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NECESSARY CONDITIONS FOR THE CONVERGENCE OF THE GENERALIZED PERIODIC OVERIMPLICIT MULTISTEP METHOD

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First of all let us recall the definition of the general periodic overimplicit multistep method (shortly GPOM method) the basic properties of which were studied in [1]. The method is used for solving the ordinary differential equation

$$y' = f(x, y), \quad x \in [a, b] \tag{1}$$

with the initial condition

$$y(a) = \eta . \tag{2}$$

The essence of this method consists in the fact that in one step of the method, the approximate solution is computed simultaneously at k points supposing that it is known at l points ($1 \leqq l \leqq k$). To be able to describe the method precisely let us introduce some notation. The points

$$x_{jk} = a + jmh, \quad j = 0, 1, \dots$$

where m is a positive integer and $h > 0$ is the integration step will be called *basic points* and the points

$$x_{jk+i} = x_{jk} + \mu_i h, \quad i = 1, \dots, k - 1$$

where μ_i are any real numbers will be called intermediate points.

The approximate solution at the point x_{jk+i} ($j = 0, 1, \dots; i = 0, \dots, k - 1$) will be denoted by $\tilde{y}(x_{jk+i})$. Further, let us define the l-dimensional vector $\boldsymbol{x}_j$ by

$$\boldsymbol{x}_j = (x_{jk}, \dots, x_{jk+l-1})^{\mathrm{T}}$$

and the l-dimensional vector $\tilde{\boldsymbol{y}}(\boldsymbol{x}_j)$ of the approximate solution at these points by

$$\tilde{\boldsymbol{y}}(\boldsymbol{x}_j) = (\tilde{y}(x_{jk}), \dots, \tilde{y}(x_{jk+l-1}))^{\mathrm{T}}$$

and analogously, the vectors z_j and $\tilde{\boldsymbol{y}}(z_j)$ by

$$z_j = (x_{jk+l}, \ldots, x_{(j+1)k+l-1})^{\mathsf{T}}$$

and

$$\tilde{\boldsymbol{y}}(z_j) = (\tilde{y}(x_{jk+l}), \ldots, \tilde{y}(x_{(j+1)k+l-1}))^{\mathsf{T}},$$

and the vector-valued functions $\boldsymbol{f}(\boldsymbol{x}_j, \tilde{\boldsymbol{y}}(\boldsymbol{x}_j))$ and $\boldsymbol{f}(z_j, \tilde{\boldsymbol{y}}(z_j))$ by

$$\boldsymbol{f}(\boldsymbol{x}_j, \tilde{\boldsymbol{y}}(\boldsymbol{x}_j)) = (f(x_{jk}, \tilde{y}(x_{jk})), \ldots, f(x_{jk+l-1}, \tilde{y}(x_{jk+l-1})))^{\mathsf{T}}$$

and

$$\boldsymbol{f}(z_j, \tilde{\boldsymbol{y}}(z_j)) = (f(x_{jk+l}, \tilde{y}(x_{jk+l})), \ldots, f(x_{(j+1)k+l-1}, \tilde{y}(x_{(j+1)k+l-1})))^{\mathsf{T}}.$$

Finally, let $\boldsymbol{B}, \boldsymbol{D}$ be any $k \times l$ matrices and $\boldsymbol{C}$ any $k \times k$ matrix. Then the system

(3) $$\tilde{\boldsymbol{y}}(z_j) = \boldsymbol{B}\,\tilde{\boldsymbol{y}}(\boldsymbol{x}_j) + h\boldsymbol{C}\boldsymbol{f}(z_j, \tilde{\boldsymbol{y}}(z_j)) + h\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_j, \tilde{\boldsymbol{y}}(\boldsymbol{x}_j)), \quad j = 0, 1, \ldots$$

is called GPOM method.

In [1] sufficient conditions for the convergence of GPOM method were formulated. The aim of the present paper is to find necessary conditions for the convergence of the method. Since these conditions depend obviously on the definition of the convergence, let us first of all recall it.

Definition 1. *The GPOM method will be called convergent if it holds*

$$\lim_{\substack{h \to 0 \\ x_{k} = x}} \tilde{y}(x_{jk+i}) = y(x), \quad i = 0, \ldots, k-1$$

where $y(x)$ is the exact solution of a differential equation of the form (1) with the right-hand term satisfying the Lipschitz condition determined by the initial condition (2) and $\tilde{y}(x)$ is any solution of the corresponding equation (3) determined by the initial conditions $\tilde{y}(x_s)$, $s = 0, \ldots, l-1$ satisfying

$$\lim_{h \to 0} \tilde{y}(x_s) = \eta \quad for \quad s = 0, \ldots, l-1.$$

The first result concerning our problem is formulated in the following theorem.

Theorem 1. *The convergent GPOM method is stable in the sense of [1], i.e., the matrix $\boldsymbol{E} = \boldsymbol{R}\boldsymbol{B}$ where $R = [\boldsymbol{O}_{l,k-l}, \boldsymbol{I}_l]$ satisfies the condition $\|\boldsymbol{E}^n\| \leqq G$ for $n = 0, 1, \ldots$ with G independent of n.*

Proof. Let a GPOM method be given which is convergent and let us investigate the initial value problem

(4) $$y' = 0, \quad x \in [a, b]; \quad y(a) = 0,$$

the exact solution of which is obviously the function $y(x) = 0$. Further, let us apply the given method to the above problem. We get

(5) $$\tilde{\boldsymbol{y}}(z_j) = \boldsymbol{B}\,\tilde{\boldsymbol{y}}(\boldsymbol{x}_j), \quad j = 0, 1, \ldots,$$

or in more detail,

(6) $[\tilde{y}(x_{jk+l}), \ldots, \tilde{y}(x_{(j+1)k+l-1})]^{\mathsf{T}} = \boldsymbol{B}[\tilde{y}(x_{jk}), \ldots, \tilde{y}(x_{jk+l-1})]^{\mathsf{T}}\,, \quad j = 0, 1, \ldots.$

Now it must hold

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} \tilde{y}(x_{jk+v}) = 0\,, \quad v = 0, \ldots, k-1 \tag{7}$$

for any solution of (5) satisfying

$$\lim_{h \to 0} \tilde{y}(x_\mu) = 0\,, \quad \mu = 0, \ldots, l-1 \tag{8}$$

since the given method is supposed to be convergent. Let us choose an arbitrary function $\phi(h)$ satisfying $\lim_{h \to 0} \phi(h) = 0$ and let us define the vector $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j) = [\tilde{y}(x_{jk+l}), \ldots$ $\ldots, \tilde{y}(x_{(j+1)k+l-1})]^{\mathsf{T}}$ by the following relations:

$$\begin{bmatrix} \tilde{y}(x_0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}(x_{l-1}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \phi(h) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \ldots s - \text{th} \quad \text{row} \\ \\ \\ \\ \end{matrix} \tag{9}$$

where s is an arbitrary number, $1 \leqq s \leqq l$,

$$[\tilde{y}(x_{jk}), \ldots, \tilde{y}(x_{jk+l-1})]^{\mathsf{T}} = \boldsymbol{E}^j[\tilde{y}(x_0), \ldots, \tilde{y}(x_{l-1})]^{\mathsf{T}}\,, \quad j = 1, 2, \ldots; \tag{10}$$

and

$$\tilde{y}(x_{jk+\mu}) = \sum_{v=1}^{l} b_{\mu-l+1,v}\, \tilde{y}(x_{jk+v-1})\,, \quad \mu = l, \ldots, k-1\,; \quad j = 0, 1, \ldots. \tag{11}$$

Let us show that the sequence of vectors $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ defined by (9) to (11) satisfies (5) (or (6)). Really, if we premultiply (6) by the matrix $\boldsymbol{R}$ we get

$$[\tilde{y}(x_{(j+1)k}), \ldots, \tilde{y}(x_{(j+1)k+l-1})]^{\mathsf{T}} = \boldsymbol{E}[\tilde{y}(x_{jk}), \ldots, \tilde{y}(x_{jk+l-1})]^{\mathsf{T}}\,, \quad j = 0, 1, \ldots \tag{12}$$

which is obviously satisfied by (10) and the remaining components of the vector $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ are defined directly by (6) as is seen from the comparison of the first $k - l$ equations (6) with (11). Moreover, the initial values $\tilde{y}(x_\mu)$, $\mu = 0, \ldots, l-1$ of the just defined vector $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ satisfy (8) in virtue of (9) and the definition of $\phi(h)$. Consequently, the vector $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ must satisfy (7). Substituting (9) into (10), we have

$$\tilde{y}(x_{jk+v}) = \varepsilon_{v+1,s}(j)\, \phi(h) \quad \text{for} \quad v = 0, \ldots, l-1$$

where $\varepsilon_{m,n}(j)$ denote the elements of the matrix $\boldsymbol{E}^j$. From (7) it follows that

(13) $$\lim_{\substack{h \to 0 \\ x_{jk} = x}} \varepsilon_{\nu+1,s}(j)\, \phi(h) = 0 \quad \text{for} \quad \nu = 0, \ldots, l-1 \,.$$

Since $x_{jk} = x$ implies that j behaves as $1/h$ for $h \to 0$ we can conclude from (13) and from the fact that $\phi(h)$ is arbitrary that $\varepsilon_{\nu+1,s}(j)$ must be bounded for any j and for $\nu = 0, \ldots, l-1$ by a constant independent of j. Since s was chosen arbitrary we can finally assert that all elements of the matrix $\boldsymbol{E}^j$ must be bounded. But the boundedness of the elements of $\boldsymbol{E}^j$ is obviously equivalent to the boundedness of $\|\boldsymbol{E}^j\|$. The theorem is proved.

Further necessary conditions for convergence are expressed in the following theorems.

Theorem 2. *The convergent GPOM method satisfies the condition*

(14) $$\sum_{j=1}^{l} b_{ij} = 1 \,, \quad i = 1, \ldots, k \,.$$

Proof. Let us investigate the problem

(15) $$y'(x) = 0 \,, \quad x \in [a, b] \,; \quad y(a) = 1$$

having the exact solution $y(x) \equiv 1$. Applying the given method to the problem (15) we obtain again (6). If we define the initial conditions $\tilde{\boldsymbol{y}}(\boldsymbol{x}_0)$ for the vectors $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$, $j = 0, 1, \ldots$ by

$$[\tilde{y}(x_0), \tilde{y}(x_1), \ldots, \tilde{y}(x_{l-1})]^{\mathsf{T}} = [1, 1, \ldots, 1]^{\mathsf{T}} \,.$$

and $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ for $j = 1, 2, \ldots$ by (6) we get obviously the solution of (6) satisfying

$$\lim_{h \to 0} \tilde{y}(x_\mu) = 1 \quad \text{for} \quad \mu = 0, \ldots, l-1 \,.$$

Consequently, (the given method being convergent) it must hold for this solution

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} \tilde{y}(x_{jk+\nu}) = 1 \quad \text{for} \quad \nu = 0, \ldots, k-1 \,,$$

which can be written in a simpler form

$$\lim_{j \to \infty} \tilde{y}(x_{jk+\nu}) = 1 \quad \text{for} \quad \nu = 0, \ldots, k-1$$

since the solution $\tilde{\boldsymbol{y}}(\boldsymbol{z}_j)$ we are dealing with does not depend on h at all.

Thus, passing to the limit for $j \to \infty$ in (6) we obtain the equation

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = B \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

i.e., (14). The theorem is proved.

To be able to formulate another necessary condition for convergence of the GPOM method we must introduce the following two lemmas with easy proofs.

Lemma 1. *Let E be a square matrix of order l satisfying the condition of the stability and further let*

$$(16) \qquad E i^{(l)} = i^{(l)}$$

where $i^{(l)}$ is the l-dimensional vector with all elements equal to unity. Finally, let T be the regular matrix transforming the matrix E to the Jordan canonical form, i.e.

$$T^{-1} E T = J = \begin{bmatrix} J_1 & & O \\ & \ddots & \\ O & & J_r \end{bmatrix}$$

and let us suppose that the Jordan blocks J_i are ordered in such a way that $J_1, \ldots, J$ correspond to eigenvalues of E equal to unity. Then

(i) $\alpha \geqq 1$,

(ii) $J_1, \ldots, J_\alpha$ *are numbers;*

(iii) *If we put $T^{-1} i^{(l)} = (v_1, \ldots, v_\alpha, v_{\text{res}})^{\text{T}}$, then $v_{\text{res}} = 0$.*

Proof. The condition (16) implies that the matrix E has at least one eigenvalue equal to unity. Consequently, $\alpha \geqq 1$. Further, from Remark 2 in [1] it follows that the elementary divisors corresponding to eigenvalues equal to unity must be linear. Thus $J_1, \ldots, J_\alpha$ are 1×1 matrices. Finally, it follows from (16) that $i^{(l)}$ is the eigenvector of E corresponding to the unit eigenvalue. Consequently, it is $J T^{-1} i^{(l)} = T^{-1} i^{(l)}$ which shows that $T^{-1} i^{(l)}$ is the eigenvector corresponding to the unity of the matrix J. On the other hand, from (i) and (ii) we have

$$J = \begin{bmatrix} I_\alpha & O \\ O & J_{\text{res}} \end{bmatrix}$$

where I_α is the identity matrix of order α and J_{res} is a (triangular) matrix with eigenvalues different from unity. Hence,

$$J T^{-1} i^{(l)} = \begin{bmatrix} v_1 \\ \vdots \\ v_\alpha \\ J_{\text{res}} v_{\text{res}} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_\alpha \\ v_{\text{res}} \end{bmatrix}$$

i.e., $J_{\text{res}} v_{\text{res}} = v_{\text{res}}$.

But the matrix $J_{\text{res}} - I$ is regular which proves (iii). The lemma is proved.

Lemma 2. *Let H be a square matrix such that the matrix $H - I$ is regular. Then*

$$\sum_{\nu=0}^{j-1} H^{\nu} = (H - I)^{-1} (H^{j} - I)$$

for any positive integer j.

Proof. The lemma can be easily proved by induction.

Theorem 3. *The convergent GPOM method satisfies the condition*

$$T \begin{bmatrix} I_{\alpha} & O \\ O & O \end{bmatrix} T^{-1} p = m i^{(l)}$$

where $p = R(Ci^{(k)} + Di^{(l)})$.

Proof. Let us investigate the initial value problem

$$y'(x) = 1 \,, \quad x \in [a, b] \,; \quad y(a) = 0 \,.$$

The exact solution of the above problem is obviously the function $y(x) = x - a$. If we apply the given method to this differential equation we get

$$\tilde{y}(z_j) = B\, \tilde{y}(x_j) + hCi^{(k)} + hDi^{(l)} \,. \tag{17}$$

Premultiplying both sides of (17) by the matrix R we have

$$\tilde{y}(x_{j+1}) = E\, \tilde{y}(x_j) + hp \,.$$

The solution of this equation can be written in the form

$$\tilde{y}(x_j) = E^{j}\, \tilde{y}(x_0) + h \sum_{\nu=0}^{j-1} E^{\nu} p \tag{18}$$

as can be easily verified. Thus the components $\tilde{y}(x_j)$ of the solution of (17) are given by (18) and the remaining components of this vector which are not used in the new step of the method are given directly by the part of equations (17).

If we assume in (18) that

$$\lim_{h \to 0} \tilde{y}(x_{\mu}) = 0 \quad \text{for} \quad \mu = 0, \ldots, l - 1 \tag{19}$$

It must hold that

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} \tilde{y}(x_j) = (x - a)\, i^{(l)} \tag{20}$$

since the given method is assumed tp be convergent. Further, the matrix E is stable as follows from Theorem 1. Thus (19), (20) and (18) imply

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} h \sum_{\nu=0}^{j-1} \boldsymbol{E}^{\nu} \boldsymbol{p} = (x - a)\, \boldsymbol{i}^{(l)} . \tag{21}$$

Using the notation of Lemma 1 the assumptions of which are satisfied as follows from Theorems 1 and 2 we can write

$$\begin{aligned} h \sum_{\nu=0}^{j-1} \boldsymbol{E}^{\nu} \boldsymbol{p} &= h \boldsymbol{T} \sum_{\nu=0}^{j=1} \begin{bmatrix} \boldsymbol{I}_{\alpha}^{\nu} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_{\mathrm{res}}^{\nu} \end{bmatrix} \boldsymbol{T}^{-1} \boldsymbol{p} \\ &= h \boldsymbol{T} \begin{bmatrix} \sum\limits_{\nu=0}^{j-1} \boldsymbol{I}_{\alpha}^{\nu} & \boldsymbol{O} \\ \boldsymbol{O} & \sum\limits_{\nu=0}^{j-1} \boldsymbol{J}_{\mathrm{res}}^{\nu} \end{bmatrix} \boldsymbol{T}^{-1} \boldsymbol{p} . \end{aligned} \tag{22}$$

Obviously

$$\sum_{\nu=0}^{j-1} \boldsymbol{I}_{\alpha}^{\nu} = j \boldsymbol{I}_{\alpha} . \tag{23}$$

Further, the matrix $\boldsymbol{J}_{\mathrm{res}} - \boldsymbol{I}$ is regular as follows from Lemma 1 and, consequently, Lemma 2 gives

$$\sum_{\nu=0}^{j-1} \boldsymbol{J}_{\mathrm{res}}^{\nu} = (\boldsymbol{J}_{\mathrm{res}} - \boldsymbol{I})^{-1} (\boldsymbol{J}_{\mathrm{res}}^{j} - \boldsymbol{I}) . \tag{24}$$

Substituting (23) and (24) into (22) we get

$$h \sum_{\nu=0}^{j-1} \boldsymbol{E}^{\nu} \boldsymbol{p} = \boldsymbol{T} \begin{bmatrix} hj\boldsymbol{I}_{\alpha} & 0 \\ 0 & h(\boldsymbol{J}_{\mathrm{res}} - \boldsymbol{I})^{-1} (\boldsymbol{J}_{\mathrm{res}}^{j} - \boldsymbol{I}) \end{bmatrix} \boldsymbol{T}^{-1} \boldsymbol{p} . \tag{25}$$

But the matrix $\boldsymbol{J}_{\mathrm{res}}$ satisfies obviously the condition of stability, consequently,

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} h (\boldsymbol{J}_{\mathrm{res}} - \boldsymbol{I})^{-1} (\boldsymbol{J}_{\mathrm{res}}^{j} - \boldsymbol{I}) = \boldsymbol{O} . \tag{26}$$

The equation $x_{jk} = x$ implies

$$hj = \frac{1}{m} (x - a) . \tag{27}$$

Thus, if we pass to the limit in (25) using (26) and (27) we get

$$\lim_{\substack{h \to 0 \\ x_{jk} = x}} h \sum_{\nu=0}^{j-1} \boldsymbol{E}^{\nu} \boldsymbol{p} = \boldsymbol{T} \begin{bmatrix} \frac{1}{m} (x - a)\, \boldsymbol{I}_{\alpha} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \boldsymbol{T}^{-1} \boldsymbol{p} .$$

From here and from (20) the assertion of the theorem follows directly.

Let us now try to construct necessary and sufficient conditions for convergence in some special cases included in the following two remarks:

Remark 1. Let the matrix $\boldsymbol{B} = \{b_{ij}\}$ of the given method satisfy

$$b_{k-l+i,j} = \delta_i^{(j)} \tag{28}$$

where $\delta_i^{(j)}$ is the Kronecker δ. Then the conditions of Theorems 2 and 3 are exactly the conditions of the consistency with respect to l from [1]. Consequently, in this special case the stability and the consistency with respect to l are necessary and sufficient conditions for the convergence.

Remark 2. The problem of necessary and sufficient conditions for the convergence is also completely solved in the special case of the GPOM method with $l = 1$ since in this case (28) is obviously satisfied. This case is very important from the practical point of view since such methods are selfstarting, i.e., one needs no special procedures for the computation of initial values.

These two remarks suggest that even in the general case the condition of consistency would be necessary for the convergence. There are, moreover, other intuitive arguments for this assertion but the definite answer is till now a problem for further investigation.

It is proved in the previous paper [1] that Dahlquist's method is a special case of our GPOM method. We can prove now that the necessary and sufficient conditions developed above give the same result as the classical theory. Before this let us give first a short survey of Dahlquist's method.

In this method the approximate solution $\tilde{y}(t_j)$ of $y' = f(x, y)$, $y(a) = \eta$ at the points $t_j = a + jh$; $j = 0, 1, \ldots$ is computed from the equation

$$\sum_{\nu=0}^{k} \alpha_\nu \, \tilde{y}(t_{n+\nu}) = h \sum_{\nu=0}^{k} \beta_\nu f(t_{n+\nu}, \tilde{y}(t_{n+\nu})) \,, \quad n = 0, 1, \ldots$$

where α_ν, β_ν are constants, $\alpha_k \neq 0$, $|\alpha_0| + |\beta_0| > 0$ and the initial values $\tilde{y}(t_\mu)$, $\mu = 0, \ldots, k - 1$ are assumed to be known. It is well known (c.f., for example [2]) that necessary and sufficient conditions for the convergence of these methods are the following:

(i) The roots ζ_i of the polynomial

$$\varrho(\zeta) = \sum_{\nu=0}^{k} \alpha_\nu \zeta^\nu \tag{29}$$

satisfy the condition $|\zeta_i| \leqq 1$ and the roots ζ_i for which $|\zeta_i| = 1$ are simple (this condition is called the Dahlquist stability condition).

(ii) The polynomial $\varrho(\zeta)$ and the polynomial $\sigma(\zeta)$ given by $\sigma(\zeta) = \sum_{\nu=0}^{k} \beta_\nu \zeta^\nu$ satisfy

$$\varrho(1) = 0 \,, \tag{30}$$

$$\varrho'(1) = \sigma(1) \,. \tag{31}$$

The equivalent GPOM method has $k = l$, $m = 1$, $\mu_i = i$ for $i = 1, \ldots, k-1$ and the matrices $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ given by

$$(32)\quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{O}_l & & \boldsymbol{I}_l \\ \\ -\dfrac{\alpha_0}{\alpha_k} & \ldots & -\dfrac{\alpha_{k-1}}{\alpha_k} \end{bmatrix}, \quad \boldsymbol{C} = \frac{1}{\alpha_k}\begin{bmatrix} 0 & \ldots & & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \beta_k \end{bmatrix}, \quad \boldsymbol{D} = \frac{1}{\alpha_k}\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \beta_0 & \ldots & \beta_{k-1} \end{bmatrix}$$

(see [1]).

Let us begin with the necessary conditions. From Theorem 1 it follows that the given matrix $\boldsymbol{B}$ must satisfy the condition of stability since $\boldsymbol{E} = \boldsymbol{B}$.

But the matrix $\boldsymbol{B}$ is in the so-called Frobenius form and, as far as such matrices are concerned, it is well known that

(i) the characteristic polynomial is the polynomial $\varrho(\zeta)$ given by (29);

(ii) the elementary divisor corresponding to any eigenvalue λ_i is $(\zeta - \lambda_i)^{r_i}$ where r_i is the multiplicity of λ_i as the root of ϱ. Now from here it follows immediately that the root of $\varrho(\xi)$ must lie in the unit circle and that those with moduli equal to unity must be simple, i.e., the GPOM stability is the Dahlquist stability.

Further, Theorem 2 implies immediately (30).

Finally, from the stability condition and from (30) it follows that the unity is a simple eigenvalue of $\boldsymbol{B}$. Consequently $\alpha = 1$ in Theorem 3 which can be written in the form

$$(33)\qquad \frac{1}{\alpha_k}\, t_{1k}^{-1}\, \sigma(1) = \sum_{\nu=1}^{k} t_{1\nu}^{-1}$$

where $\boldsymbol{T}^{-1} = \{t_{ij}^{-1}\}$ since all components of the vector $\boldsymbol{T}^{-1}\boldsymbol{i}^{(k)}$ except the first one are equal to zero as follows from Lemma 1, and since the last component of the vector $\boldsymbol{p}$ is equal to $(1/\alpha_k)\,\sigma(1)$ and all other components are equal to zero as follows immediately from (32).

The matrix $\boldsymbol{T}$ satisfies

$$\boldsymbol{T}^{-1}\boldsymbol{B} = \boldsymbol{J}\boldsymbol{T}^{-1}$$

and we know that the first row of the matrix $\boldsymbol{J}$ is $(1, 0, \ldots, 0)$. Consequently, it is possible to write the first row of $\boldsymbol{J}\boldsymbol{T}^{-1}$ as $\boldsymbol{t}^{\mathrm{T}}$ where $\boldsymbol{t}^{\mathrm{T}} = (t_{11}^{-1}, \ldots, t_{1k}^{-1})$.

On the other hand, the first row of $\boldsymbol{T}^{-1}\boldsymbol{B}$ has obviously the form $\boldsymbol{t}^{\mathrm{T}}\boldsymbol{B}$. Thus we have

$$\boldsymbol{t}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{t}^{\mathrm{T}}$$

or

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{t} = \boldsymbol{t}\,,$$

i.e., the vector $\boldsymbol{t}$ is the eigenvector corresponding to the eigenvalue equal to unity of the transpose to $\boldsymbol{B}$. If we write the last vector equation in more detail we get

$$(34)\qquad \begin{aligned} -\frac{\alpha_0}{\alpha_k} t_{1k}^{-1} &= t_{11}^{-1} \\ t_{11}^{-1} - \frac{\alpha_1}{\alpha_k} t_{1k}^{-1} &= t_{12}^{-1} \\ &\cdots\cdots\cdots \\ t_{1,k-1}^{-1} - \frac{\alpha_{k-1}}{\alpha_k} t_{1k}^{-1} &= t_{1k}^{-1} \end{aligned}$$

Adding the last $k - 1$ equations we get

$$t_{11}^{-1} - t_{1k}^{-1} = \frac{\alpha_1 + \ldots + \alpha_{k-1}}{\alpha_k} t_{1k}^{-1}$$

i.e.

$$t_{11}^{-1} = \frac{\alpha_1 + \ldots + \alpha_k}{\alpha_k} t_{1k}^{-1} = -\frac{\alpha_0}{\alpha_k} t_{1k}^{-1}$$

since $\varrho(1) = \alpha_0 + \ldots + \alpha_k = 0$. Thus, the first equation of (34) is a consequence of the other equations and it may be, therefore, ommitted. The remaining equations give

$$\begin{aligned} t_{1,k-1}^{-1} &= \left(1 + \frac{\alpha_{k-1}}{\alpha_k}\right) t_{1k}^{-1} \\ t_{1,k-2}^{-1} &= \frac{\alpha_{k-2}}{\alpha_k} t_{1k}^{-1} + t_{1,k-1}^{-1} = \left(1 + \frac{\alpha_{k-1} + \alpha_{k-2}}{\alpha_k}\right) t_{1k}^{-1} \\ &\vdots \\ t_{11}^{-1} &= \left(1 + \frac{\alpha_{k-1} + \ldots + \alpha_1}{\alpha_k}\right) t_{1k}^{-1}\,. \end{aligned}$$

From here we first conclude that $t_{1k}^{-1} \neq 0$ since in the opposite case all $t_{1\nu}^{-1}$ would be equal to zero and the matrix $\boldsymbol{T}^{-1}$ would be singular.

Secondly, adding these last equations we get

$$\sum_{\nu=1}^{k-1} t_{1\nu}^{-1} = \left(k - 1 + \frac{(k-1)\,\alpha_{k-1} + \ldots + 1\,.\,\alpha_1}{\alpha_k}\right) t_{1k}^{-1}$$

what can be obviously rewritten in the form

$$(35)\qquad \sum_{\nu=1}^{k} t_{1\nu}^{-1} = \left(\frac{k\alpha_k + \ldots + 1\,.\,\alpha_1}{\alpha_k}\right) t_{1k}^{-1} = \frac{1}{\alpha_k}\, \varrho'(1)\, t_{1k}^{-1}\,.$$

Comparing (33) and (35) we get (31) since $t_{1k}^{-1} \neq 0$ and $\alpha_k \neq 0$.

Thus our results imply that the Dahlquist stability and the Dahlquist consistency are necessary for the convergence. But our results imply also that these conditions are sufficient, as well. The proof of this part is very easy and it is omitted. Hence we see that in the case of Dahlquist's method our theory and the classical theory give the same results.

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Souhrn

NUTNÉ PODMÍNKY KONVERGENCE ZOBECNĚNÝCH PERIODICKÝCH SILNĚ IMPLICITNÍCH MNOHOKROKOVÝCH METOD

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Článek je přímým pokračováním práce ,,Zobecněné periodické silně implicitní mnohokrokové metody" téhož autora a je věnován studiu nutných a v některých speciálních případech nutných a postačujících podmínek konvergence zobecněných periodických silně implicitních mnohokrokových metod.

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