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A CONTRIBUTION TO THE RUNGE-KUTTA FORMULAS OF THE 7TH ORDER WITH RATIONAL COEFFICIENTS FOR THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE 1ST ORDER

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In the articles [1] and [2] Runge-Kutta formulas of the 4th and the 5th order are given, respectively, while in [3] and [4] the Runge-Kutta formulas of the 6th order are presented, whereby the solutions of the condition equations are rational numbers. In this paper we give the 7th order Runge-Kutta formulas for the system of differential equations.

Let us consider the differential equation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \tag{1}$$

where the functions $\mathbf{f}(x, \mathbf{y})$ and $\partial \mathbf{f}(x, \mathbf{y})/\partial \mathbf{y}$ are continuous in the domain $\mathbf{G}$ of $\mathbf{R}_{xy}$. Let the differential equation (1) satisfy the initial condition

$$\mathbf{y}(x_n) = \mathbf{y}_n . \tag{2}$$

Therefore the increment $\mathbf{K}$ of the function $\mathbf{y}(x)$ can be written in the form

$$\mathbf{K} = \mathbf{y}(x_n + h) - \mathbf{y}(x_n) . \tag{3}$$

The development of the function $\mathbf{y}(x_n + h)$ into the Taylor series is

$$\mathbf{K} = \sum_{i=1}^{\infty} \frac{h^i}{i!} \mathbf{y}^{(i)}(x)$$

and in virtue of

$$\mathbf{y}^{(i)}(x) = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}} \mathbf{f}[x, \mathbf{y}(x)]$$

we have

$$\mathbf{K} = \sum_{i=1}^{\infty} \frac{h^i}{i!} \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}} \mathbf{f}[x, \mathbf{y}(x)] . \tag{4}$$

Now we introduce the following simplifications:

1) If the function $\mathbf{f}(x + h, \mathbf{y} + \mathbf{k})$ the both increments h, $\mathbf{k}$ are zero, we write only $\mathbf{f}$ instead of $\mathbf{f}(x, \mathbf{y})$;

2) for the partial derivative we use the symbol

$$ {}_p\mathbf{f}_q = \frac{\partial^{p+q}\mathbf{f}}{\partial x^p\, \partial \mathbf{y}^q} \quad \text{where} \quad \mathbf{f}_q = \frac{\partial^q \mathbf{f}}{\partial \mathbf{y}^q}\,; $$

3) we introduce the operator D^n and its derivatives as follows:

$$ (5) \qquad D^n\mathbf{f} = \sum_{j=0}^{n} \binom{n}{j} \mathbf{f}^j \cdot {}_{n-j}\mathbf{f}_j\,, \quad (D^n\mathbf{f})' = D^{n+1}\mathbf{f} + n \,.\, D^{n-1}\mathbf{f}_1 \,.\, D\mathbf{f}\,, $$

$$ (6) \qquad D^n\mathbf{f}_i = \sum_{j=0}^{n} \binom{n}{j} \mathbf{f}^j \cdot {}_{n-j}\mathbf{f}_{i+j}\,, \quad (D^n\mathbf{f}_i)' = D^{n+1}\mathbf{f}_i + n \,.\, D^{n-1}\mathbf{f}_{i+1} \,.\, D\mathbf{f}\,. $$

Using these relations we can write

$$ \begin{aligned} (7) \qquad \mathbf{K} = \mathbf{y}(x_n + h) - \mathbf{y}(x_n) = \mathbf{f} \,.\, h + \frac{1}{2!} D\mathbf{f} \,.\, h^2 + \frac{1}{3!}(D^2\mathbf{f} + \mathbf{f}_1 \,.\, D\mathbf{f})\, h^3 + \\ + \frac{1}{4!}(D^3\mathbf{f} + 3D\mathbf{f} \,.\, D\mathbf{f}_1 + \mathbf{f}_1 \,.\, D^2\mathbf{f} + \mathbf{f}_1^2 \,.\, D\mathbf{f})\, h^4 + \\ + \frac{1}{5!}[D^4\mathbf{f} + 6D\mathbf{f} \,.\, D^2\mathbf{f}_1 + 4D^2\mathbf{f} \,.\, D\mathbf{f}_1 + 7\mathbf{f}_1 \,.\, D\mathbf{f} \,.\, D\mathbf{f}_1 + \\ + 3\mathbf{f}_2 \cdot (D\mathbf{f})^2 + \mathbf{f}_1 \,.\, D^3\mathbf{f} + \mathbf{f}_1^2 \,.\, D^2\mathbf{f} + \mathbf{f}_1^3 \,.\, D\mathbf{f}]\, h^5 + \\ + \frac{1}{6!}[D^5\mathbf{f} + 10D\mathbf{f} \,.\, D^3\mathbf{f}_1 + 5D^3\mathbf{f} \,.\, D\mathbf{f}_1 + 9\mathbf{f}_1 \,.\, D^2\mathbf{f} \,.\, D\mathbf{f}_1 + \\ + 12\mathbf{f}_1^2 \,.\, D\mathbf{f} \,.\, D\mathbf{f}_1 + 15D\mathbf{f} \,.\, (D\mathbf{f}_1)^2 + 16\mathbf{f}_1 \,.\, D\mathbf{f} \,.\, D^2\mathbf{f}_1 + \\ + 10D^2\mathbf{f} \,.\, D^2\mathbf{f}_1 + 15(D\mathbf{f})^2 \,.\, D\mathbf{f}_2 + 10\mathbf{f}_2 \,.\, D\mathbf{f} \,.\, D^2\mathbf{f} + \\ + 13\mathbf{f}_1 \cdot \mathbf{f}_2 \cdot (D\mathbf{f})^2 + \mathbf{f}_1 \,.\, D^4\mathbf{f} + \mathbf{f}_1^2 \,.\, D^3\mathbf{f} + \mathbf{f}_1^3 \,.\, D^2\mathbf{f} + \mathbf{f}_1^4 \,.\, D\mathbf{f}]\, h^6 + \\ + \frac{1}{7!}[D^6\mathbf{f} + 15D\mathbf{f} \,.\, D^4\mathbf{f}_1 + 20D^2\mathbf{f} \,.\, D^3\mathbf{f}_1 + 30\mathbf{f}_1 \,.\, D\mathbf{f} \,.\, D^3\mathbf{f}_1 + \\ + 45(D\mathbf{f})^2 \,.\, D^2\mathbf{f}_2 + 6D^4\mathbf{f} \,.\, D\mathbf{f}_1 + 81D\mathbf{f} \,.\, D\mathbf{f}_1 \,.\, D^2\mathbf{f}_1 + 15D^3\mathbf{f} \,.\, D^2\mathbf{f}_1 + \\ + 15\mathbf{f}_2 \,.\, D\mathbf{f} \,.\, D^3\mathbf{f} + 24D^2\mathbf{f} \,.\, (D\mathbf{f}_1)^2 + 11\mathbf{f}_1 \,.\, D^3\mathbf{f} \,.\, D\mathbf{f}_1 + \\ + 57\mathbf{f}_1 \,.\, D\mathbf{f} \,.\, (D\mathbf{f}_1)^2 + 25\mathbf{f}_1 \,.\, D^2\mathbf{f} \,.\, D^2\mathbf{f}_1 + 45\mathbf{f}_1 \cdot \mathbf{f}_2 \,.\, D\mathbf{f} \,.\, D^2\mathbf{f} + \\ + 15\mathbf{f}_1^2 \,.\, D^2\mathbf{f} \,.\, D\mathbf{f}_1 + 18\mathbf{f}_1^3 \,.\, D\mathbf{f} \,.\, D\mathbf{f}_1 + 31\mathbf{f}_1^2 \,.\, D\mathbf{f} \,.\, D^2\mathbf{f}_1 + \\ + 38\mathbf{f}_1^2 \cdot \mathbf{f}_2 \cdot (D\mathbf{f})^2 + 63\mathbf{f}_2 \cdot (D\mathbf{f})^2 \,.\, D\mathbf{f}_1 + 60D\mathbf{f} \,.\, D^2\mathbf{f} \,.\, D\mathbf{f}_2 + \end{aligned} $$

$$+ 75\mathbf{f}_1 \cdot (D\mathbf{f})^2 \cdot D\mathbf{f}_2 + 15\mathbf{f}_3 \cdot (D\mathbf{f})^3 + 10\mathbf{f}_2 \cdot (D^2\mathbf{f})^2 + \mathbf{f}_1 \cdot D^5\mathbf{f} +$$
$$+ \mathbf{f}_1^2 \cdot D^4\mathbf{f} + \mathbf{f}_1^3 \cdot D^3\mathbf{f} + \mathbf{f}_1^4 \cdot D^2\mathbf{f} + \mathbf{f}_1^5 \cdot D\mathbf{f}]\, h^7 + O(h^8)\,.$$

According to Runge's idea we can express the right-hand side of (7) as a linear combination of the values of the function $\mathbf{f}(x, \mathbf{y})$. Then the order of the method is the greatest exponent of h on the right-hand side of (7), i.e., 7. If we denote

$$\mathbf{k}_0 = h \cdot \mathbf{f}(x_n, \mathbf{y}_n)\,, \tag{8}$$

$$\mathbf{k}_i = h \cdot \mathbf{f}(x_n + a_i \cdot h,\ \mathbf{y}_n + \sum_{j=1}^{i} b_{ij} \cdot \mathbf{k}_{j-1})\,, \quad i = 1, 2, \ldots, m - 1\,,$$

then the approximate increment $\mathbf{k}$ has the value

$$\mathbf{k} = \sum_{i=0}^{m-1} p_i \cdot \mathbf{k}_i\,, \tag{9}$$

where Runge's conditions must be satisfied:

$$a_i = \sum_{j=1}^{i} b_{i,j}, \quad i = 1, 2, \ldots, m - 1\,.$$

The number m in the relation (9) denotes how many times it is necessary to substitute into the function $\mathbf{f}(x, \mathbf{y})$ (the so-called stage). It is necessary to substitute into the function $\mathbf{f}(x, \mathbf{y})$ so many times (by comparing indeterminate coefficients) as to obtain a system of condition equations, in which the number of the unknowns is at least equal to the number of the equations. In general, only in this case it is possible to obtain the values of the unknowns as rational numbers.

The development of $\mathbf{k}_i$ into the Taylor series is

$$\mathbf{k}_i = h \cdot \mathbf{f} + h \cdot \sum_{t=1}^{\infty} \frac{1}{t!} \left[h \cdot a_i \cdot D + \sum_{j=1}^{i-1} b_{i-1,j+1}(\mathbf{k}_j - h\mathbf{f}) \frac{\partial}{\partial \mathbf{y}} \right]^t \mathbf{f}\,. \tag{10}$$

By using the equation (10) up to h^7 for $\mathbf{k}_i$ we have

$$(11_1)\quad \mathbf{k}_1 = \mathbf{f} \cdot h + a_1 \cdot D\mathbf{f} \cdot h^2 + \tfrac{1}{2}a_1^2 \cdot D^2\mathbf{f} \cdot h^3 + \tfrac{1}{6}a_1^3 \cdot D^3\mathbf{f} \cdot h^4 +$$
$$+ \tfrac{1}{24}a_1^4 \cdot D^4\mathbf{f} \cdot h^5 + \tfrac{1}{120}a_1^5 \cdot D^5\mathbf{f} \cdot h^6 + \tfrac{1}{720}a_1^6 \cdot D^6\mathbf{f} \cdot h^7$$

$$(11_2)\quad \mathbf{k}_2 = \mathbf{f} \cdot h + a_2 \cdot D\mathbf{f} \cdot h^2 + \tfrac{1}{2}(a_2^2 \cdot D^2\mathbf{f} + 2c_{12}\mathbf{f}_1 \cdot D\mathbf{f})\, h^3 +$$
$$+ \tfrac{1}{6}(a_2^3 \cdot D^3\mathbf{f} + 6a_2c_{12} \cdot D\mathbf{f} \cdot D\mathbf{f}_1 + 3c_{22} \cdot \mathbf{f}_1 \cdot D^2\mathbf{f})\, h^4 +$$
$$+ \tfrac{1}{24}[a_2^4 \cdot D^4\mathbf{f} + 12a_2^2c_{12} \cdot D\mathbf{f} \cdot D^2\mathbf{f}_1 + 12a_2c_{22} \cdot D^2\mathbf{f} \cdot D\mathbf{f}_1 +$$
$$+ 12c_{12}^2 \cdot \mathbf{f}_2(D\mathbf{f})^2 + 4c_{32} \cdot \mathbf{f}_1 \cdot D^3\mathbf{f}]\, h^5 + \tfrac{1}{120}[a_2^5 \cdot D^5\mathbf{f} +$$
$$+ 20a_2^3c_{12} \cdot D\mathbf{f} \cdot D^3\mathbf{f}_1 + 30a_2^2c_{22} \cdot D^2\mathbf{f} \cdot D^2\mathbf{f}_1 + 60a_2c_{12}^2(D\mathbf{f})^2 \cdot D\mathbf{f}_2 +$$

$$+ 20a_2c_{32} . D^3\mathbf{f} . D\mathbf{f}_1 + 5c_{42} . \mathbf{f}_1 . D^4\mathbf{f} + 60c_{12}c_{22} . \mathbf{f}_2 . D\mathbf{f} . D^2\mathbf{f}]\, h^6 +$$

$$+ \tfrac{1}{720}[a_2^6 . D^6\mathbf{f} + 30a_2^4c_{12} . D\mathbf{f} . D^4\mathbf{f}_1 + 60a_2^3c_{22} . D^2\mathbf{f} . D^3\mathbf{f}_1 +$$

$$+ 90c_{22}^2\mathbf{f}_2(D^2\mathbf{f})^2 + 180a_2^2c_{12}^2 . (D\mathbf{f})^2 . D^2\mathbf{f}_2 + 60a_2^2c_{32} . D^3\mathbf{f} . D^2\mathbf{f}_1 +$$

$$+ 360a_2c_{12}c_{22} . D\mathbf{f} . D^2\mathbf{f} . D\mathbf{f}_2 + 120c_{12}^3 . \mathbf{f}_3 . (D\mathbf{f})^3 + 6c_{52} . \mathbf{f}_1 . D^5\mathbf{f} +$$

$$+ 30a_2c_{42} . D^4\mathbf{f} . D\mathbf{f}_1 + 120c_{12}c_{32} . \mathbf{f}_2 . D\mathbf{f} . D^3\mathbf{f}]\, h^7$$

$$(11_3)\quad \mathbf{k}_3 = \mathbf{f} . h + a_3 . D\mathbf{f} . h^2 + \tfrac{1}{2}(a_3^2 . D^2\mathbf{f} + 2c_{13} . \mathbf{f}_1 . D\mathbf{f})\, h^3 +$$

$$+ \tfrac{1}{6}(a_3^3 . D^3\mathbf{f} + 6a_3c_{13} . D\mathbf{f} . D\mathbf{f}_1 + 3c_{23} . \mathbf{f}_1 . D^2\mathbf{f} + 6d_{13} . \mathbf{f}_1^2 . D\mathbf{f})\, h^4 +$$

$$+ \tfrac{1}{24}[a_3^4 . D^4\mathbf{f} + 12a_3^2c_{13} . D\mathbf{f} . D^2\mathbf{f}_1 + 12a_3c_{23} . D^2\mathbf{f} . D\mathbf{f}_1 +$$

$$+ 24(a_3d_{13} + d_{13}^{(1)})\, \mathbf{f}_1 . D\mathbf{f} . D\mathbf{f}_1 + 12c_{13}^2 . \mathbf{f}_2 . (D\mathbf{f})^2 +$$

$$+ 4c_{33} . \mathbf{f}_1 . D^3\mathbf{f} + 12d_{23} . \mathbf{f}_1^2 . D^2\mathbf{f}]\, h^5 + \tfrac{1}{120}[a_3^5 . D^5\mathbf{f} +$$

$$+ 20a_3^3c_{13} . D\mathbf{f} . D^3\mathbf{f}_1 + 30a_3^2c_{23} . D^2\mathbf{f} . D^2\mathbf{f}_1 +$$

$$+ 60(a_3^2d_{13} + d_{13}^{(2)})\, \mathbf{f}_1 . D\mathbf{f} . D^2\mathbf{f}_1 + 60a_3c_{13}^2(D\mathbf{f})^2 . D\mathbf{f}_2 +$$

$$+ 60(a_3d_{23} + d_{23}^{(1)})\, \mathbf{f}_1 . D^2\mathbf{f} . D\mathbf{f}_1 + 20a_3c_{33} . D^3\mathbf{f} . D\mathbf{f}_1 +$$

$$+ 120a_3d_{13}^{(1)} . D\mathbf{f} . (D\mathbf{f}_1)^2 + 60c_{13}c_{23} . \mathbf{f}_2 . D\mathbf{f} . D^2\mathbf{f} +$$

$$+ 60(2c_{13}d_{13} + E_{13})\, \mathbf{f}_1 . \mathbf{f}_2 . (D\mathbf{f})^2 + 5c_{43} . \mathbf{f}_1 . D^4\mathbf{f} +$$

$$+ 20d_{33}\mathbf{f}_1^2 . D^3\mathbf{f}]\, h^6 + \tfrac{1}{720}[a_3^6 . D^6\mathbf{f} + 30a_3^4c_{13} . D\mathbf{f} . D^4\mathbf{f}_1 +$$

$$+ 60a_3^3c_{23} . D^2\mathbf{f} . D^3\mathbf{f}_1 + 60a_3^2c_{33} . D^3\mathbf{f} . D^2\mathbf{f}_1 +$$

$$+ 120(a_3^3d_{13} + d_{13}^{(3)})\, \mathbf{f}_1 . D\mathbf{f} . D^3\mathbf{f}_1 + 180a_3^2c_{13}^2(D\mathbf{f})^2 . D^2\mathbf{f}_2 +$$

$$+ 360(a_3^2d_{13}^{(1)} + a_3d_{13}^{(2)})\, D\mathbf{f} . D\mathbf{f}_1 . D^2\mathbf{f}_1 + 180(a_3^2d_{23} +$$

$$+ d_{23}^{(2)})\mathbf{f}_1 . D^2\mathbf{f} . D^2\mathbf{f}_1 + 360a_3c_{13}c_{23} . D\mathbf{f} . D^2\mathbf{f} . D\mathbf{f}_2 +$$

$$+ 720(a_3c_{13}d_{13} + \tfrac{1}{2}E_{13}^{(1)})\, \mathbf{f}_1(D\mathbf{f})^2\, D\mathbf{f}_2 + 120c_{13}^3 . \mathbf{f}_3(D\mathbf{f})^3 +$$

$$+ 30a_3c_{43} . D^4\mathbf{f} . D\mathbf{f}_1 + 360a_3d_{23}^{(1)} . D^2\mathbf{f} . (D\mathbf{f}_1)^2 + 360(a_3E_{13} +$$

$$+ 2c_{13}d_{13}^{(1)})\, \mathbf{f}_2 . (D\mathbf{f})^2 . D\mathbf{f}_1 + 120c_{13}c_{33} . \mathbf{f}_2 . D\mathbf{f} . D^3\mathbf{f} +$$

$$+ 120(a_3d_{33} + d_{33}^{(1)})\, \mathbf{f}_1 . D^3\mathbf{f} . D\mathbf{f}_1 + 360(c_{13}d_{23} + c_{23}d_{13} +$$

$$+ \mathscr{C}_{23})\, \mathbf{f}_1 . \mathbf{f}_2 . D\mathbf{f} . D^2\mathbf{f} + 90c_{23}^2 . \mathbf{f}_2(D^2\mathbf{f})^2 + 360d_{13}^2 . \mathbf{f}_1^2 . \mathbf{f}_2(D\mathbf{f})^2 +$$

$$+ 6c_{53} . \mathbf{f}_1 . D^5\mathbf{f} + 30d_{43} . \mathbf{f}_1^2 . D^4\mathbf{f}]\, h^7$$

$$(11_4)\quad \mathbf{k}_4 = \mathbf{f} . h + a_4 . D\mathbf{f} . h^2 + \tfrac{1}{2}(a_4^2 . D^2\mathbf{f} + 2c_{14} . \mathbf{f}_1 . D\mathbf{f})\, h^3 +$$

$$+ \tfrac{1}{6}(a_4^3 . D^3\mathbf{f} + 6a_4c_{14} . D\mathbf{f} . D\mathbf{f}_1 + 3c_{24} . \mathbf{f}_1 . D^2\mathbf{f} + 6d_{14} . \mathbf{f}_1^2 . D\mathbf{f})\, h^4 +$$

$$+ \tfrac{1}{24}[a_4^4 . D^4\mathbf{f} + 12a_4^2c_{14} . D\mathbf{f} . D^2\mathbf{f}_1 + 12a_4c_{24} . D^2\mathbf{f} . D\mathbf{f}_1 +$$
$$+ 24(a_4d_{14} + d_{14}^{(1)})\mathbf{f}_1 . D\mathbf{f} . D\mathbf{f}_1 + 12c_{14}^2 . \mathbf{f}_2 . (D\mathbf{f})^2 + 4c_{34} . \mathbf{f}_1 . D^3\mathbf{f} +$$
$$+ 12d_{24} . \mathbf{f}_1^2 . D^2\mathbf{f} + 24e_{14} . \mathbf{f}_1^3 . D\mathbf{f}]\, h^5 + \tfrac{1}{120}[a_4^5 . D^5\mathbf{f} +$$
$$+ 20a_4^3c_{14} . D\mathbf{f} . D^3\mathbf{f}_1 + 30a_4^2c_{24} . D^2\mathbf{f} . D^2\mathbf{f}_1 + 60a_4c_{14}^2(D\mathbf{f})^2\, D\mathbf{f}_2 +$$
$$+ 60(a_4^2d_{14} + d_{14}^{(2)})\mathbf{f}_1 . D\mathbf{f} . D^2\mathbf{f}_1 + 20a_4c_{34} . D^3\mathbf{f} . D\mathbf{f}_1 +$$
$$+ 60(a_4d_{24} + d_{24}^{(1)})\mathbf{f}_1 . D^2\mathbf{f} . D\mathbf{f}_1 + 120a_4d_{14}^{(1)}\, D\mathbf{f}(D\mathbf{f}_1)^2 +$$
$$+ 120(a_4e_{14} + e_{14}^{(1)} + \varepsilon_{14}^{(1)})\mathbf{f}_1^2 . D\mathbf{f} . D\mathbf{f}_1 + 60c_{14}c_{24} . \mathbf{f}_2 . D\mathbf{f} . D^2\mathbf{f} +$$
$$+ 60(2c_{14}d_{14} + E_{14})\mathbf{f}_1 . \mathbf{f}_2 . (D\mathbf{f})^2 +$$
$$+ 5c_{44} . \mathbf{f}_1 . D^4\mathbf{f} + 20d_{34} . \mathbf{f}_1^2 . D^3\mathbf{f} + 60e_{24} . \mathbf{f}_1^3 . D^2\mathbf{f}]\, h^6 +$$
$$+ \tfrac{1}{720}[a_4^6 . D^6\mathbf{f} + 30a_4^4c_{14} . D\mathbf{f} . D^4\mathbf{f}_1 + 60a_4^3c_{24} . D^2\mathbf{f} . D^3\mathbf{f}_1 +$$
$$+ 180a_4^2c_{14}^2(D\mathbf{f})^2 . D^2\mathbf{f}_2 + 120(a_4^3d_{14} + d_{14}^{(3)})\mathbf{f}_1 . D\mathbf{f} . D^3\mathbf{f}_1 +$$
$$+ 60a_4^2c_{34} . D^3\mathbf{f} . D^2\mathbf{f}_1 + 180(a_4^2d_{24} + d_{24}^{(2)})\mathbf{f}_1 . D^2\mathbf{f} . D^2\mathbf{f}_1 +$$
$$+ 360a_4c_{14}c_{24} . D\mathbf{f} . D^2\mathbf{f} . D\mathbf{f}_2 + 360(a_4^2d_{14}^{(1)} + a_4d_{14}^{(2)})\, D\mathbf{f} . D\mathbf{f}_1 . D^2\mathbf{f}_1 +$$
$$+ 360(a_4^2e_{14} + e_{14}^{(2)} + \varepsilon_{14}^{(2)})\mathbf{f}_1^2 . D\mathbf{f} . D^2\mathbf{f}_1 +$$
$$+ 360(2a_4c_{14}d_{14} + E_{14}^{(1)})\mathbf{f}_1 . (D\mathbf{f})^2 . D\mathbf{f}_2 + 120c_{14}^3 . \mathbf{f}_3 . (D\mathbf{f})^3 +$$
$$+ 30a_4c_{44} . D^4\mathbf{f} . D\mathbf{f}_1 + 360a_4d_{24}^{(1)} . D^2\mathbf{f} . (D\mathbf{f}_1)^2 +$$
$$+ 120(a_4d_{34} + d_{34}^{(1)})\mathbf{f}_1 . D^3\mathbf{f} . D\mathbf{f}_1 + 120c_{14}c_{34} . \mathbf{f}_2 . D\mathbf{f} . D^3\mathbf{f} +$$
$$+ 720(a_4e_{14}^{(1)} + a_4\varepsilon_{14}^{(1)} + {}^1H_{14}^{(1)})\mathbf{f}_1 . D\mathbf{f} . (D\mathbf{f}_1)^2 +$$
$$+ 360(a_4e_{24} + e_{24}^{(1)} + \varepsilon_{24}^{(1)})\mathbf{f}_1^2 . D^2\mathbf{f} . D\mathbf{f}_1 +$$
$$+ 360(2c_{14}d_{14}^{(1)} + a_4E_{14})\mathbf{f}_2 . (D\mathbf{f})^2 . D\mathbf{f}_1 + 360(c_{14}d_{24} + c_{24}d_{14} +$$
$$+ \mathscr{C}_{24})\mathbf{f}_1 . \mathbf{f}_2 . D\mathbf{f} . D^2\mathbf{f} + 360(2c_{14}e_{14} + d_{14}^2 + 2\mathscr{D}_{14} +$$
$$+ \mathscr{E}_{14})\mathbf{f}_1^2 . \mathbf{f}_2 . (D\mathbf{f})^2 + 90c_{24}^2 . \mathbf{f}_2(D^2\mathbf{f})^2 + 6c_{54} . \mathbf{f}_1 . D^5\mathbf{f} +$$
$$+ 30d_{44} . \mathbf{f}_1^2 . D^4\mathbf{f} + 120e_{34} . \mathbf{f}_1^3 . D^3\mathbf{f}]\, h^7$$

$$(11_5)\quad \mathbf{k}_5 = \mathbf{f} . h + a_5 . D\mathbf{f} . h^2 + \tfrac{1}{2}(a_5^2 . D^2\mathbf{f} + 2c_{15} . \mathbf{f}_1 . D\mathbf{f})\, h^3 +$$
$$+ \tfrac{1}{6}(a_5^3 . D^3\mathbf{f} + 6a_5c_{15} . D\mathbf{f} . D\mathbf{f}_1 + 3c_{25} . \mathbf{f}_1 . D^2\mathbf{f} + 6d_{15} . \mathbf{f}_1^2 . D\mathbf{f})\, h^4 +$$
$$+ \tfrac{1}{24}[a_5^4 . D^4\mathbf{f} + 12a_5^2c_{15} . D\mathbf{f} . D^2\mathbf{f}_1 + 12a_5c_{25} . D^2\mathbf{f} . D\mathbf{f}_1 +$$
$$+ 24(a_5d_{15} + d_{15}^{(1)})\mathbf{f}_1 . D\mathbf{f} . D\mathbf{f}_1 + 12c_{15}^2 . \mathbf{f}_2 . (D\mathbf{f})^2 + 4c_{35} . \mathbf{f}_1 . D^3\mathbf{f} +$$
$$+ 12d_{25} . \mathbf{f}_1^2 . D^2\mathbf{f} + 24e_{15} . \mathbf{f}_1^3 . D\mathbf{f}]\, h^5 + \tfrac{1}{120}[a_5^5 . D^5\mathbf{f} +$$

$$+ 20a_5^3c_{15} \cdot Df \cdot D^3\mathbf{f}_1 + 30a_5^2c_{25} \cdot D^2\mathbf{f} \cdot D^2\mathbf{f}_1 + 60a_5c_{15}^2(Df)^2 \cdot D\mathbf{f}_2 +$$

$$+ 60(a_5^2d_{15} + d_{15}^{(2)})\,\mathbf{f}_1 \cdot Df \cdot D^2\mathbf{f}_1 + 20a_5c_{35} \cdot D^3\mathbf{f} \cdot D\mathbf{f}_1 +$$

$$+ 120a_5d_{15}^{(1)} \cdot Df(D\mathbf{f}_1)^2 + 60(a_5d_{25} + d_{25}^{(1)})\,\mathbf{f}_1 \cdot D^2\mathbf{f} \cdot D\mathbf{f}_1 +$$

$$+ 60c_{15}c_{25} \cdot \mathbf{f}_2 \cdot Df \cdot D^2\mathbf{f} + 120(a_5e_{15} + e_{15}^{(1)} + \varepsilon_{15}^{(1)})\,\mathbf{f}_1^2 \cdot Df \cdot D\mathbf{f}_1 +$$

$$+ 60(2c_{15}d_{15} + E_{15})\,\mathbf{f}_1 \cdot \mathbf{f}_2 \cdot (Df)^2 + 5c_{45} \cdot \mathbf{f}_1 \cdot D^4\mathbf{f} +$$

$$+ 20d_{35} \cdot \mathbf{f}_1^2 \cdot D^3\mathbf{f} + 60e_{25} \cdot \mathbf{f}_1^3 \cdot D^2\mathbf{f} + 120g_{15} \cdot \mathbf{f}_1^4 \cdot Df]\,h^6 +$$

$$+ \tfrac{1}{720}[a_5^6 \cdot D^6\mathbf{f} + 30a_5^4c_{15} \cdot Df \cdot D^4\mathbf{f}_1 + 60a_5^3c_{25} \cdot D^2\mathbf{f} \cdot D^3\mathbf{f}_1 +$$

$$+ 120(a_5^3d_{15} + d_{15}^{(3)})\,\mathbf{f}_1 \cdot Df \cdot D^3\mathbf{f}_1 + 180a_5^2c_{15}^2(Df)^2 \cdot D^2\mathbf{f}_2 +$$

$$+ 60a_5^2c_{35} \cdot D^3\mathbf{f} \cdot D^2\mathbf{f}_1 + 360(a_5^2d_{15}^{(1)} + a_5d_{15}^{(2)})\,Df \cdot D\mathbf{f}_1 \cdot D^2\mathbf{f}_1 +$$

$$+ 180(a_5^2d_{25} + d_{25}^{(2)})\,\mathbf{f}_1 \cdot D^2\mathbf{f} \cdot D^2\mathbf{f}_1 + 360(a_5^2e_{15} + e_{15}^{(2)} +$$

$$+ \varepsilon_{15}^{(2)})\,\mathbf{f}_1^2 \cdot Df \cdot D^2\mathbf{f}_1 + 360(2a_5c_{15}d_{15} + E_{15}^{(1)})\,\mathbf{f}_1 \cdot (Df)^2 \cdot D\mathbf{f}_2 +$$

$$+ 360a_5c_{15}c_{25} \cdot Df \cdot D^2\mathbf{f} \cdot D\mathbf{f}_2 + 120c_{15}^3 \cdot \mathbf{f}_3 \cdot (Df)^3 +$$

$$+ 30a_5c_{45} \cdot D^4\mathbf{f} \cdot D\mathbf{f}_1 + 360(2c_{15}d_{15}^{(1)} + a_5E_{15})\,\mathbf{f}_2 \cdot (Df)^2 \cdot D\mathbf{f}_1 +$$

$$+ 360a_5d_{25}^{(1)} \cdot D^2\mathbf{f} \cdot (D\mathbf{f}_1)^2 + 120(a_5d_{35} + d_{35}^{(1)})\,\mathbf{f}_1 \cdot D^3\mathbf{f} \cdot D\mathbf{f}_1 +$$

$$+ 720(a_5e_{15}^{(1)} + a_5\varepsilon_{15}^{(1)} + {}^1H_{15}^{(1)})\,\mathbf{f}_1 \cdot Df \cdot (D\mathbf{f}_1)^2 +$$

$$+ 360(a_5e_{25} + e_{25}^{(1)} + \varepsilon_{25}^{(1)})\,\mathbf{f}_1^2 \cdot D^2\mathbf{f} \cdot Df + 720(a_5g_{15} + g_{15}^{(1)} +$$

$$+ G_{15}^{(1)} + \mathscr{G}_{15}^{(1)})\,\mathbf{f}_1^3 \cdot D\mathbf{f}_1 \cdot Df + 120c_{14}c_{34} \cdot \mathbf{f}_2 \cdot Df \cdot D^3\mathbf{f} +$$

$$+ 90c_{25}^2 \cdot \mathbf{f}_2(D^2\mathbf{f})^2 + 360(c_{15}d_{25} + c_{25}d_{15} + \mathscr{C}_{25})\,\mathbf{f}_1\mathbf{f}_2 \cdot Df \cdot D^2\mathbf{f} +$$

$$+ 360(2c_{15}e_{15} + d_{15}^2 + 2\mathscr{D}_{15} + \mathscr{E}_{15})\,\mathbf{f}_1^2\mathbf{f}_2(Df)^2 +$$

$$+ 6c_{55} \cdot \mathbf{f}_1 \cdot D^5\mathbf{f} + 30d_{45} \cdot \mathbf{f}_1^2 \cdot D^4\mathbf{f} + 120e_{35} \cdot \mathbf{f}_1^3 \cdot D^3\mathbf{f} +$$

$$+ 360g_{25} \cdot \mathbf{f}_1^4 \cdot D^2\mathbf{f}]\,h^7\,,$$

$$(11_i)\quad \mathbf{k}_i = \mathbf{f} \cdot h + a_i \cdot Df \cdot h^2 + \tfrac{1}{2}(a_i^2 \cdot D^2\mathbf{f} + 2c_{1i} \cdot \mathbf{f}_1 \cdot Df)\,h^3 +$$

$$+ \tfrac{1}{6}(a_i^3 \cdot D^3\mathbf{f} + 6a_ic_{1i} \cdot Df \cdot D\mathbf{f}_1 + 3c_{2i} \cdot \mathbf{f}_1 \cdot D^2\mathbf{f} + 6d_{1i} \cdot \mathbf{f}_1^2 \cdot Df)\,h^4 +$$

$$+ \tfrac{1}{24}[a_i^4 \cdot D^4\mathbf{f} + 12a_i^2c_{1i} \cdot Df \cdot D^2\mathbf{f}_1 + 12a_ic_{2i} \cdot D^2\mathbf{f} \cdot D\mathbf{f}_1 +$$

$$+ 24(a_id_{1i} + d_{1i}^{(1)})\,\mathbf{f}_1 \cdot Df \cdot D\mathbf{f}_1 + 12c_{1i}^2 \cdot \mathbf{f}_2(Df)^2 + 4c_{3i} \cdot \mathbf{f}_1 \cdot D^3\mathbf{f} +$$

$$+ 12d_{2i} \cdot \mathbf{f}_1^2 \cdot D^2\mathbf{f} + 24e_{1i} \cdot \mathbf{f}_1^3 \cdot Df]\,h^5 + \tfrac{1}{120}[a_i^5 \cdot D^5\mathbf{f} +$$

$$+ 20a_i^3c_{1i} \cdot Df \cdot D^3\mathbf{f}_1 + 30a_i^2c_{2i} \cdot D^2\mathbf{f} \cdot D^2\mathbf{f}_1 + 60a_ic_{1i}^2(Df)^2 \cdot D\mathbf{f}_2 +$$

$$+ 60(a_i^2d_{1i} + d_{1i}^{(2)})\,\mathbf{f}_1 \cdot Df \cdot D^2\mathbf{f}_1 + 20a_ic_{3i} \cdot D^3\mathbf{f} \cdot D\mathbf{f}_1 +$$

$$+ 120a_i d_{1i}^{(1)} \cdot \mathbf{Df}(\mathbf{Df}_1)^2 + 60(a_i d_{2i} + d_{2i}^{(1)}) \mathbf{f}_1 \cdot \mathbf{D}^2\mathbf{f} \cdot \mathbf{Df}_1 +$$
$$+ 120(a_i e_{1i} + e_{1i}^{(1)} + \varepsilon_{1i}^{(1)}) \mathbf{f}_1^2 \cdot \mathbf{Df} \cdot \mathbf{Df}_1 + 60c_{1i}c_{2i} \cdot \mathbf{f}_2 \cdot \mathbf{Df} \cdot \mathbf{D}^2\mathbf{f} +$$
$$+ 60(2c_{1i}d_{1i} + E_{1i}) \mathbf{f}_1\mathbf{f}_2(\mathbf{Df})^2 + 5c_{4i} \cdot \mathbf{f}_1 \cdot \mathbf{D}^4\mathbf{f} +$$
$$+ 20d_{3i} \cdot \mathbf{f}_1^2 \cdot \mathbf{D}^3\mathbf{f} + 60e_{2i} \cdot \mathbf{f}_1^3 \cdot \mathbf{D}^2\mathbf{f} + 120g_{1i} \cdot \mathbf{f}_1^4 \cdot \mathbf{Df}] h^6 +$$
$$+ \tfrac{1}{720}[a_i^6 \cdot \mathbf{D}^6\mathbf{f} + 30a_i^4 c_{1i} \cdot \mathbf{Df} \cdot \mathbf{D}^4\mathbf{f}_1 + 60a_i^3 c_{2i} \cdot \mathbf{D}^2\mathbf{f} \cdot \mathbf{D}^3\mathbf{f}_1 +$$
$$+ 120(a_i^3 d_{1i} + d_{1i}^{(3)}) \mathbf{f}_1 \cdot \mathbf{Df} \cdot \mathbf{D}^3\mathbf{f}_1 + 180a_i^2 c_{1i}^2 (\mathbf{Df})^2 \cdot \mathbf{D}^2\mathbf{f}_2 +$$
$$+ 60a_i^2 c_{3i} \cdot \mathbf{D}^3\mathbf{f} \cdot \mathbf{D}^2\mathbf{f}_1 + 360(a_i^2 d_{1i}^{(1)} + a_i d_{1i}^{(2)}) \mathbf{Df} \cdot \mathbf{Df}_1 \cdot \mathbf{D}^2\mathbf{f}_1 +$$
$$+ 180(a_i^2 d_{2i} + d_{2i}^{(2)}) \mathbf{f}_1 \cdot \mathbf{D}^2\mathbf{f} \cdot \mathbf{D}^2\mathbf{f}_1 + 120c_{1i}^3 \cdot \mathbf{f}_3(\mathbf{Df})^3 +$$
$$+ 360(a_i^2 e_{1i} + e_{1i}^{(2)} + \varepsilon_{1i}^{(2)}) \mathbf{f}_1^2 \cdot \mathbf{Df} \cdot \mathbf{D}^2\mathbf{f}_1 + 360a_i c_{1i} c_{2i} \cdot \mathbf{Df} \cdot \mathbf{D}^2\mathbf{f} \cdot \mathbf{Df}_2 +$$
$$+ 360(2a_i c_{1i} d_{1i} + E_{1i}^{(1)}) \mathbf{f}_1(\mathbf{Df})^2 \cdot \mathbf{Df}_2 + 30a_i c_{4i} \cdot \mathbf{D}^4\mathbf{f} \cdot \mathbf{Df}_1 +$$
$$+ 360a_i d_{2i}^{(1)} \cdot \mathbf{D}^2\mathbf{f}(\mathbf{Df}_1)^2 + 360(a_i E_{1i} + 2c_{1i} d_{1i}^{(1)}) \mathbf{f}_2(\mathbf{Df})^2 \cdot \mathbf{Df}_1 +$$
$$+ 90c_{2i}^2 \cdot \mathbf{f}_2(\mathbf{D}^2\mathbf{f})^2 + 360(a_i e_{2i} + e_{2i}^{(1)} + \varepsilon_{2i}^{(1)}) \mathbf{f}_1^2 \cdot \mathbf{D}^2\mathbf{f} \cdot \mathbf{Df}_1 +$$
$$+ 120(a_i d_{3i} + d_{3i}^{(1)}) \mathbf{f}_1 \cdot \mathbf{D}^3\mathbf{f} \cdot \mathbf{Df}_1 + 120c_{1i}c_{3i} \cdot \mathbf{f}_2 \cdot \mathbf{Df} \cdot \mathbf{D}^3\mathbf{f} +$$
$$+ 720(a_i e_{1i}^{(1)} + a_i \varepsilon_{1i}^{(1)} + {}^1H_{1i}^{(1)}) \mathbf{f}_1 \cdot \mathbf{Df}(\mathbf{Df}_1)^2 + 720(a_i g_{1i} + g_{1i}^{(1)} +$$
$$+ G_{1i}^{(1)} + \mathcal{G}_{1i}^{(1)}) \mathbf{f}_1^3 \cdot \mathbf{Df} \cdot \mathbf{Df}_1 + 360(c_{1i}d_{2i} + c_{2i}d_{1i} + \mathcal{C}_{2i}) \mathbf{f}_1\mathbf{f}_2 \cdot \mathbf{Df} \cdot \mathbf{D}^2\mathbf{f} +$$
$$+ 360(2c_{1i}e_{1i} + d_{1i}^2 + 2\mathcal{D}_{1i} + \mathcal{E}_{1i}) \mathbf{f}_1^2\mathbf{f}_2(\mathbf{Df})^2 + 6c_{5i} \cdot \mathbf{f}_1 \cdot \mathbf{D}^5\mathbf{f} +$$
$$+ 30d_{4i} \cdot \mathbf{f}_1^2 \cdot \mathbf{D}^4\mathbf{f} + 120e_{3i} \cdot \mathbf{f}_1^3 \cdot \mathbf{Df} + 360g_{2i} \cdot \mathbf{f}_1^4 \cdot \mathbf{D}^2\mathbf{f} +$$
$$+ 720l_{1i} \cdot \mathbf{f}_1^5 \cdot \mathbf{Df}] h^7 .$$

As we can see the coefficients $\mathbf{k}_i$ for $i=6, 7, 8, 9, 10$ have both the same form and the same number of terms. Therefore we could write the expressions from (11_6) to (11_{10}) as (11_i) for $i = 6, 7, 8, 9, 10$.

In the formulas (11_1) to (11_{10}) we have used the following notation:

(12)
$$c_{1i} = \sum_{k=2}^{i} a_{k-1} b_{ik} = c(i, 2/1)$$
$$c_{2i} = \sum_{k=2}^{i} a_{k-1}^2 b_{ik} = c(i, 2/2)$$
$$c_{3i} = \sum_{k=2}^{i} a_{k-1}^3 b_{ik} = c(i, 2/3)$$
$$c_{4i} = \sum_{k=2}^{i} a_{k-1}^4 b_{ik} = c(i, 2/4)$$

$$c_{5i} = \sum_{k=2}^{i} a_{k-1}^{5} b_{ik} = c(i, 2/5)$$

$$d_{1i} = \sum_{k=3}^{i} b_{ik} c_{1,k-1} = c(i, 3/0/1, 1)$$

$$d_{2i} = \sum_{k=3}^{i} b_{ik} c_{2,k-1} = c(i, 3/0/1, 2)$$

$$d_{3i} = \sum_{k=3}^{i} b_{ik} c_{3,k-1} = c(i, 3/0/1,3)$$

$$d_{4i} = \sum_{k=3}^{i} b_{ik} c_{4,k-1} = c(i, 4/0/1, 4)$$

$$d_{1i}^{(1)} = \sum_{k=3}^{i} a_{k-1} b_{ik} c_{1,k-1} = c(i, 3/1/1,1)$$

$$d_{2i}^{(1)} = \sum_{k=3}^{i} a_{k-1} b_{ik} c_{2,k-1} = c(i, 3/1/1, 2)$$

$$d_{3i}^{(1)} = \sum_{k=3}^{i} a_{k-1} b_{ik} c_{3,k-1} = c(i, 3/1/1, 3)$$

$$d_{1i}^{(2)} = \sum_{k=3}^{i} a_{k-1}^{2} b_{ik} c_{1,k-1} = c(i, 3/2/1,1)$$

$$d_{2i}^{(2)} = \sum_{k=3}^{i} a_{k-1}^{2} b_{ik} c_{2,k-1} = c(i, 3/2/1, 2)$$

$$d_{1i}^{(3)} = \sum_{k=3}^{i} a_{\kappa-1}^{3} b_{ik} c_{1,k-1} = c(i, 3/3/1, 1)$$

$$E_{1i} = \sum_{k=3}^{i} b_{i,k} c_{1,k-1}^{2} = c(i, 3/0/2,1)$$

$$E_{1i}^{(1)} = \sum_{k=3}^{i} a_{k-1} b_{ik} c_{1,k-1}^{2} = c(i, 3/1/2, 1)$$

$$\mathscr{C}_{2i} = \sum_{k=3}^{i} b_{ik} c_{1,k-1} c_{2,k-1} = c(i, 3/0/1, 1 : 1, 2)$$

$$e_{1i} = \sum_{k=4}^{i} b_{ik} d_{1,k-1} = c(i, 4/0/0, 0/1, 0, 1, 1)$$

$$e_{2i} = \sum_{k=4}^{i} b_{i,k} d_{2,k-1} = c(i, 4/0/0, 0/1, 0, 1, 2)$$

$$e_{3i} = \sum_{k=4}^{i} b_{ik} d_{3,k-1} = c(i, 4/0/0, 0/1, 0, 1, 3)$$

$$e_{1i}^{(1)} = \sum_{k=4}^{i} a_{k-1} b_{ik} d_{1,k-1} = c(i, 4/1/0, 0/1, 0, 1, 1)$$

$$e_{2i}^{(1)} = \sum_{k=4}^{i} a_{k-1} b_{ik} d_{2,k-1} = c(i, 4/1/0, 0/1, 0, 1, 2)$$

$$e_{1i}^{(2)} = \sum_{k=4}^{i} a_{k-1}^2 b_{ik} d_{1,k-1} = c(i, 4/2/0, 0/1, 0, 1, 1)$$

$$\varepsilon_{1i}^{(1)} = \sum_{k=4}^{i} b_{ik} d_{1,k-1}^{(1)} = c(i, 4/0/0, 0/1, 1, 1, 1)$$

$$\varepsilon_{2i}^{(1)} = \sum_{k=4}^{i} b_{ik} d_{2,k-1}^{(1)} = c(i, 4/0/0, 0/1, 1, 1, 2)$$

$$\varepsilon_{1i}^{(2)} = \sum_{k=4}^{i} b_{ik} d_{1,k-1}^{(2)} = c(i, 4/0/0, 0/1, 2, 1, 1)$$

$${}^1H_{1i}^{(1)} = \sum_{k=4}^{i} a_{k-1} b_{ik} d_{1,k-1}^{(1)} = c(i, 4/1/0, 0/1, 1, 1, 1)$$

$$\mathscr{E}_{1i} = \sum_{k=4}^{i} b_{ik} E_{1,k-1} = c(i, 4/0/0, 0/1, 0, 2, 1)$$

$$\mathscr{D}_{1i} = \sum_{k=4}^{i} b_{ik} c_{1,k-1} d_{1,k-1} = c(i, 4/0/1, 1/1, 0, 1, 1)$$

$$g_{1i} = \sum_{k=5}^{i} b_{ik} e_{1,k-1} = c(i, 5/0/0/0/1, 0, 0^2, 1, 0, 1, 1)$$

$$g_{2i} = \sum_{k=5}^{i} b_{ik} e_{2,k-1} = c(i, 5/0/0/0/1, 0, 0^2, 1, 0, 1, 2)$$

$$\mathscr{G}_{1i}^{(1)} = \sum_{k=5}^{i} b_{ik} \varepsilon_{1,k-1}^{(1)} = c(i, 5/0/0/0/1, 0, 0^2, 1, 1, 1, 1)$$

$$g_{1i}^{(1)} = \sum_{k=5}^{i} a_{k-1} b_{ik} e_{1,k-1} = c(i, 5/1/0/0/1, 0, 0^2, 1, 0, 1, 1)$$

$$G_{1i}^{(1)} = \sum_{k=5}^{i} b_{ik} e_{1,k-1}^{(1)} = c(i, 5/0/0/0/1, 1, 0^2, 1, 0, 1, 1)$$

$$l_{1i} = \sum_{k=6}^{i} b_{ik} g_{1,k-1} = c(i, 6/0/0/0/0/1, 0, 0^2, 1, 0^3, 1, 0, 1, 1)$$

By inserting $\boldsymbol{k}_i$'s into the formula (9) we obtain a series that we compare with the series (7). We define the indefinite coefficients p_i, a_i and b_{ij} so that the first 7 terms ($r = 7$) of both series coincide. In this way we get the condition equations of the 7th order and the 11th stage, which were presented in the article [5]:

$$\sum_{i=0}^{10} p_i = 1\,, \quad \sum_{i=1}^{10} p_i a_i^k = \frac{1}{k+1}\,; \quad k = 1, 2, 3, 4, 5, 6\,; \tag{13}$$

$$\sum_{i=2}^{10} p_i a_i^k c(i, 2/n) = \frac{1}{(n+1)(n+k+2)}\,; \quad k = 0, 1, 2, 3, 4\,; \quad n = 1, 2, 3, 4, 5\,;$$

$$n + k \leqq 5;$$

$$\sum_{i=2}^{10} p_i a_i^k c(i, 2/n_1)\, c(i, 2/n) = \frac{1}{(n_1+1)(n+1)(n_1+n+k+3)}$$

$$n_1 = 1, 2\,; \quad n = 1, 2, 3\,; \quad k = 0, 1, 2\,; \quad n_1 + n + k \leqq 4\,;$$

$$\sum_{i=2}^{10} p_i c^3(i, 2/1) = \tfrac{1}{56}$$

$$\sum_{i=3}^{10} p_i a_i^k c(i, 3/\alpha/\beta, n) = \frac{1}{(n+1)^\beta (n+\alpha+\beta+2)(n+\alpha+\beta+k+3)}$$

$$n = 1, 2, 3, 4\,; \quad k = 0, 1, 2, 3\,; \quad \alpha = 0, 1, 2, 3\,; \quad \beta = 1, 2\,; \quad n + \alpha + \beta + k \leqq 4\,;$$

$$\sum_{i=3}^{10} p_i a_i^k c(i, 2/\gamma)\, c(i, 3/\alpha/1, n) = \frac{1}{(\gamma+1)(n+1)(n+\alpha+2)(n+\alpha+\gamma+k+4)}$$

$$n = 1, 2\,; \quad k = 0, 1\,; \quad \gamma = 1, 2\,; \quad \alpha = 0, 1\,; \quad n + \alpha + \gamma + k \leqq 3\,;$$

$$\sum_{i=3}^{10} p_i c^2(i, 3/0/1, 1) = \tfrac{1}{252}\,, \quad \sum_{i=3}^{10} p_i c(i, 3/0/1, 1 : 1, 2) = \tfrac{1}{252}$$

$$\sum_{i=4}^{10} p_i a_i^k c(i, 4/\alpha/0, 0/1, \beta, 1, n) =$$

$$= \frac{1}{(n+1)(n+\beta+2)(n+\alpha+\beta+3)(n+\alpha+\beta+k+4)}$$

$$n = 1, 2, 3\,; \quad k = 0, 1, 2\,; \quad \beta = 0, 1, 2\,; \quad \alpha = 0, 1, 2\,; \quad n + \alpha + \beta + k \leqq 3\,;$$

$$\sum_{i=4}^{10} p_i c(i, 4/0/0, 0/1, 0, 2, 1) = \tfrac{1}{420}\,, \quad \sum_{i=4}^{10} p_i c(i, 4/0/1, 1/1, 0, 1, 1) = \tfrac{1}{720}$$

$$\sum_{i=4}^{10} p_i c(i, 2/1)\, c(i, 4/0/0, 0/1, 0, 1, 1) = \tfrac{1}{336}$$

$$\sum_{i=5}^{10} p_i a_i^k c(i, 5/\alpha/0/0/1, 0, 0^2, 1, 0, 1, n) =$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+\alpha+4)(n+\alpha+k+5)}$$

$$n = 1, 2; \quad k = 0, 1\,; \quad \alpha = 0, 1\,; \quad n + \alpha + k \leqq 2\,;$$

$$\sum_{i=5}^{10} p_i c(i, 5/0/0/0/1, 1, 0^2, 1, 0, 1, 1) = \tfrac{1}{1260}$$

$$\sum_{i=5}^{10} p_i c(i, 5/0/0/0/1, 0, 0^2, 1, 1, 1, 1) = \tfrac{1}{1680}$$

$$\sum_{i=6}^{10} p_i c(i, 6/0/0/0/0/1, 0, 0^2, 1, 0^3, 1, 0, 1, 1) = \tfrac{1}{5040}$$

As we can see, in the condition equations (13) we used a more effective notation (introduced in the paper [6]), which makes it possible to write the 59 equations of this method more briefly.

In order to derive the Runge-Kutta formulas we need to find the solution of the above mentioned condition equations. We seek the solution in the form of rational numbers, because then the residues of the equations after introducing the values are zeros and the solution is exact up to the term h^7 (there are no round-off errors). For finding the solution of the condition equations we have used the linearization of the condition equations, which was described in the paper [6]. In this way we obtain the following formulas:

$$(14)\qquad \begin{aligned}
\mathbf{k}_0 &= h \,.\, \mathbf{f}[x_n; \mathbf{y}_n]\\
\mathbf{k}_1 &= h \,.\, \mathbf{f}[x_n + \tfrac{2}{36}h; \mathbf{y}_n + \tfrac{1}{18}\mathbf{k}_0]\\
\mathbf{k}_2 &= h \,.\, \mathbf{f}[x_n + \tfrac{3}{36}h; \mathbf{y}_n + \tfrac{1}{60}(4\mathbf{k}_0 + \mathbf{k}_1)]\\
\mathbf{k}_3 &= h \,.\, \mathbf{f}[x_n + \tfrac{4}{36}h; \mathbf{y}_n + \tfrac{1}{180}(-181\mathbf{k}_0 + 171\mathbf{k}_1 + 30\mathbf{k}_2)]\\
\mathbf{k}_4 &= h \,.\, \mathbf{f}[x_n + \tfrac{5}{36}h; \mathbf{y}_n + \tfrac{1}{180}(-902\mathbf{k}_0 + 2937\mathbf{k}_1 - 2040\mathbf{k}_2 + 30\mathbf{k}_3)]\\
\mathbf{k}_5 &= h \,.\, \mathbf{f}[x_n + \tfrac{1}{6}h; \mathbf{y}_n + \tfrac{1}{24}(-15\mathbf{k}_0 + 48\mathbf{k}_1 - 31\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)]\\
\mathbf{k}_6 &= h \,.\, \mathbf{f}[x_n + \tfrac{2}{6}h; \mathbf{y}_n + \tfrac{1}{30}(17\mathbf{k}_0 - 48\mathbf{k}_1 + 31\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 + 12\mathbf{k}_5)]\\
\mathbf{k}_7 &= h \,.\, \mathbf{f}[x_n + \tfrac{3}{6}h; \mathbf{y}_n + \tfrac{1}{80}(192\mathbf{k}_0 - 528\mathbf{k}_1 + 341\mathbf{k}_2 - 11\mathbf{k}_3 -\\
&\quad - 11\mathbf{k}_4 + 32\mathbf{k}_5 + 25\mathbf{k}_6)]\\
\mathbf{k}_8 &= h \,.\, \mathbf{f}[x_n + \tfrac{4}{6}h; \mathbf{y}_n + \tfrac{1}{66}(54\mathbf{k}_0 - 144\mathbf{k}_1 + 93\mathbf{k}_2 - 3\mathbf{k}_3 - 3\mathbf{k}_4 +\\
&\quad + 32\mathbf{k}_5 - 17\mathbf{k}_6 + 32\mathbf{k}_7)]\\
\mathbf{k}_9 &= h \,.\, \mathbf{f}[x_n + \tfrac{5}{6}h; \mathbf{y}_n + \tfrac{1}{3960}(-22\,876\mathbf{k}_0 + 64\,464\mathbf{k}_1 - 41\,633\mathbf{k}_2 +\\
&\quad + 1\,343\mathbf{k}_3 + 1\,343\mathbf{k}_4 - 656\mathbf{k}_5 - 460\mathbf{k}_6 - 40\mathbf{k}_7 + 1\,815\mathbf{k}_8)]\\
\mathbf{k}_{10} &= h \,.\, \mathbf{f}[x_n + h; \mathbf{y}_n + \tfrac{1}{902}(16\,139\mathbf{k}_0 - 45\,120\mathbf{k}_1 + 29\,140\mathbf{k}_2 - 940\mathbf{k}_3 -\\
&\quad - 940\mathbf{k}_4 + 1\,828\mathbf{k}_5 - 769\mathbf{k}_6 + 2\,752\mathbf{k}_7 - 1\,980\mathbf{k}_8 + 792\mathbf{k}_9)]\\
\mathbf{k} &= \tfrac{1}{840}(41\mathbf{k}_0 + 216\mathbf{k}_5 + 27\mathbf{k}_6 + 272\mathbf{k}_7 + 27\mathbf{k}_8 + 216\mathbf{k}_9 + 41\mathbf{k}_{10})\,.
\end{aligned}$$

This procedure yields the Runge-Kutta formula of the 7th order and 11th stage for the solution of the ordinary differential equation of the 1st order, which is easily programmable.

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Súhrn

PRÍSPEVOK K RUNGEHO-KUTTOVÝM VZORCOM 7. RÁDU S RACIONÁLNYMI KOEFICIENTAMI PRE SYSTÉM DIFERENCIÁLNYCH ROVNÍC PRVÉHO RÁDU

ANTON HUŤA, VLADIMÍR PENJAK

Cieľom tohto článku je nájdenie Rungeho-Kuttových vzorcov 7. rádu s racionálnymi parametrami. Vzorce sú 11. stupňa. Ak porovnáme koeficienty rozvoja

$$\sum_{i=1}^{\infty} \frac{h^i}{i!} \frac{d^{i-1}}{dx^{i-1}} \mathbf{f}[x, \mathbf{y}(x)]$$

až po h^7 s rozvojom získaným postupným dosadzovaním do vzorca $h \cdot \mathbf{f}_i(\mathbf{k}_0, \mathbf{k}_1, \ldots, \ldots, \mathbf{k}_{i-1}) = \mathbf{k}_i$ pre $i = 1, 2, \ldots, 10$ a $\mathbf{k} = \sum_{i=0}^{10} p_i \mathbf{k}_i$ dostaneme sústavu 59 podmienkových rovníc o 65 neznámych (okrem prvej všetky rovnice sú nelineárne). Riešením tejto sústavy dostávame hodnoty parametrov Rungových-Kuttových vzorcov 7. rádu ako racionálne čísla.

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