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# QUALITATIVE ANALYSIS OF BASIC NOTIONS IN PARAMETRIC CONVEX PROGRAMMING, II

(Parameters in the objective function)

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A short survey of recent results in the field of parametric convex programming from the qualitative point of view can be found in [4].

In this paper the same notions as those introduced in [4], i.e. the notions of the solvability set, the stability set of the first kind and the stability set of the second kind, are defined and analyzed qualitatively for the problem

$$\text{(II)} \qquad \min \sum_{a=1}^{m} \lambda_a \, \Phi_a(x) \, ,$$

subject to

$$\boldsymbol{M} = \{x \in \mathrm{R}^n / g_r(x) \leqq 0, \; r = 1, 2, \ldots, l\} \, ,$$

where $\Phi_a(x)$, $a = 1, 2, \ldots, m$; $g_r(x)$, $r = 1, 2, \ldots, l$ are convex functions possessing continuous first order partial derivatives on $\mathrm{R}^n$ (the vector space of all ordered $n$-tuples of real numbers) and $\lambda_a$, $a = 1, 2, \ldots, m$ are arbitrary nonnegative real numbers. The restriction set $\boldsymbol{M}$ is supposed to be nonempty and fixed.

## 1. CHARACTERIZATION OF THE SOLVABILITY SET

**Definition 1.** *The solvability set of problem* (II) *denoted by* $\boldsymbol{B}$, *is defined by*

$$(1) \qquad \boldsymbol{B} = \{\lambda \in {}'\mathrm{R}^m_+ / \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \lambda_a \, \Phi_a(x) \text{ exists}\} \, ,$$

*where* $'\mathrm{R}^m_+$ *denotes the nonnegative orthant of the* $'\mathrm{R}^m$ *vector space of parameters.*

**Lemma 1.** *If the set* $\boldsymbol{B}$ *is defined by* (1), *then it is a cone with vertex at* $\lambda = 0$.

Proof. It is clear that $\lambda = 0$ is a point in $\boldsymbol{B}$. Let us assume that $\bar{\lambda} \in \boldsymbol{B}$, $\bar{\lambda} \neq 0$, then there exists $\bar{x} \in \boldsymbol{M}$ such that

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x)$$

and therefore, for all $0 < t < \infty$ we have

$$\sum_{a=1}^{m} t\bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} t\bar{\lambda}_a \Phi_a(x),$$

i.e. $\lambda^* \in \boldsymbol{B}$, where $\lambda^* = t\bar{\lambda}$, $0 < t < \infty$ and hence the result.

**Lemma 2.** *If problem* (II) *is solvable for* $\lambda^1, \lambda^2$ $(\lambda^1 \neq \lambda^2)$, *then it is solvable for all* $\lambda = \mu_1\lambda^1 + \mu_2\lambda^2$, $\mu_1 + \mu_2 = 1$ $(\mu_1 \geqq 0, \mu_2 \geqq 0)$ *iff for the problem*

$$\text{(II)}' \qquad \min_{x \in \boldsymbol{M}} [\mu_1 \mathsf{H}_1(x) + \mu_2 \mathsf{H}_2(x)], \quad \mu_1 + \mu_2 = 1\,(\mu_1 \geqq 0, \mu_2 \geqq 0),$$

*where*

$$\mathsf{H}_i(x) = \sum_{a=1}^{m} \lambda_a^i \Phi_a(x), \quad i = 1, 2$$

*the solvability set* $\boldsymbol{B}^{\sim}$ *is convex in* $'\mathsf{R}^2$, *where*

$$\boldsymbol{B}^{\sim} = \{(\mu_1, \mu_2) \in {'\mathsf{R}^2} / \min_{x \in \boldsymbol{M}} [\mu_1 \mathsf{H}_1(x) + \mu_2 \mathsf{H}_2(x)] \text{ exists}, \mu_1 + \mu_2 = 1(\mu_1 \geqq 0, \mu_2 \geqq 0)\}.$$

Proof. i) Suppose that if problem (II) is solvable for $\lambda^1, \lambda^2$ $(\lambda^1 \neq \lambda^2)$, then it is solvable for all $\lambda = \mu_1\lambda^1 + \mu_2\lambda^2$, $\mu_1 + \mu_2 = 1$ $(\mu_1 \geqq 0,\ \mu_2 \geqq 0)$ and let $(\mu_1^*, \mu_2^*) \in \boldsymbol{B}^{\sim}$; then there exists $x^* \in \boldsymbol{M}$ such that

$$\text{(2)} \qquad \sum_{a=1}^{m} (\mu_1^*\lambda_a^1 + \mu_2^*\lambda_a^2)\, \Phi_a(x^*) \leqq \sum_{a=1}^{m} (\mu_1^*\lambda_a^1 + \mu_2^*\lambda_a^2)\, \Phi_a(x), \quad \forall x \in \boldsymbol{M}.$$

Further let $(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, then there exists $\hat{x} \in \boldsymbol{M}$ such that

$$\text{(3)} \qquad \sum_{a=1}^{m} (\hat{\mu}_1\lambda_a^1 + \hat{\mu}_2\lambda_a^2)\, \Phi_a(\hat{x}) \leqq \sum_{a=1}^{m} (\hat{\mu}_1\lambda_a^1 + \hat{\mu}_2\lambda_a^2)\, \Phi_a(x), \quad \forall x \in \boldsymbol{M},$$

where $\mu_1^*; \mu_2^*; \hat{\mu}_1; \hat{\mu}_2 \geqq 0$, $\mu_1^* + \mu_2^* = 1$, $\hat{\mu}_1 + \hat{\mu}_2 = 1$. Let us denote $\gamma_1 = \mu_1^*\lambda^1 + \mu_2^*\lambda^2$, $\gamma_2 = \hat{\mu}_1\lambda^1 + \hat{\mu}_2\lambda^2$. From (2), (3) it follows that problem (II) is solvable for $\gamma_1, \gamma_2$ and by the assumptions of the lemma it is solvable for all $\gamma = (1 - \omega)\gamma_1 + \omega\gamma_2$, $0 \leqq \omega \leqq 1$, and hence $(1 - \omega)(\mu_1^*, \mu_2^*) + \omega(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, $0 \leqq \omega \leqq 1$, i.e. the set $\boldsymbol{B}^{\sim}$ is convex.

ii) Assume that the set $\boldsymbol{B}^{\sim}$ is convex and let $(\mu_1^*, \mu_2^*) \in \boldsymbol{B}^{\sim}$, $(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, then it follows that $(1 - \omega)(\mu_1^*, \mu_2^*) + \omega(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, $0 \leqq \omega \leqq 1$, therefore, if $\gamma_1 \in \boldsymbol{B}$, $\gamma_2 \in \boldsymbol{B}$, $\gamma_1 \neq \gamma_2$, then $(1 - \omega)\gamma_1 + \omega\gamma_2 \in \boldsymbol{B}$, $0 \leqq \omega \leqq 1$, where $\gamma_1, \gamma_2$ are defined in i).

Remark 1. If problem (II)′ is solvable for $\mu_1 = 0$; $\mu_2 = 1$, then

$$\min_{x\in M} [\mu_1 H_1(x) + \mu_2 H_2(x)] = \min_{x\in M} H_2(x) = \min_{x\in M} \sum_{a=1}^{m} \lambda_a^2 \Phi_a(x),$$

and if it solvable for $\mu_1 = 1$; $\mu_2 = 0$, then

$$\min_{x\in M} [\mu_1 H_1(x) + \mu_2 H_2(x)] = \min_{x\in M} H_1(x) = \min_{x\in M} \sum_{a=1}^{m} \lambda^1 \Phi_a(x).$$

**Lemma 3.** *If* $f_1(x)$; $f_2(x)$ *are convex functions on* $M$ *such that* $f_1(x) \geqq 0$; $f_2(x) \geqq 0$ *for all* $x \in M$, *then*

$$\max [f_1(x), f_2(x)] \leqq f_1(x) + f_2(x), \quad \forall x \in M,$$

*and the functions* $\max [f_1(x), f_2(x)]$; $f_1(x) + f_2(x)$ *are convex on* $M$, *where* $M$ *is defined in problem* (II).

Proof. Let

$$A_1 = \{x \in M / f_1(x) \geqq f_2(x)\},$$
$$A_2 = \{x \in M / f_1(x) \leqq f_2(x)\},$$

then

$$\max [f_1(x), f_2(x)] = f_1(x) \leqq f_1(x) + f_2(x), \quad \forall x \in A_1,$$
$$\max [f_1(x), f_2(x)] = f_2(x) \leqq f_1(x) + f_2(x), \quad \forall x \in A_2,$$

which implies that $\max [f_1(x), f_2(x)] \leqq f_1(x) + f_2(x)$, $\forall x \in M$. The convexity of the function $\max [f_1(x), f_2(x)]$ follows from the fact that

$$\max \{f_1[(1-\omega) x^1 + \omega x^2], f_2[(1-\omega) x^1 + \omega x^2]\} \leqq$$
$$\leqq \max \{[(1-\omega) f_1(x^1) + \omega f_1(x^2)], [(1-\omega) f_2(x^1) + \omega f_2(x^2)]\} \leqq$$
$$\leqq \max [(1-\omega) f_1(x^1), (1-\omega) f_2(x^1)] + \max [\omega f_1(x^2) + \omega f_2(x^2] =$$
$$= (1-\omega) \max [f_1(x^1), f_2(x^1)] + \omega \max [f_1(x^2), f_2(x^2)]$$

for all $0 \leqq \omega \leqq 1$.
The convexity of $f_1(x) + f_2(x)$ is clear [3], [5].

**Lemma 4.** *If* $f_1(x)$, $f_2(x)$ *are strictly convex and closed functions on* $M$ [6] *and* $\min_{x\in M} [f_i(x)]$, $i = 1, 2$ *exists, then both the sets* $\Lambda_1(k)$, $\Lambda_2(k)$ *defined by*

$$\Lambda_1(k) = \{x \in M / f_1(x) \leqq k\}, \tag{4}$$

$$\Lambda_2(k) = \{x \in M / f_2(x) \leqq k\} \tag{5}$$

*are bounded for all* $k \in {}'R$ *and such that* $\Lambda_1(k) \neq \emptyset$, $\Lambda_2(k) \neq \emptyset$.

Proof. Let $\min_{x \in \mathbf{M}} f_i(x) = f_i(x^i) = k_i$, $i = 1, 2$, where $x^1 \in \mathbf{M}$, $x^2 \in \mathbf{M}$.

Then the sets $\Lambda_1(k_1)$, $\Lambda_2(k_2)$ given by

$$\Lambda_1(k_1) = \{x \in \mathbf{M}/f_1(x) \leqq k_1\} ;$$
$$\Lambda_2(k_2) = \{x \in \mathbf{M}/f_2(x) \leqq k_2\}$$

are clearly bounded since $\Lambda_1(k_1) = x^1$, $\Lambda_2(k_2) = x^2$ (which follows from the strict convexity of the functions $f_1(x)$, $f_2(x)$ on $\mathbf{M}$). Therefore, a lemma given in [6] (this lemma states: "The nonvoid level sets $\mathbf{S}(\alpha) = \{x \in R^n/f(x) \leqq \alpha\}$ of a closed convex function f are either all bounded or all unbounded") implies directly the results.

Remark 2. The nonvoid level sets [6] $\{x \in \mathbf{M}/f(x) \leqq k, k \in 'R\}$ are bounded iff the nonvoid level sets $\{x \in \mathbf{M}/f(x) + a \leqq k, k \in 'R\}$ are bounded for any constant $a \in R$.

**Lemma 5.** *If the assumptions of Lemma 4 are satisfied, then the sets $\Gamma(k)$ defined by*

$$\Gamma(k) = \{x \in \mathbf{M}/f_1(x) + f_2(x) \leqq k\} \tag{6}$$

*are bounded for all $k \in 'R$ such that $\Gamma(k) \neq \emptyset$.*

Proof. From the assumptions it follows that there exist constants $a_i \in R$, $i = 1, 2$ with $a_i > |\min_{x \in \mathbf{M}} f_i(x)|$, $i = 1, 2$ such that

$$f_i(x) + a_i \geqq 0 , \quad i = 1, 2 \quad \text{for all} \quad x \in \mathbf{M} .$$

From Lemma 3 we have

$$\max \{[f_1(x) + a_1], [f_2(x) + a_2]\} \leqq f_1(x) + f_2(x) + a_1 + a_2$$

and therefore

$$\{x \in \mathbf{M}/f_1(x) + f_2(x) + a_1 + a_2 \leqq k\} \subset$$
$$\subset \{x \in \mathbf{M}/\max \{[f_1(x) + a_1], [f_2(x) + a_2]\} \leqq k\} .$$

It is clear from (4), (5) that

$$\{x \in \mathbf{M}/\max [f_1(x), f_2(x)] \leqq k\} = \Lambda_1(k) \cap \Lambda_2(k) \tag{7}$$

and hence the result follows from Lemma 4, Remark 2.

**Theorem 1.** *If $f_1(x), f_2(x)$ are strictly convex and closed functions on $\mathbf{M}$ [6] and $\min_{x \in \mathbf{M}} f_i(x)$, $i = 1, 2$ exists, then*

$$\min_{x \in \mathbf{M}} [f_1(x) + f_2(x)] \quad \textit{exists.}$$

Proof. Let us define the sets denoted by $\mathbf{C}$, $\mathbf{D}$ as follows:

$$\mathbf{C} = \{k \in {}'\mathrm{R} / \Lambda_1(k) \cap \Lambda_2(k) \neq \emptyset\} \quad \text{(see (7))},$$
$$\mathbf{D} = \{k \in {}'\mathrm{R} / \Gamma(k) \neq \emptyset\} \quad \text{(see (6))}.$$

It is clear (see [4]) that $\mathbf{C} \neq \emptyset$, $\mathbf{D} \neq \emptyset$. It follows from Lemma 3 that $\mathbf{D} \subset \mathbf{C}$. From the assumptions and from Lemma 1, Lemma 2 it follows that the sets $\mathbf{C}$; $\mathbf{D}$ are convex, closed and unbounded subsets of the real line and $\mathbf{C}$ has the form $\mathbf{C} = [k_0, \infty)$ where $k_0 = \min_{x \in \mathbf{M}} \{\max_{x \in \mathbf{M}} [\mathsf{f}_1(x), \mathsf{f}_2(x)]\}$. Hence $\min_{x \in \mathbf{M}} [\mathsf{f}_1(x) + \mathsf{f}_2(x)]$ exists.

**Corollary 1.** *If all the assumptions of Theorem 1 are satisfied, then all problems of the form*

$$\min_{x \in \mathbf{M}} [\mu_1 \mathsf{f}_1(x) + \mu_2 \mathsf{f}_2(x)], \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0$$

*are solvable.*

Remark 3. It should be noted that Theorem 1 can be proved under the assumptions that the functions $\mathsf{f}_1(x)$, $\mathsf{f}_2(x)$ are closed, convex on $\mathbf{M}$ and $\min \mathsf{f}_i(x)$, $i = 1, 2$ exists such that both the sets

$$\boldsymbol{m}^1_{\mathrm{opt}} = \{x^* \in \mathbf{M} / \mathsf{f}_1(x^*) = \min_{x \in \mathbf{M}} \mathsf{f}_1(x)\},$$
$$\boldsymbol{m}^2_{\mathrm{opt}} = \{x^* \in \mathbf{M} / \mathsf{f}_2(x^*) = \min_{x \in \mathbf{M}} \mathsf{f}_2(x)\}$$

are bounded (see the proof of Lemma 4).

**Theorem 2.** *If the set $\mathbf{U}$ is defined by*

$$\mathbf{U} = \{\lambda \in \mathbf{B} / \boldsymbol{m}_{\mathrm{opt}}(\lambda) \text{ is bounded}\}, \tag{8}$$

*where $\mathbf{B}$ is given by (1), and*

$$\boldsymbol{m}_{\mathrm{opt}}(\lambda) = \{\tilde{x} \in \mathbf{M} / \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x)\}, \tag{9}$$

*then $\mathbf{U}$ is a convex set.*

Proof. Let $\lambda^1 \in U$, $\lambda^2 \in U(\lambda^1 \neq \lambda^2)$, then

$$\min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^1_a \Phi_a(x) = \min_{x \in \mathbf{M}} \mathsf{H}_1(x) \quad \text{exists},$$

and

$$\min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^2_a \Phi_a(x) = \min_{x \in \mathbf{M}} \mathsf{H}_2(x) \quad \text{exists}.$$

Since the functions $\mathsf{H}_1(x)$, $\mathsf{H}_2(x)$ are continuous and convex on $\mathsf{R}^n$, they are convex and closed on $\boldsymbol{M}$ (since lower semicontinuity is equivalent to closedness over $\mathsf{R}^n$) and hence from Corollary 1, Remark 3 it follows that

$$\min_{x \in \boldsymbol{M}} [\mu_1 \mathsf{H}_1(x) + \mu_2 \mathsf{H}_2(x)] \quad \text{exists}, \quad \mu_1 + \mu_2 = 1, \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0,$$

i.e. $$\min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} (\mu_1 \lambda_a^1 + \mu_2 \lambda_a^2)\, \Phi_a(x) \quad \text{exists}, \quad \mu_1 + \mu_2 = 1, \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0,$$

and hence $\mu_1 \lambda^1 + \mu_2 \lambda^2 \in \boldsymbol{U}$ for all $\mu_1 + \mu_2 = 1$, $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, therefore $\boldsymbol{U}$ is convex.

Remark 4. If $\boldsymbol{B} = \boldsymbol{U}$, then the solvability set of problem (II) $\boldsymbol{B}$ is convex.

**Corollary 2.** *If the set $\boldsymbol{M}$ is bounded, then* (8) *implies that $\boldsymbol{B} = \boldsymbol{U}$ and therefore $\boldsymbol{B}$ is convex by Remark* 4.

**Corollary 3.** *If the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex on $\boldsymbol{M}$, then* (8) *implies $\boldsymbol{B} = \boldsymbol{U}$, and therefore $\boldsymbol{B}$ is convex by Remark* 4.

**Lemma 6.** *If for problem* (II) $\boldsymbol{m}_{\text{opt}}(\lambda)$ *is defined by* (9), *then it is convex and closed in* $\mathsf{R}^n$.

Proof. If $\boldsymbol{m}_{\text{opt}}(\lambda)$ is a one-point set, or the empty set, or the whole $\mathsf{R}^n$-space, the result is clear. Suppose that $x^1$, $x^2$ are two points in $\boldsymbol{m}_{\text{opt}}(\lambda)$, then the convexity of the set $\boldsymbol{M}$ and the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$, yields

$$\sum_{a=1}^{m} \lambda_a \Phi_a[(1 - \omega)\, x^1 + \omega x^2] \leqq (1 - \omega) \sum_{a=1}^{m} \lambda_a \Phi_a(x^1) + \omega \sum_{a=1}^{m} \lambda_a \Phi_a(x^2) =$$
$$= \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad 0 \leqq \omega \leqq 1$$

and hence $(1 - \omega)\, x^1 + \omega x^2 \in \boldsymbol{m}_{\text{opt}}(\lambda)$ for all $0 \leqq \omega \leqq 1$, i.e. the set $\boldsymbol{m}_{\text{opt}}(\lambda)$ is convex. Assume that $\tilde{x}_n \in \boldsymbol{m}_{\text{opt}}(\lambda)$, $n = 1, 2, \ldots$ is a sequence of points which converges to $\tilde{x}$. Then

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}_n) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)],$$

$$\lim_{n \to \infty} \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}_n) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)].$$

From the finiteness of the sum and the continuity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$, we have

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\lim_{n \to \infty} \tilde{x}_n) = \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)].$$

Hence $\tilde{x} \in \boldsymbol{m}_{\text{opt}}(\lambda)$ and the set $\boldsymbol{m}_{\text{opt}}(\lambda)$ is therefore closed.

Remark 5. If $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex functions on $\mathbf{M}$ and $\min_{x \in \mathbf{M}} \Phi_a(x)$, $a = 1, 2, \ldots, m$ exists, then the solvability set of problem (II) $\mathbf{B}$ is given by $\mathbf{B} = {}'\mathrm{R}^m_+$.

**Theorem 3.** *If the solvability fuction of problem* (II) *denoted by* $\xi(\lambda)$ *is defined by*

$$\xi(\lambda) = \min_{x \in \mathbf{M}} \left[ \sum_{a=1}^{m} \lambda_a \Phi_a(x) \right], \tag{10}$$

*then it is concave on* $\mathbf{U}$, *where* $\mathbf{U}$ *is given by* (8).

Proof. If $\lambda^1, \lambda^2$ are any two points in $\mathbf{U}$, then by Theorem 2, $(1 - \omega) \lambda^1 + \omega\lambda^2 \in \mathbf{U}$ for all $0 \leqq \omega \leqq 1$, and therefore

$$\xi[(1 - \omega) \lambda^1 + \omega\lambda^2] = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} [(1 - \omega) \lambda^1_a + \omega\lambda^2_a] \Phi_a(x) \geqq$$

$$\geqq (1 - \omega) \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^1_a \Phi_a(x) + \omega \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^2_a \Phi_a(x) =$$

$$= (1 - \omega) \xi(\lambda^1) + \omega \xi(\lambda^2), \quad 0 \leqq \omega \leqq 1 .$$

Hence the function $\xi(\lambda)$ is concave on the set $\mathbf{U}$.

**Corollary 4.** *If the functions* $\Phi_a(x)$, $a = 1, 2, \ldots, m$ *are strictly convex on* $\mathbf{M}$, *or if the set* $\mathbf{M}$ *is bounded, then the solvability function* $\xi(\lambda)$ *is concave on* $\mathbf{B}$ (*see Corollaries* 2 *and* 3).

## 2. CHARACTERIZATION OF THE STABILITY SET OF THE FIRST KIND

**Definition 2.** *Suppose that* $\bar{\lambda} \in \mathbf{B}$ *with a corresponding optimal point* $\bar{x}$, *then the stability set of the first kind of problem* (II) *corresponding to* $\bar{x}$ *denoted by* $\mathbf{S}(\bar{x})$ *is defined by*

$$\mathbf{S}(\bar{x}) = \{\lambda \in \mathbf{B} / \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)]\} . \tag{11}$$

**Lemma 7.** *If the set* $\mathbf{S}(\bar{x})$ *is defined by* (11), *then it is a cone in* $'\mathrm{R}^m$ *with vertex at* $\lambda = 0$.

Proof. It is clear that $0 \in \mathbf{S}(\bar{x})$. Suppose that $\lambda^* \in \mathbf{S}(\bar{x})$, $\lambda^* \neq 0$, then $\sum_{a=1}^{m} \lambda^*_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} \lambda^*_a \Phi_a(x)]$ and therefore $\sum_{a=1}^{m} t\lambda^*_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} t\lambda^*_a \Phi_a(x)]$ for all $t > 0$, i.e. $t\lambda^* \in \mathbf{S}(\bar{x})$ for all $t > 0$. Hence the result.

**Theorem 4.** *If the functions* $g_r(x)$, $r = 1, 2, \ldots, l$ *(see problem* (II)) *satisfy any one of the constraint qualifications* [1], [3] *(for example Slater), then the set* $\mathbf{S}(\bar{x})$ *is convex and closed in* $'R^m$.

Proof. If $\mathbf{S}(\bar{x})$ is a one-point set, or the empty set, or the whole nonnegative orthant of the $'R^m$ space, it is convex and closed. Suppose that $\lambda^1 \in \mathbf{S}(\bar{x})$, $\lambda^2 \in \mathbf{S}(\bar{x})$, $\lambda^1 \neq \lambda^2$, then there exist $u^1 \in R^l$, $u^2 \in R^l$ such that $(\bar{x}, u^1)$ and $(\bar{x}, u^2)$ solve the Kuhn-Tucker problem [1], [3], i.e.

$$\sum_{a=1}^{m} \lambda_a^1 \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin I_1} u_r^1 \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0\,, \quad \alpha = 1, 2, \ldots, n\,,$$

$$g_r(\bar{x}) \leqq 0\,; \quad u_r^1 g_r(\bar{x}) = 0, \quad r = 1, 2, \ldots, l\,,$$

$$u_r^1 = 0\,, \quad r \in I_1 \subset \{1, 2, \ldots, l\}\,, \quad u_r^1 \geqq 0\,, \quad r \in \{1, 2, \ldots, l\} - I_1\,,$$

and

$$\sum_{a=1}^{m} \lambda_a^2 \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin I_2} u_r^2 \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0\,, \quad \alpha = 1, 2, \ldots, n\,,$$

$$g_r(\bar{x}) \leqq 0\,; \quad u_r^2\, g_r(\bar{x}) = 0\,, \quad r = 1, 2, \ldots, l\,,$$

$$u_r^2 = 0\,, \quad r \in I_2 \subset \{1, 2, \ldots, l\}\,, \quad u_r^2 \geqq 0\,, \quad r \in \{1, 2, \ldots, l\} - I_2\,.$$

Hence we deduce that for all $0 \leqq \omega \leqq 1$,

$$\sum_{a=1}^{m} [(1 - \omega)\, \lambda_a^1 + \omega \lambda_a^2] \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin (I_1 \cap I_2)} u_r^* \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0\,, \quad \alpha = 1, 2, \ldots, n\,,$$

$$g_r(\bar{x}) \leqq 0\,; \quad u_r^*\, g_r(\bar{x}) = 0\,, \quad r = 1, 2, \ldots, l\,,$$

$$u_r^* = 0\,, \quad r \in I_1 \cap I_2\,, \quad u_r^* \geqq 0\,, \quad r \in \{1, 2, \ldots, l\} - (I_1 \cap I_2)\,,$$

where

$$\begin{aligned} u_r^* &= (1 - \omega)\, u_r^1\,, & & r \in [\{1, 2, \ldots, l\} - I_1] \cap I_2\,, \\ &= \omega u_r^2\,, & & r \in I_1 \cap [\{1, 2, \ldots, l\} - I_2]\,, \\ &= (1 - \omega)\, u_r^1 + \omega u_r^2\,, & & r \in [\{1, 2, \ldots, l\} - I_1] \cap [\{1, 2, \ldots, l\} - I_2]\,, \\ &= 0\,, & & r \in I_1 \cap I_2\,. \end{aligned}$$

Therefore it follows from the sufficient optimality theorem of Kuhn-Tucker [1], [3] that $(1 - \omega)\, \lambda^1 + \omega \lambda^2 \in \mathbf{S}(\bar{x})$ for all $0 \leqq \omega \leqq 1$. Hence the set $\mathbf{S}(\bar{x})$ is convex in $'R^m$. Assume that $\hat{\lambda}$ is a boundary point of $\mathbf{S}(\bar{x})$, then for any interior point $\lambda^0$ of $\mathbf{S}(\bar{x})$

the open line segment $(\lambda^0, \hat{\lambda})$ lies in $\mathbf{S}(\bar{x})$ due to the convexity of $\mathbf{S}(\bar{x})$. For any $\lambda \in (\lambda^0, \hat{\lambda})$ we have

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad \forall x \in \mathbf{M}$$

and therefore

$$\lim_{\lambda \to \hat{\lambda}} \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) \leqq \lim_{\lambda \to \hat{\lambda}} \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad \forall x \in \mathbf{M}.$$

From the finiteness of the sum and the continuity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots$ $\ldots, m$ on $\mathbf{M}$ it follows that

$$\sum_{a=1}^{m} \lim_{\lambda \to \hat{\lambda}} [\lambda_a \Phi_a(\bar{x})] \leqq \sum_{a=1}^{m} \lim_{\lambda \to \hat{\lambda}} [\lambda_a \Phi_a(x)], \quad \forall x \in \mathbf{M}.$$

The limiting process concerns the path directed from $\lambda^0$ to $\hat{\lambda}$ as a straight line, and since $\lambda^0$ is an arbitrary point in int $\mathbf{S}(\bar{x})$, this path is considered to be arbitrary, and therefore

$$\sum_{a=1}^{m} \hat{\lambda}_a \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \hat{\lambda}_a \Phi_a(x), \quad \forall x \in \mathbf{M}.$$

Hence $\hat{\lambda} \in \mathbf{S}(\bar{x})$, and therefore the set $\mathbf{S}(\bar{x})$ is closed.

**Theorem 5.** *If* int $[\mathbf{S}(x^1) \cap \mathbf{S}(x^2)] \neq \emptyset$, *then* $\mathbf{S}(x^1) = \mathbf{S}(x^2)$, *where* $\mathbf{S}(x^1)$, $\mathbf{S}(x^2)$ *are the stability sets of the first kind of problem* (II) *corresponding to* $x^1, x^2$ *respectively* $(x^1 \neq x^2)$.

Proof. Let $\lambda^0 \in \text{int}\,[\mathbf{S}(x^1) \cap \mathbf{S}(x^2)]$, then

$$\sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^1) = \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^2). \tag{12}$$

Assume that $\lambda^1 \in \mathbf{S}(x^1)$, $\lambda^1 \neq \lambda^0$, then there exists $0 < \omega < 1$ such that $\lambda^* =$ $= (1 - \omega)\,\lambda^1 + \omega\lambda^0 \in \mathbf{S}(x^2)$, and therefore

$$\sum_{a=1}^{m} \lambda_a^* \Phi_a(x^2) \leqq \sum_{a=1}^{m} \lambda_a^* \Phi_a(x^1), \quad \text{i.e.}$$

$$(1 - \omega) \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^2) + \omega \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^2) \leqq (1 - \omega) \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^1) + \omega \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^1).$$

Using (12) we get

$$\sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^2) \leqq \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^1) \leqq \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x), \quad \forall x \in \mathbf{M},$$

therefore $\lambda^1 \in \mathbf{S}(x^2)$, and hence $\mathbf{S}(x^1) \subseteq \mathbf{S}(x^2)$. Similarly it can be shown that $\mathbf{S}(x^2) \subseteq$ $\subseteq \mathbf{S}(x^1)$. Hence $\mathbf{S}(x^1) = \mathbf{S}(x^2)$.

In order to have an analytic description for the set $\mathbf{S}(\bar{x})$ defined by (11), let us proceed in the following way: We order the functions $g_r(x)$, $r = 1, 2, \ldots, l$ in such a way that

$$r \in \{1, 2, \ldots, s\} \quad \text{if} \quad g_r(\bar{x}) = 0\,, \quad s \geqq 1\,,$$
$$r \in \{s + 1, \ldots, l\} \quad \text{if} \quad g_r(\bar{x}) < 0\,.$$

Consider the system of equations

$$(13) \qquad \sum_{a=1}^{m} \lambda_a \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r=1}^{s} u_r \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0\,, \quad \alpha = 1, 2, \ldots, n\,.$$

It represents $n$ linear homogeneous equations in $m + s$ unknowns $\lambda_a$, $a = 1, 2, \ldots, m$ and $u_r$, $r = 1, 2, \ldots, s$, which can be solved explicitly.

Suppose that $\lambda_a^* \geqq 0$, $a = 1, 2, \ldots, m$; $u_r^* \geqq 0$, $r = 1, 2, \ldots, s$ solve the system (13), then it is clear that $(\bar{x}, \bar{u})$ solves the Kuhn-Tucker problem [1], [3], where $\bar{u}_r = u_r^*$, $r = 1, 2, \ldots, s$, $\bar{u}_r = 0$, $r = s + 1, \ldots, l$ and hence $\lambda^* \in \mathbf{S}(\bar{x})$. Let us define the set denoted by $\mathbf{p}(\lambda, u)$ as follows:

$$(14) \qquad \mathbf{p}(\lambda, u) = \{(\lambda, u) \in {}'\mathrm{R}_+^m \times \mathrm{R}_+^s / (\lambda, u) \text{ solves } (13)\}\,,$$

where ${}'\mathrm{R}_+^m$; $\mathrm{R}_+^s$ are the nonnegative orthants of the ${}'\mathrm{R}^m$ vector $\lambda$-space, and $\mathrm{R}^s$ vector $u$-space, respectively. Then

$$(15) \qquad \mathbf{S}(\bar{x}) = \{\lambda \in {}'\mathrm{R}^m / (\lambda, u) \in \mathbf{p}(\lambda, u)\}\,.$$

The representation of $\mathbf{S}(\bar{x})$ by (15) can be used to prove the convexity and closedness of the set $\mathbf{S}(\bar{x})$. If $g_r(\bar{x}) < 0$, $r = 1, 2, \ldots, l$, then it is easy to see that $\mathbf{S}(\bar{x})$ can be written in the form

$$\mathbf{S}(\bar{x}) = \left\{\lambda \in {}'\mathrm{R}_+^m / \sum_{a=1}^{m} \lambda_a \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) = 0\,, \ \alpha = 1, 2, \ldots, n\right\}$$

and it is clear that this representation proves the convexity and the closedness of the set $\mathbf{S}(\bar{x})$.

It may happen that for some problems, the system (13) has only the trivial solution, and for such cases $\mathbf{S}(\bar{x})$ is a one-point set, namely $\mathbf{S}(\bar{x}) = \{0\}$.

## 3. CHARACTERIZATION OF THE STABILITY SET OF THE SECOND KIND

**Definition 3.** *Suppose that $\bar{\lambda} \in \mathbf{B}$ (see (1)) with a corresponding optimal point $\bar{x}$ and $\Sigma(\bar{\lambda}, \mathrm{J})$ denotes either the unique side of $\mathbf{M}$ from those given by $\{x \in \mathrm{R}^n / g_r(x) = 0,$ $r \in \mathrm{J};\ g_r(x) < 0,\ r \notin \mathrm{J}\}$ which contains $\bar{x}$, or* int $\mathbf{M}$. Then *the stability set of the second kind of problem* (II) *corresponding to $\Sigma(\bar{\lambda}, \mathrm{J})$ denoted by $\mathbf{Q}(\Sigma(\bar{\lambda}, \mathrm{J}))$, is defined by*

$$(16) \qquad \mathbf{Q}(\Sigma(\bar{\lambda}, \mathrm{J})) = \{\lambda \in \mathbf{B} / \mathbf{m}_{\mathrm{opt}}(\lambda) \cap \Sigma(\bar{\lambda}, \mathrm{J}) \neq \emptyset\}\,,$$

*where $\mathbf{m}_{\mathrm{opt}}(\lambda)$ is defined by* (9).

(16) gives a definition for the stability set of the second kind corresponding to a side rather than to an index set as was done in [4], and this is due to the assumption that the set $\boldsymbol{M}$ is fixed and independent of parameters.

Let us adjoin to problem (II) the following problem

$$\text{(II)}' \qquad \min \left[ \sum_{a=1}^{m} \lambda_a \Phi_a(x) \right],$$

subject to

$$'\boldsymbol{M} = \{ x \in \mathrm{R}^n / g_r(x) \leqq 0 \,, \quad r \in \mathrm{J} \}$$

where J is the index set given in the definition of $\Sigma(\bar{\lambda}, \mathrm{J})$.

**Lemma 8.** *If* $\bar{\lambda} \in \boldsymbol{B}$ *with* $\boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) \subseteq \Sigma(\bar{\lambda}, \mathrm{J})$, $\Phi_k$ *is strictly convex on* $\mathrm{R}^n$ *for at least one* $k \in \{1, 2, \ldots, m\}$ *for which* $\bar{\lambda}_k > 0$, *then*

$$\bar{x} \in \boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) \Leftrightarrow \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) = \min_{x \in 'M} \left[ \sum_{a=1}^{m} \lambda_a \Phi_a(x) \right],$$

*where* $'\boldsymbol{M}$ *is the same as in problem* (II)$'$ *and*

$$\boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) = \{ x^* \in \mathrm{R}^n / \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^*) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) \} \,.$$

Proof. i) Let $\bar{x} \in \boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda})$, then $g_r(\bar{x}) = 0$, $r \in \mathrm{J}$, $g_r(\bar{x}) < 0$, $r \notin \mathrm{J}$ and hence $\bar{x} \in {}'\boldsymbol{M}$. Assume that there exists $x^* \in {}'\boldsymbol{M}$ such that $\sum_{a=1}^{m} \bar{\lambda}_a \Phi(\bar{x}) > \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^*)$. It is easy to prove that there exists $\omega$ with $0 < \omega \leqq 1$ such that $\hat{x} = (1 - \omega) \bar{x} + \omega x^* \in \boldsymbol{M}$. From the convexity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ we obtain

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\hat{x}) \leqq (1 - \omega) \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^*) <$$
$$< (1 - \omega) \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) =$$
$$= \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x})$$

which contradicts our assumption, and hence

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) \,, \quad \forall x \in {}'\boldsymbol{M} \,,$$

i.e.

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in '\boldsymbol{M}} \left[ \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) \right].$$

ii) Let

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in '\boldsymbol{M}} \left[ \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) \right].$$

If $\bar{x} \in \mathbf{M}$, the result is clear. Suppose that $\bar{x} \notin \mathbf{M}$ and let $x^0 \in \Sigma(\bar{\lambda}, \mathrm{J})$ be an optimal point corresponding to $\bar{\lambda}(x^0 \neq \bar{x})$ with $\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0) = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x)$.

There exists a point $\tilde{x} = (1 - \omega)\bar{x} + \omega x^0 \in \mathbf{M}$, $0 < \omega \leqq 1$. Therefore, from the convexity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$; $a \neq k$ and the strict convexity of $\Phi_k(x)$, we obtain

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\tilde{x}) < (1 - \omega) \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0)$$

and by the assumption

$$\sum_{a=1}^{m} \lambda_a \Phi_a(x^0) < \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) .$$

Therefore

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0) < \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) ,$$

which contradicts our assumption, and therefore $\bar{x} = x^0$ which follows from the strict convexity of $\Phi_k(x)$. Hence $\bar{x} \in \mathbf{m}_{\mathrm{opt}}(\bar{\lambda})$.

**Lemma 9.** *If the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex on $\mathbf{M}$ and $\Sigma(\lambda^1, \mathrm{J}_1)$; $\Sigma(\lambda^2, \mathrm{J}_2)$ are two distinct sides of $\mathbf{M}$ then*

$$\mathbf{Q}(\Sigma(\lambda^1, \mathrm{J}_1)) \cap \mathbf{Q}(\Sigma(\lambda^2, \mathrm{J}_2)) = \{0\} .$$

Proof. It is clear that $\lambda = 0$ belongs to all stability sets of the second kind corresponding to different sides of $\mathbf{M}$. Suppose that $\lambda^* \in \mathbf{Q}(\Sigma(\lambda^1, \mathrm{J}_1)) \cap \mathbf{Q}(\Sigma(\lambda^2, \mathrm{J}_2))$, $\lambda^* \neq 0$, then (16) yields

$$\mathbf{m}_{\mathrm{opt}}(\lambda^*) \cap \Sigma(\lambda^1, \mathrm{J}_1) \neq \emptyset ,$$

$$\mathbf{m}_{\mathrm{opt}}(\lambda^*) \cap \Sigma(\lambda^2, \mathrm{J}_2) \neq \emptyset .$$

This leads to a contradiction, since $\mathbf{m}_{\mathrm{opt}}(\lambda^*)$ by the assumption consists only of a single point. Hence the result.

In order to have more properties concerning the stability set of the second kind, let us concentrate our attention to the problem

$$(\mathrm{II})_q \qquad \min \left[ \sum_{i,j=1}^{n} \tfrac{1}{2} c_{ij} x_i x_j + \sum_{i=1}^{n} p_i x_i \right] ,$$

subject to the restriction set $\mathbf{M}$,

where $[c_{ij}]$, $i, j = 1, 2, \ldots, n$ is a real symmetric positive semidefinite matrix, $p_i$, $i = 1, 2, \ldots, n$ are arbitrary parameters and $\mathbf{M}$ is the same set as in problem (II).

**Lemma 10.** *If* $\Sigma(\bar{p}, J_L)$ *denotes either a linear side of* $\boldsymbol{M}$ *or* int $\boldsymbol{M}$, *then the stability set of the second kind of problem* $(\mathrm{II})_q$ *corresponding to* $\Sigma(\bar{p}, J_L)$ *denoted by* $\mathbf{Q}_q(\Sigma(\bar{p}, J_L))$ *is convex in* $'\mathrm{R}^n$ *(the vector space of* $p_\alpha$, $\alpha = 1, 2, \ldots, n$*).*

Proof. The proof will be done for the case of a linear side of $\boldsymbol{M}$, the proof for the case of int $\boldsymbol{M}$ being similar. Suppose that $p^1$, $p^2$ are two points in $\mathbf{Q}_q(\Sigma(\bar{p}, J_L))$, then there exist $u^1$, $u^2$ in $\mathrm{R}^l$ such that $(x^1, u^1)$ and $(x^2, u^2)$ solve the Kuhn-Tucker problem [1], [3], where

$$x^1 \in \boldsymbol{m}_{\mathrm{opt}}(p^1) \cap \Sigma(\bar{p}, J_L), \quad x^2 \in \boldsymbol{m}_{\mathrm{opt}}(p^2) \cap \Sigma(\bar{p}, J_L),$$
$$\Sigma(\bar{p}, J_L) = \{x \in \mathrm{R}^n / g_r(x) = 0, \quad r \in J_L, \quad g_r(x) < 0, r \notin J_L\},$$

and the functions $g_r(x)$, $r \in J_L$ are linear over $\boldsymbol{M}$. Therefore,

$$\sum_{j=1}^{n} c_{\alpha j} x_j^1 + p_\alpha^1 + \sum_{r \in J_L} u_j^1 \frac{\partial g_r}{\partial x_\alpha}(x^1) = 0, \qquad \alpha = 1, 2, \ldots, n,$$
$$g_r(x^1) = 0, \quad r \in J_L, \quad g_r(x^1) < 0, \qquad r \notin J_L,$$
$$u_r^1 \, g_r(x^1) = 0, \qquad r = 1, 2, \ldots, l,$$
$$u_r^1 = 0, \quad r \notin J_L, \quad u_r^1 \geqq 0, \qquad r \in J_L$$

and

$$\sum_{j=1}^{n} c_{\alpha j} x_j^2 + p_\alpha^2 + \sum_{r \in J_L} u_r^2 \frac{\partial g_r}{\partial x_\alpha}(x^2) = 0, \qquad \alpha = 1, 2, \ldots, n,$$
$$g_r(x^2) = 0, \quad r \in J_L, \quad g_r(x^2) < 0, \qquad r \notin J_L,$$
$$u_r^2 \, g_r(x^2) = 0, \qquad r = 1, 2 \ldots, l,$$
$$u_r^2 = 0, \quad r \notin J_L, \quad u_r^2 \geqq 0, \qquad r \in J_L.$$

Hence it follows from the linearity of the functions $g_r(x)$, $r \in J_L$ that for all $0 \leqq \omega \leqq 1$ we have

$$\sum_{j=1}^{n} c_{\alpha j} x_j^* + p_\alpha^* + \sum_{r \in J_L} u_r^* \frac{\partial g_r}{\partial x_\alpha}(x^*) = 0, \qquad \alpha = 1, 2, \ldots, n,$$
$$g_r(x^*) = 0, \quad r \in J_L, \quad g_r(x^*) < 0, \qquad r \notin J_L.$$
$$u_r^* \, g_r(x^*) = 0, \qquad r = 1, 2, \ldots, l,$$
$$u_r^* = 0, \quad r \notin J_L, \quad u_r^* \geqq 0, \qquad r \in J_L,$$
$$x^* = (1 - \omega) x^1 + \omega x^2,$$
$$p^* = (1 - \omega) p^1 + \omega p^2,$$
$$u^* = (1 - \omega) u^1 + \omega u^2.$$

This together with the Kuhn-Tucker sufficient optimality theorem [1], [3] implies that

$$x^* \in \boldsymbol{m}_{\text{opt}}(p^*) \cap \Sigma(\bar{p}, J_L)$$

for all $0 \leqq \omega \leqq 1$. Hence the set $\mathbf{Q}_q(\Sigma(\bar{p}, J_L))$ is convex.

Remark 6. It is easy to prove that (see Lemma 9) if $[c_{ij}]$, $i; j = 1, 2, \ldots, n$ is a real symmetric positive definite matrix, then the nonempty stability sets of the second kind of problem (II) corresponding to certain sides of $\boldsymbol{M}$, int $\boldsymbol{M}$ are mutually disjoint and all together exhaust the solvability set of problem $(\text{II})_q$.

Example. Consider the problem
Minimize

$$[x_1^2 + x_2^2 + p_1 x_1 + p_2 x_2],$$

subject to

$$\boldsymbol{M} = \{x \in \mathrm{R}^2 / x_1^2 + x_2^2 \leqq 1, \quad x_1 + x_2 \leqq 1\}.$$

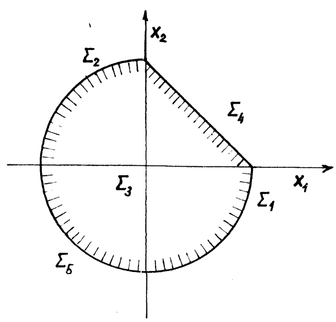


Fig. a. The set $\boldsymbol{M}$.

The set $\boldsymbol{M}$ is compact, and therefore $\mathbf{B} = {}'\mathrm{R}_2$. $\boldsymbol{M}$ consists of four distinct sides and int $\boldsymbol{M}$ (see Fig. a). Let $\mathbf{Q}_i$ denote the stability sets of the second kind corresponding to the sides $\Sigma_i$, $i = 1, 2, 4, 5$ while $\mathbf{Q}_3$ is the stability set of the second kind corresponding to $\Sigma_3 \equiv$ int $\boldsymbol{M}$. Then the sets $\mathbf{Q}_i$, $i = 1, 2, \ldots, 5$ are obtained in the form (see Fig. b)

$$\mathbf{Q}_1 = \{p \in {}'\mathrm{R}^2 / p_2 \leqq 0, \ p_2 - p_1 - 2 \geqq 0\},$$

$$\mathbf{Q}_2 = \{p \in {}'\mathrm{R}^2 / p_1 \leqq 0, \ p_1 - p_2 - 2 \geqq 0\},$$

$$\mathbf{Q}_3 = \{p \in {}'\mathrm{R}^2 / p_1^2 + p_2^2 < 4, \ p_1 + p_2 > -2\},$$

$$\mathbf{Q}_4 = \{p \in {}'\mathrm{R}^2 / p_1 + p_2 \leqq -2, -2 < p_2 - p_1 < 2\},$$

$$\mathbf{Q}_5 = \{p \in {}'\mathrm{R}^2 / p_1 > 0, \ p_2 > 0, \ p_1^2 + p_2^2 \geqq 4\} \cup$$
$$\cup \{p \in {}'\mathrm{R}^2 / p_1 < 0, \ p_2 > 0, \ p_1^2 + p_2^2 \geqq 4\} \cup$$
$$\cup \{p \in {}'\mathrm{R}^2 / p_1 > 0, \ p_2 < 0, \ p_1^2 + p_2^2 \geqq 4\}.$$

The set $\mathbf{B}$ is decomposed into the sets $\mathbf{Q}_i$, $i = 1, 2, 3, 4, 5$, and $\mathbf{Q}_i \cap \mathbf{Q}_j = \emptyset$, $i \neq j$, $i; j = 1, 2, 3, 4, 5$. The sets $\mathbf{Q}_i$, $i = 1, 2, 3, 4$ are convex. The convexity and the closedness of the sets $\mathbf{Q}_1$, $\mathbf{Q}_2$ follows from the fact that

$$\mathbf{Q}_1 = \mathbf{S}(1, 0),$$

$$\mathbf{Q}_2 = \mathbf{S}(0, 1),$$

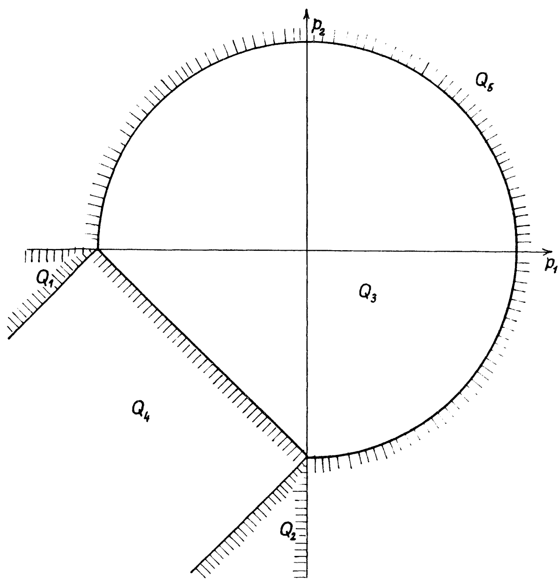


Fig. b. The nonempty stability sets of the second kind.

where $\mathbf{S}(1, 0)$, $\mathbf{S}(0, 1)$ are the stability sets of the first kind of our problem corresponding to the points $(1, 0)$, $(0, 1)$ respectively.

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## Souhrn

# KVALITATIVNÍ ANALÝZY ZÁKLADNÍCH POJMŮ PARAMETRICKÉHO KONVEXNÍHO PROGRAMOVÁNÍ, II

## (Parametry v cílové funkci)

MOHAMED SAYED ALI OSMAN

V článku je podána kvalitativní analýza základních pojmů parametrického konvexního programování pro konvexní programy s parametry v cílové funkci. Jsou to pojmy množiny přípustných parametrů, množiny řešitelnosti a množin stability prvního a druhého druhu. Předpokládá se, že vyšetřované funkce mají spojité parciální derivace prvního řádu v $R^n$ a že parametry nabývají libovolných reálných hodnot. Výsledky mohou být použity pro širokou třídu konvexních programů.


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