq \sigma \leqq \sigma_2$, $\sigma_1, \sigma_2 \geqq 0$.

Keywords: Iterative process, overrelaxation, AOR method.

AMS Classification: 65F10.

1. INTRODUCTION

We shall consider a system of n linear equations with n unknowns, written in the matrix form

$$Ax = b .$$

where A is a nonsingular complex matrix with nonvanishing diagonal elements and b is a complex n-vector. For the numerical solution of this system one can use the accelerated overrelaxation (AOR) method, which was introduced by Hadjidimos in [5], and which is a two-parameter generalization of the SOR (successive overrelaxation) method. In certain cases the AOR method has better rate of convergence than Jacobi, JOR, Gauss-Seidel or SOR method, [1], [5]. Sufficient conditions for the convergence of the AOR method have been considered by many authors including [1], [2], [3], [4], [7]. If the matrix A has some particular properties, such as strict or irreducible diagonal dominance, positive definiteness or consistent orderedness, then it is known that AOR method converges if the relaxation parameter ω belongs to $(0, \omega_1)$, $1 \leqq \omega_1 < 2$ and the acceleration parameter σ belongs to $(-\sigma_1, \sigma_2)$, σ_1 and σ_2 being positive real numbers, see [1], [2], [3], [7].

In this paper, motivated by Haque's paper [6], we show that the interval of convergence of the AOR method need not be of the above form. Also, we give values of parameters σ and ω for which the AOR method converges in an example for which the SOR method does not converge. Our two theorems are generalizations of the results from [6].

In Theorem 1 it is shown that, if all eigenvalues of the Jacobi matrix have real parts less than unity, then the AOR method converges for $\omega \in (0, \omega_1)$, $\sigma \in (-\sigma_1, \sigma_2)$, where σ_1, σ_2 are positive and $0 < \omega_1 \leqq 2$. In general, it is possible that the Jacobi method does not converge.

In Theorem 2, the assumption that some eigenvalues of the Jacobi matrix have real parts greater than unity, it is proved that if the AOR method converges for some σ and ω, then there exist σ_1 and σ_2, both positive, such that $\sigma \leqq -\sigma_1$ or $\sigma \geqq \sigma_2$. It means that in this case the JOR method ($\sigma = 0$) does not converge for any choice of parameter ω.

2. CONVERGENCE OF THE AOR METHOD

Let $A = D - T - S$ be the decomposition of the matrix A into its diagonal, strictly lower and strictly upper triangular parts respectively and let $\omega, \sigma \in \mathbb{R}$, $\omega \neq 0$. The AOR method for the numerical solution of the system $Ax = b$ can be written as

$$x^0 \in \mathbb{C}^n, \quad x^{k+1} = M_{\sigma,\omega} x^k + d, \quad k = 0, 1, \ldots,$$

where

$$M_{\sigma,\omega} = (D - \sigma T)^{-1} ((1 - \omega) D + (\omega - \sigma) T + \omega S), \quad d = \omega (D - \sigma T)^{-1} b.$$

The matrix $M_{\sigma,\omega}$ can be expressed in the form

$$M_{\sigma,\omega} = I - \omega (D - \sigma T)^{-1} A.$$

For specific values of parameters ω and σ the AOR method reduces to well-known methods. So, for $\sigma = 0$, $\omega = 1$ we have the Jacobi method, for $\sigma = \omega = 1$ the Gauss-Seidel method, for $\sigma = 0$ the JOR method and for $\sigma = \omega$ the SOR method.

Theorem 1. *If all eigenvalues of the Jacobi matrix have real parts less than unity, then there exist intervals $\Omega = (0, \omega_1)$ and $\Sigma = [-\sigma_1, \sigma_2]$, $0 \in \Sigma$, such that the AOR method converges for all $\sigma \in \Sigma$ and $\omega \in \Omega$. ω_1 is not greater than* 2.

Proof. The condition of the theorem implies that the real parts of the eigenvalues of $D^{-1}A$ are all positive. Since the eigenvalues of a matrix are continuous functions of its elements, the eigenvalues of $(D - \sigma T)^{-1} A$ are continuous functions of σ. Hence, we can choose an interval $\Sigma = [-\sigma_1, \sigma_2]$ such that $0 \in \Sigma$ and the real parts of the eigenvalues of $(D - \sigma T)^{-1} A$ are all positive for all $\sigma \in \Sigma$. Let $N = \{1, 2, \ldots, n\}$. If we denote by λ_i, $i \in N$, the eigenvalues of $(D - \sigma T)^{-1} A$ and by μ_i, $i \in N$, the eigenvalues of the AOR matrix $M_{\sigma,\omega}$, then we have

$$\mu_i = 1 - \omega \lambda_i.$$

Now, since $Re\lambda_i > 0$, $|\mu_i| < 1$ requires $1 - 2\omega Re\lambda_i + \omega^2 |\lambda_i|^2 < 1$, i.e.

$$0 < \omega < \frac{2 Re\lambda_i}{|\lambda_i|^2}.$$

Let

$$\omega_0(\sigma) = \min \left\{ \frac{2Re\lambda_i}{|\lambda_i|^2} : i \in N \right\} > 0 ,$$

$$\omega_1 = \min \{\omega_0(\sigma) : \sigma \in \Sigma\} > 0 .$$

Then $|\mu_i| < 1$, $i \in N$, holds for all $\sigma \in \Sigma$ and $\omega \in \Omega = (0, \omega_1)$.

If we suppose that $\omega_1 > 2$, we can choose $\omega = 2$, $\sigma = 0$ from the intervals Ω and Σ. For this values the AOR (i.e. JOR) method is not convergent. ■

In the following theorem we show that the interval of convergence for the parameter σ need not contain 0.

Theorem 2. *Let some eigenvalues of the Jacobi matrix have real parts greater than unity. Then there exist positive numbers σ_1 and σ_2 such that if the AOR method converges for some σ and ω $(\omega > 0)$, then $\sigma \leqq -\sigma_1$ or $\sigma \geqq \sigma_2$.*

Proof. The condition of the theorem implies that some eigenvalues of $D^{-1}A$, have negative real parts. Hence, we can choose an interval $\Sigma = [-\sigma_1, \sigma_2]$, $\sigma_1, \sigma_2 > 0$- such that some eigenvalues of $(D - \sigma T)^{-1} A$ have negative real parts for all $\sigma \in \Sigma$. Consequently, because of $\omega > 0$, $\varrho(M_{\sigma,\omega}) \geqq 1$ for all $\sigma \in \Sigma$. ■

3. NUMERICAL EXAMPLES

In this section we apply our theorems to examples from [6]. All computations have been carried out on the ATARI 1040 ST with 48 bits accuracy in floating point. point.

Example 1. We consider a system of linear equations whose Jacobi matrix is

$$M_{0,1} = \begin{bmatrix} 0 & d & 0 \\ a & 0 & e \\ b & c & 0 \end{bmatrix},$$

where a, b, c, d and e satisfy $ad + ec = 2$ and $bed = -4$, so that two complex eigenvalues of $M_{0,1}$ have their real part equal to unity. The eigenvalues of $M_{0,1}$ are $1 \pm i$ and 2. The characteristic equation of the associated AOR matrix is

$$p(\lambda) = \lambda^3 + (3\omega - 3 - 2\omega\sigma)\lambda^2 + (\omega^2 - 6\omega + 3 + 2\omega^2\sigma + 4\omega\sigma)\lambda +$$
$$+ (\omega - 1)^3 + 2\omega(\omega + 1)(\omega - \sigma) .$$

In [6] it was shown that the SOR method diverges for all $\omega \in (0, 2)$, hence for all real ω. The AOR method converges for some values ω and σ, as we can see from Table 1.

Table 1. Spectral radii of the SOR and AOR matrices for $\omega = 0{\cdot}0625$

σ	Spectral radius of the SOR $M_{\sigma,\sigma}$ matrix	Spectral radius of the AOR $M_{\sigma,\omega}$ matrix
0·0625	1·00126	1·00126
0·1250	1·00547	1·00057
0·1875	1·01335	0·99988
0·2500	1·02584	0·99921
0·3125	1·04411	0·99858
0·3750	1·06956	0·99803
0·4375	1·10383	0·99759
0·5000	1·14870	0·99731
0·5625	1·20594	0·99726
0·6250	1·27703	0·99748
0·6875	1·36291	0·99802
0·7500	1·46377	0·99891
0·8125	1·57910	1·00016
0·8750	1·70783	1·00176

Example 2. The matrix

Table 2. Spectral radii of the SOR and AOR matrices for $\omega = 0{\cdot}70972$

σ	Spectral radius of the SOR $M_{\sigma,\sigma}$ matrix	Spectral radius of the AOR $M_{\sigma,\omega}$ matrix
0·0625	1·00511	1·27267
0·1250	1·00766	1·25700
0·1875	1·00714	1·23729
0·2500	1·00301	1·21291
0·3125	0·99468	1·18317
0·3750	0·98148	1·14726
0·4375	0·96264	1·10417
0·5000	0·93727	1·05259
0·5625	0·90430	0·99081
0·6250	0·86238	0·91635
0·6875	0·80972	0·82539
0·7500	0·74371	0·71119
0·8125	0·66024	0·55926
0·8750	0·55154	0·31898
0·891471	**0·51684**	**0·20769**
0·910402	0·99998	0·55912
0·910403	$>1{\cdot}00000$	0·55913
0·9375	1·39506	0·81314
1·0000	2·23200	1·29382

$$M_{0,1} = \begin{bmatrix} 0 & 0 & d \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix},$$

where $bd = 2{\cdot}63$, $acd = -4{\cdot}862$, has the eigenvalues $1{\cdot}1 \pm i$ and $-2{\cdot}2$. We shall use the matrix $M_{0,1}$ as the Jacobi matrix for a system of linear equations. Table 2 ilustrates Theorem 2.

As we can see, the "optimal" AOR method ($\omega = 0{\cdot}70972$, $\sigma = 0{\cdot}891471$, $\varrho(M_{\sigma,\omega}) = 0{\cdot}20769$) is better that the "optimal" SOR method ($\sigma = 0{\cdot}891471$, $\varrho(M_{\sigma,\sigma}) = 0{\cdot}516836855$) from [6].

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Souhrn

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Autoři vyšetřují zrychlenou superrelaxační metodu. Ukazují, že intervaly konvergence pro parametry σ a ω nejsou vždy tvaru $0 \leqq \omega \leqq \omega_1$, $-\sigma_1 \leqq \sigma \leqq \sigma_2$, $\sigma_1, \sigma_2 \geqq 0$.

Резюме

СХОДИМОСТЬ УСКОРЕННОГО МЕТОДА ВЕРХНЕЙ РЕЛАКСАЦИИ

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Авторы исследуют ускоренный метод верхней релаксации и показывают, что интервалы сходимости для параметров σ и ω не всегда имеют вид $0 \leqq \omega \leqq \omega_1$, $-\sigma_1 \leqq \sigma \leqq \sigma_2$, σ_1, $\sigma_2 \geqq 0$.

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