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SOME REMARKS ABOUT THE MONOTONE INCLUSION FOR SOLUTIONS OF NONLINEAR EQUATIONS BY REGULA-FALSI-LIKE METHODS

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INTRODUCTION

In [5] and [6] Regula-falsi-like methods are considered in order to obtain sequences of upper and lower bounds for solutions of nonlinear equations. By using a generalization of Schmidt's concept of a divided difference operator the results of [6] are generalized in [7].

The method which is considered in [5], has the advantage that the order of convergence is $1 + \sqrt{2}$ instead of the typical order of $(1 + \sqrt{5})/2$ for Regula-falsi methods. In this paper the enclosure results of [5] are generalized in the same manner as it was done for [6] in [7]. But if this method is realized by operators which are not divided difference operators, it can only be shown that the order of convergence is 2. This fact was also mentioned in [5].

We also consider a new iteration method in this paper. It is derived from Schmidt's method, but it works with much less effort and it is quadratically convergent. It is more effective than other known methods in the sense of Ostrowski [4].

2. PRELIMINARIES

The definitions and properties used in connection with the cone which introduces a partial ordering in a Banach space B, are found in [8].

For $x, y \in \mathbb{R}^n$, $x \leqq y$ if and only if $x^{(i)} \leqq y^{(i)}$, $i = 1(1)\,n$. $S(B)$ means the set of all continuous linear operators $(B \to B)$.

Kantorovich Lemma [1]: Let B be a Banach space, which is partially ordered by a closed regular cone, and $A([x, y] \subset B \to B)$ a continuous, isotone maping. If $x \leqq A(x)$ and $A(y) \leqq y$, then A has a fixed point in $[x, y]$.

As a measure of the rate of convergence of iterative processes we use the R-order defined in [3].

In this paper B denotes a Banach space which is partially ordered by a closed regular cone.

In order to enclose the solutions of $F(x) = 0$, $F(V \subset B \to B)$ in an interval $[x_1, y_1] \subset V$, Schmidt [5] considers the following iteration method

$$\left\{ \begin{array}{l} F(y_k) + \delta F(y_{k-1}, y_k)(x_k - y_k) = 0 \\ F(y_k) + \delta F(x_k, y_k)(y_{k+1} - y_k) = 0 . \end{array} \right. \tag{3.1}$$

$\delta F(V \times V \subset B \times B \to S(B))$ denotes a mapping which satisfies

$$\delta F(x; y)(x - y) = F(x) - F(y) . \tag{3.2}$$

This concept of a divided difference operator can be generalized in the following manner.

Definition 3.1. *Given an operator $F(V \subset B \to B)$. A mapping $A(M := \{(x, y) \mid x, y \in V; x, y$ comparable$\} \to S(B))$ is called generalized divided difference operator ("verallgemeinerte Steigung") of F if*

$$A(x, y)(x - y) \geqq F(x) - F(y) , \quad (x, y) \in M . \tag{3.3}$$

A realization of this concept in the case $B = \mathbb{R}^N$ is given in [7]. By using this generalization the following theorem can be proved. It generalizes an enclosure theorem proved in [5].

Theorem 3.1. *Let $F(V \subset B \to B)$ be a continuous mapping. Suppose there are $y_0, y_1 \in V$, such that*

$$y_0 \geqq y_1, F(y_1) \leqq 0 .$$

The mapping A is a generalized divided difference operator with the property

$$A(u_1, v_1) \geqq A(u_2, v_2) , \quad u_1 \geqq u_2 , \quad v_1 \geqq v_2 . \tag{3.4}$$

Assume there exist a nonnegative, injective mapping $T \in S(B)$ and a mapping $G(B \to B)$ such that

$$\left\{ \begin{array}{l} TG \leqq I , \\ T\{G + A(u, v)\} \geqq 0 \quad \textit{for all} \quad (u, v) \in M , \end{array} \right. \tag{3.5}$$

and an $x_1 \in V$; $[x_1, x_1] \subset V$, which is a solution of the equation

$$F(y_1) + A(y_0, y_1)(x_1 - y_1) = 0. \tag{3.6}$$

Then the iteration method

$$\left\{ \begin{array}{l} F(y_k) + A(y_{k-1}, y_k)(x_k - y_k) = 0 , \\ F(y_k) + A(x_k, y_k)(y_{k+1} - y_k) = 0 \end{array} \right. \tag{3.6a}$$

is well defined. This means there exist solutions x_{k+1}, y_{k+1} of the linear equations (3.6).

The monotone sequences (x_k), (y_k) have limits x^*. y^* , y^* is a solution of $F(x) = 0$ and we get the monotone enclosure

$$(3.7) \qquad x_1 \leqq \ldots \leqq x_k \leqq x_{k+1} \leqq \ldots \leqq x^* \leqq z^* \leqq y^* \leqq \ldots \leqq y_{k+1} \leqq y_k \leqq \ldots y_1$$

for any solution $z^* \in [x_1, y_1]$ of $F(x) = 0$.

Moreover, if there exists an operator $S(B \to B)$ with the properties

$$(3.8) \qquad S \leqq -A(u, v), \quad u \geqq v, \quad 0 \leqq S^{-1} \in S(B),$$

then $x^* = y^*$.

Proof. Let $x_1 \in V$ be a solution of the equation (3.6) such that $[x_1, y_1] \subset V$. Then we get

$$F(x_1) \geqq F(y_1) + A(y_1, x_1)(x_1 - y_1) \geqq F(y_1) + A(y_0, y_1)(x_1 - y_1) = 0.$$

We consider the continuous operator H defined by

$$H(z) := z + TF(z), \quad z \in [x_1, y_1].$$

$z_1 \geqq z_2$ implies

$$H(z_1) - H(z_2) = z_1 - z_2 + T\{F(z_1) - F(z_2)\} \geqq z_1 - z_2 + TA(z_1, z_2)(z_1 - z_2) \geqq$$
$$\geqq T\{G + A(z_1, z_2)\}(z_1 - z_2) \geqq 0,$$

so H is isotone. Together with

$$H(x_1) = x_1 + TF(x_1) \geqq x_1,$$
$$H(y_1) = y_1 + TF(y_1) \leqq y_1,$$

Kantorovich Lemma guarantees the existence of a fixed point $z^* \in [x_1, y_1]$, which is a solution of $F(x) = 0$.

It will be proved by induction that

$$(3.9) \qquad x_k \leqq z^* \leqq y_k \leqq y_{k-1}, F(y_k) \leqq 0.$$

(3.9) is correct for $k = 1$.

We define the continuous operator

$$H_k(z) := z + T\{F(y_k) + A(x_k, y_k)(z - y_k)\}, \quad z \in [z^*, y_k].$$

$z_1 \geqq z_2$ implies

$$H_k(z_1) - H_k(z_2) = z_1 - z_2 + TA(x_k, y_k)(z_1 - z_2) \geqq$$
$$\geqq T\{G + A(x_k, y_k)\}(z_1 - z_2) \geqq 0,$$

so H_k is isotone.

Because of

$$H_k(z^*) = z^* + T\{F(y_k) + A(x_k, y_k)(z^* - y_k)\} \geqq$$
$$\geqq z^* + T\{F(y_k) + A(z^*, y_k)(z^* - y_k)\} \geqq z^* + TF(z^*) = z^*,$$
$$H_k(y_k) = y_k + TF(y_k) \leqq y_k,$$

there exists a fixed point y_{k+1} of H_k, which is a solution of the equation

$$F(y_k) + A(x_k, y_k)(y_{k+1} - y_k) = 0$$

and has the property

$$F(y_{k+1}) \leqq F(y_k) + A(y_{k+1}, y_k)(y_{k+1} - y_k) \leqq F(y_k) + A(x_k, y_k)(y_{k+1} - y_k) = 0.$$

The operator

$$\bar{H}_k(z) := z + T\{F(y_{k+1}) + A(y_k, y_{k+1})(z - y_{k+1})\}, \quad z \in [x_k, z^*]$$

is continuous and isotone.

Since

$$\bar{H}_k(x_k) = x_k + T\{F(y_{k+1}) + A(y_k, y_{k+1})(x_k - y_{k+1})\} =$$
$$= x_k + T\{F(y_{k+1}) + A(y_k, y_{k+1})(x_k - y_k) + A(y_k, y_{k+1})(y_k - y_{k+1})\} \geqq$$
$$\geqq x_k + T\{F(y_k) + A(y_k, y_{k+1})(x_k - y_k)\} \geqq$$
$$\geqq x_k + T\{F(y_k) + A(y_{k-1}, y_k)(x_k - y_k)\} = x_k,$$
$$\bar{H}_k(z^*) = z^* + T\{F(y_{k+1}) + A(y_k, y_{k+1})(z^* - y_{k+1})\} \leqq$$
$$\leqq z^* + T\{F(y_{k+1}) + A(y_{k+1}, z^*)(z^* - y_{k+1})\} \leqq z^* + TF(z^*) = z^*,$$

the operator $\bar{H}_k$ has a fixed point x_{k+1} with the property

$$F(y_{k+1}) + A(y_k, y_{k+1})(x_{k+1} - y_{k+1}) = 0.$$

The statement (3.9) is now proved.

By the regularity of the cone the limits $x^* = \lim x_k$, $y^* = \lim y_k$ exist and it is obvious that the monotone enclosure (3.7) holds.

We have

$$0 \geqq TF(y_k) = -TA(x_k, y_k)(y_{k+1} - y_k) \geqq TG(y_{k+1} - y_k) \geqq y_{k+1} - x_k$$

and therefore by the continuity of the operators F and T we obtain

$$TF(y^*) = 0.$$

Using the injectivity of T we conclude $F(y^*) = 0$.

If we assume that there exists a mapping $S(B \to B)$ with the properties (3.8), we obtain

$$S(y^* - x_k) \leqq -A(y_{k-1}, y_k)(y^* - x_k) =$$
$$= -A(y_{k-1}, y_k)(y^* - y_k) - A(y_{k-1}, y_k)(y_k - x_k) =$$
$$= -A(y_{k-1}, y_k)(y^* - y_k) - F(y_k) \leqq S(y^* - y_k) - F(y_k)$$

and therefore

$$0 \leqq y^* - x_k \leqq y^* - y_k - S^{-1} F(y_k)\,.$$

It follows that $\lim x_k = y^*$. □

The following lemma gives a sufficient condition for the existence of a solution $x_1 \in V$ of the equation (3.6) with the property $[x_1, y_1] \subset V$.

Lemma 3.1. *Let $F(V - B \to B)$ be a mapping. We have $y_0, y_1 \in V$ such that $F(y_1) \leqq 0$.*

Assume there exist mappings $0 \leqq T, A(y_0, y_1) \in S(B)$ and $G, S(B \to B)$ such that

$$\text{(3.10)} \qquad \begin{cases} TG \leqq I\,, \quad T\{G + A(y_0, y_1)\} \geqq 0\,, \\ S \leqq -A(y_0, y_1)\,. \end{cases}$$

If there exists a solution y of the equation

$$\text{(3.11)} \qquad F(y_1) - S(y - y_1) = 0$$

with the property $[y, y_1] \subset V$, then the equation

$$F(y_1) + A(y_0, y_1)(x_1 - y_1) = 0$$

has a solution $x_1 \in V$ with the property $[x_1, y_1] \subset V$.

Proof. Let y be a solution of the equation

$$F(y_1) + S(y - y_1) = 0$$

such that $[y, y_1] \subset V$. We define the continuous and isotone operator

$$H(z) := z + T\{F(y_1) + A(y_0, y_1)(z - y_1)\}\,, \quad z \in [y, y_1]\,.$$

Since

$$H(y) = y + T\{F(y_1) + A(y_0, y_1)(y - y_1)\} \geqq$$
$$\geqq y + T\{F(y_1) - S(y - y_1)\} = y$$

and

$$H(y_1) = y_1 + TF(y_1) \leqq y_1\,,$$

there exists a fixed point $x_1 \in [y, y_1]$ of H, which is a solution of the equation

$$F(y_1) + A(y_0, y_1)(x_1 - y_1) = 0\,. \;\square$$

In [5] a convergence order of $1 + \sqrt{2}$ can be proved for the iteration method (3.6). If A is not a divided difference operator in the sense of Schmidt, under the same assumptions only a convergence order of 2 can be proved for this method. In the following chapter we will consider an iteration method, which is derived from the method (3.6). This method is more efficient in the sense of Ostrowski [4] than the other methods which are considered in [5], [6], [7].

4. A MORE EFFICIENT METHOD

We assume in this chapter that each subset of B which contains two elements, has a supremum [2].

Then we consider the iteration method

$$(4.1)\qquad \begin{cases} F(y_k) + A(x_k, y_k)(y_{k+1} - y_k) = 0\,, \\ z_{k+1} = y_{k+1} + Q\,F(y_{k+1})\,, \\ x_{k+1} = \sup\{z_{k+1}, x_1\}\,. \end{cases}$$

The sequence (y_k) is constructed in the same way as for (3.6). Nonetheless, the sequence (x_k) can be computed with less effort. Using a specific operator $Q \in S(B)$, the following theorem shows that by (4.1) we can get sequences (x_k), (y_k) which converge monotonously and enclose a solution of $F(x) = 0$.

Theorem 4.1. *Let $F(V := [x_1, y_1] \subset B \to B)$ be a continuous operator such that $F(y_1) \leqq 0$ and assume that there exists a solution $z^* \in V$ of $F(x) = 0$. A is generalized divided difference operator with the property*

$$A(u_1, v_1) \geqq A(u_2, v_2)\,, \quad u_1 \geqq u_2\,, \quad v_1 \geqq v_2\,.$$

If there exist an injective mapping $T \in S(B)$ and mappings $Q \in S(B)$, G, $S(B \to B)$ with the propertis

$$(4.2)\qquad \begin{cases} T\{G + A(u, v)\} \geqq 0 \quad \text{for all} \quad (u, v) \in M\,, \\ TG \leqq I\,, \quad QS \geqq I\,, \quad S \leqq -A(y_1, y_2)\,, \\ 0 \leqq T\,, \quad 0 \leqq Q\,, \end{cases}$$

then the iteration method (4.1) is well defined. z^* is the only solution of $F(x) = 0$ in the interval $[x_1, y_1]$. The monotone sequences (x_k), (y_k) have the same limit z^* and the monotone inclusion

$$(4.3)\qquad x_1 \leqq \ldots \leqq x_k \leqq x_{k+1} \leqq \ldots \leqq z^* \leqq \ldots \leqq y_{k+1} \leqq y_k \leqq \ldots \leqq y_1$$

holds.

If in addition

$$(4.4)\qquad \|A(x, z) - A(z, y)\| \leqq \alpha\{\|x - z\| + \|x - y\| + \|z - y\|\}$$

holds, if $x \leqq z \leqq y$ or $y \leqq z \leqq x$, $x, y, z, \in V$, the R-order of convergence of (4.1) is not less than 2.

Proof. The assumption that $F(x) = 0$ has a solution $z^* \in V$ is fulfilled if $F(x_1) \geqq 0$.

(See the proof of Theorem 3.1.) We show by induction: $x_k \leqq z^* \leqq y_k$, $F(y_k) \leqq 0$. This is correct for $k = 1$.

Now we define the operator

$$H_k(z) := z + T\{F(y_k) + A(x_k, y_k)(z - y_k)\}\,, \quad z \in [z^*, y_k]\,.$$

Since this operator is continuous and isotone, it follows from

$$H_k(z^*) = z^* + T\{F(y_k) + A(x_k, y_k)(z^* - y_k)\} \geqq$$
$$\geqq z^* + T\{F(y_k) + A(z^*, y_k)(z^* - y_k)\} \geqq z^* + TF(z^*) = z^*$$

and

$$H_k(y_k) = y_k + TF(y_k) \leqq y_k$$

that there exists a fixed point $y_{k+1} \in [z^*, y_k]$. y_{k+1} is a solution of the equation

$$F(y_k) + A(x_k, y_k)(y_{k+1} - y_k) = 0$$

with the property

$$F(y_{k+1}) \leqq A(y_{k+1}, y_k)(y_{k+1} - y_k) + F(y_k) \leqq$$
$$\leqq A(x_k, y_k)(y_{k+1} - y_k) + F(y_k) = 0\,.$$

We get

$$z^* - z_{k+1} = z^* + Q\,F(z^*) - \{y_{k+1} + Q\,F(y_{k+1})\} =$$
$$= z^* - y_{k+1} + Q\{F(z^*) - F(y_{k+1})\} \geqq z^* - y_{k+1} + Q\,A(y_{k+1}, z^*)(z^* - y_{k+1}) \geqq$$
$$\geqq \{I + Q\,A(y_{k+1}, y_{k+1})\}(z^* - y_{k+1}) \geqq \{I + Q\,A(y_1, y_2)\}(z^* - y_{k+1}) \geqq$$
$$\geqq \{I - QS\}(z^* - y_{k+1}) \geqq 0\,,$$

so that $x_{k+1} = \sup\{z_{k+1}, x_1\} \leq z^*$.

Because of $y_{k+1} \leqq y_k$ we obtain

$$z_{k+1} - z_k = y_{k+1} + Q\,F(y_{k+1}) - \{y_k + Q\,F(y_k)\} =$$
$$= y_{k+1} - y_k + Q\{F(y_{k+1}) - F(y_k)\} \geqq \{I + Q\,A(y_k, y_{k+1})\}(y_{k+1} - y_k) \geqq$$
$$\geqq \{I + Q\,A(y_1, y_2)\}(y_{k+1} - y_k) \geqq \{I - QS\}(y_{k+1} - y_k) \geqq 0\,.$$

Now we have proved that the sequences (x_k), (y_k) are monotonous and bounded, so that the limits $x^* = \lim x_k$, $y^* = \lim y_k$ exist by the regularity of the cone. Using the continuity of the operators F and T and the injectivity of T it follows from

$$0 \geqq TF(y_k) = T\{-A(x_k, y_k)(y_{k+1} - y_k)\} \geqq$$
$$\geqq TG(y_{k+1} - y_k) \geqq y_{k+1} - y_k$$

that $F(y^*) = 0$.

Since

$$z_{k+1} = y_{k+1} + Q\,F(y_{k+1}) \leqq x_{k+1} \leqq y_{k+1}$$

we get $x^* = y^*$.

In order to prove the statements about the order of convergence we define $r_k :=$ $:= \max\{\|x^* - x_k\|, \|y_k - x^*\|\}$.

Then

$$S(y_{k+1} - x^*) \leqq -A(x_k, y_k)(y_{k+1} - x^*) =$$
$$= -A(x_k, y_k)(y_{k+1} - y_k) - A(x_k, y_k)(y_k - x^*) =$$
$$= F(y_k) - F(x^*) - A(x_k, y_k)(y_k - x^*) \leqq \{A(y_k, x^*) - A(x_k, y_k)\}(y_k - x^*)$$

implies

$$0 \leqq y_{k+1} - x^* \leqq Q \,.\, S(y_{k+1} - x^*) \leqq Q\{A(y_k, x^*) - A(x_k, y_k)\}(y_k - x^*) =$$
$$= Q\{A(y_k, x^*) - A(x^*, x_k) + A(x^*, x_k) - A(x_k, y_k)\}(y_k - x^*)\,.$$

Since B is a partially ordered Banach space, which means that $\|x\| \leqq \beta\|y\|$ holds if $0 \leqq x \leqq y$, we obtain by the continuity of Q and (4.4)

$$\|y_{k+1} - x^*\| \leqq \gamma\{\|y_k - x^*\| + \|x^* - x_k\|\}\,\|y_k - x^*\|\,. \tag{4.5}$$

In the same way we get from

$$0 \leqq x^* - x_{k+1} \leqq x^* + Q\,F(x^*) - (y_{k+1} + Q\,F(y_{k+1})) \leqq$$
$$\leqq \{I + Q\,A(x^*, y_{k+1})\}(x^* - y_{k+1})\,,$$

using (4.4), the following estimate

$$\|x^* - x_{k+1}\| \leqq \delta\{\|y_k - x^*\| + \|x^* - x_k\|\}\,\|y_k - x^*\|\,. \tag{4.6}$$

From the estimates (4.5) and (4.6) we obtain

$$r_{k+1} \leqq \max\{\gamma, \delta\}\, r_k^2\,.$$

This guarantees that the R-order of our iteration method is not less than 2. □

The iteration method (4.1) has the advantage that we have to determine only one linear operator and to solve only one linear equation per iteration step. Since the convergence order of (4.1) is not less than 2, this method is more effective than the methods which are considered in [5], [6], [7].

Now we will give an answer how to find a suitable operator Q. In order to ensure the equality of the limits $\lim x_k$, $\lim y_k$ it is assumed in [5] and [6] that there exists an operator S such that the conditions

$$S \leqq -\delta\,F(u, v)\,, \quad S^{-1} \geqq 0 \tag{4.7}$$

hold. Then we can choose $Q := S^{-1}$. If $B = \mathbb{R}^N$ we consider the important case that $S = (s_{ij})$ has the property $\sum_i s_{ij} > 0$ for all j. Then it is not necessary to compute the inverse of S. We can set $Q = (q_{ij})$ with

$$q_{ij} = \left(\sum_{l=1}^{N} s_{li}\right)^{-1} \quad \text{or} \quad q_{ij} = \max\left(\sum_{l=1}^{N} s_{lk}\right)^{-1}\,.$$

If this matrix Q is used the condition (4.2) holds.

5. REMARKS

There are other versions of Theorem 4.1 corresponding to various sign configurations. We indicate these versions schematically in Table 5.1, where the first row represents

Theorem 4.1. *We define*: $(i = 0, 1)$

$$A_i: \begin{cases} (-1)^i x_1 \leqq (-1)^i z^* \leqq (-1)^i y_1 , \\ (-1)^i x_{k+1} = \sup \{(-1)^i z_{k+1}, (-1)^i x_k\} , \end{cases}$$

$$B_i: \begin{cases} (-1)^i F(y_1) \leqq 0 \; A(M \to S(B)) \text{ is a mapping with the properties} \\ (-1)^i A(u_1, v_1) \geqq (-1)^i A(u_2, v_2), u_1 \geqq u_2, v_1 \geqq v_2, \\ (-1)^i \{F(u) - F(v)\} \leqq (-1)^i A(u, v)(u - v). \end{cases}$$

$$C_i: \begin{cases} \text{There exist a nonnegative injective mapping } T \in S(B) \text{ and mappings } Q \in S(B). \\ G, S(B \to B) \text{ with the properties} \\ T\{G + (-1)^i A(u, v)\} \geqq 0 \quad \text{for all} \quad (u, v) \in M , \\ TG \leqq I, (-1)^i S \leqq (-1)^{i+1} A(y_2, y_2), \\ 0 \leqq (-1)^i Q, QS \leqq I, \end{cases}$$

$$D_i: \quad (-1)^i x_k \leqq (-1)^i x_{k+1} \leqq (-1)^i z^* \leqq (-1)^i y_{k+1} \leqq (-1)^i y_k.$$

Table 5.1

I	A_0,	B_0,	$C_0 \Rightarrow D_0$
II	A_1,	B_0,	$C_1 \Rightarrow D_1$
III	A_0,	B_1,	$C_1 \Rightarrow D_0$
IV	A_1,	B_1,	$C_0 \Rightarrow D_1$

Table 5.1 means: If we replace the corresponding assumptions of Theorem 4.1 as indicated in Table 5.1 the enclosure statements D_i follow.

6. NUMERICAL RESULTS

We consider the problem

$$(6.1) \qquad \begin{cases} x'' = g(t, x), \quad x(0) = x(1) = 0 \\ g(t, x) := \begin{cases} t^2 x + 2, & -x < t^2 \\ -x^2 + 2, & -x \geqq t^2. \end{cases} \end{cases}$$

This problem possesses a unique solution in the interval $[z_1, z_2]$, where $z_1(t) := t(1 - t)$, $z_2(t) = 0$. In order to compute a numerical approximation to the solution of (5.1) we consider the following discrete analog.

Let

$$t^{(j)} = jh\,, \quad h = 1/(n+1)\,, \quad j = 0, \ldots, n+1$$

be a uniform subdivision of the interval $[0, 1]$. We approximate $x''(t^{(j)})$ at each point $t^{(j)}$ by the second central difference quotient. Using this approximation in (5.1) we obtain an approximate solution of (5.1) by solving the nonlinear equation

$$F(x) = Cx - h^2 \,.\, \gamma(x) = 0 \tag{6.2}$$

with

$$C := \begin{bmatrix} -2 & 1 & & \\ & 1 & & \\ & & & 1 \\ & & 1 & -2 \end{bmatrix}, \quad \gamma(x) := \begin{bmatrix} g(t^{(1)}, & x^{(1)}) \\ & \vdots \\ & \vdots \\ g(t^{(n)}, & x^{(n)}) \end{bmatrix}.$$

We consider the interval

$$V := [x_1, y_1] \subset \mathbb{R}^N, x_1^{(j)} = t^{(j)}(t^{(j)} - 1)\,, \quad y_1^{(j)} = 0\,, \quad j = 1, \ldots, n$$

and set

$$A(x, y) := \begin{bmatrix} \alpha^{(1)}(x, y) & 1 & \\ 1 & & 1 \\ & & \alpha^{(n)}(x, y) \\ & & 1 \end{bmatrix},$$

$$\alpha^{(j)}(x, y) := \begin{cases} -2 - h^2(t^{(j)})^2, & -x^{(i)} < (t^{(i)})^2\,, \\ -2 + h^2(x^{(j)} + y^{(j)})\,, & -x^{(i)} \geqq (t^{(i)})^2\,, \end{cases}$$

$$G := -A(x_1, x_1)\,, \quad T = G^{-1}\,, \quad S = -A(y_2, y_1)\,.$$

Since $S = (s_{ij})$ has the property

$$\sum_i s_{ij} > 0 \quad \text{for all} \quad j$$

we can choose $Q = (q_{ij})$ defined by $q_{ij} = \max\limits_k \left(\sum\limits_l s_{lk}\right)^{-1}$. Then the assumptions of Theorem 4.1 are fulfilled. (6.2) is solved iteratively by the iteration (4.1).

For $x_k^{(5)}$ we get the following enclosing intervals

Table 6.1

Number of Iterations	Enclosing interval
1	[−0·250000000, −0·241945613]
2	[−0·242097229, −0·242002224]
3	[−0·242002573, −0·242002573].

The numerical results were obtained on the CD computer of the Technical University of Berlin.

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Souhrn

NĚKOLIK POZNÁMEK O MONOTONNÍ INKLUZI PRO ŘEŠENÍ NELINEÁRNÍCH ROVNIC METODAMI TYPU REGULA FALSI

NORBERT SCHNEIDER

V článku je zobecněn výsledek J. W. Smidta o monotonní inkluzi řešení nelineárních rovnic (tj. o existenci klesající posloupnosti intervalů obsahujících řešení) a je udána iterační metoda efektivnější než metody dosud známé. Pro tuto metodu jsou rovněž dokázána tvrzení o monotonní inkluzi.

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