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A METHOD OF CONSTRUCTING GENERAL CONTACT TANGENTIAL CHARTS

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1. Introduction

Concerning the contact tangential charts of three variables, Edward Otto [1] showed a method of obtaining the nomograms consisting of two curvilinear scales and a family of envelopes.

Recently, furthermore, Evžen Jokl [2] and Jaroslav Záhora [3] researched, respectively, the methods of constructing the contact tangential charts of three variables or more.

In this paper, the authors give another method of constructing general contact tangential charts, and show some examples of them.

2. General theory of contact tangential charts of three variables

In this article we consider a method of constructing the contact tangential chart of the general functional relation

$$F_{123}(t_1, t_2, t_3) = 0\,, \tag{1}$$

where F_{123} is a real function of three real variables t_1, t_2 and t_3.

Firstly, we give the following two pairs of equations, involving the parameters α and β, respectively:

$$(t_1): \quad x = f_1(t_1, \alpha)\,, \quad y = g_1(t_1, \alpha)\,; \tag{2}$$

$$(t_2): \quad x = f_2(t_2, \beta)\,, \quad y = g_2(t_2, \beta)\,, \tag{3}$$

where we assume that f_1, g_1 and f_2, g_2 are continuous functions of t_1 and t_2, respectively, and also that they are of class C^1 with respect to α and β.

Then, the equation of the family of tangents of a t_1-curve of (t_1)-curves is expressed by

$$(4)\qquad Y - g_1(t_1, \alpha) = \left(\frac{\partial g_1}{\partial \alpha}\Big/\frac{\partial f_1}{\partial \alpha}\right)\{X - f_1(t_1, \alpha)\},$$

where X, Y are current coordinates; and the equation of the family of tangents of a t_2-curve of (t_2)-curves is also, similarly, expressed by

$$(5)\qquad Y - g_2(t_2, \beta) = \left(\frac{\partial g_2}{\partial \beta}\Big/\frac{\partial f_2}{\partial \beta}\right)\{X - f_2(t_2, \beta)\}.$$

From (4) and (5), we have

$$(4')\qquad \frac{\partial g_1}{\partial \alpha}X - \frac{\partial f_1}{\partial \alpha}Y + \frac{\partial f_1}{\partial \alpha}g_1 - \frac{\partial g_1}{\partial \alpha}f_1 = 0,$$

$$(5')\qquad \frac{\partial g_2}{\partial \beta}X - \frac{\partial f_2}{\partial \beta}Y + \frac{\partial f_2}{\partial \beta}g_2 - \frac{\partial g_2}{\partial \beta}f_2 = 0.$$

Therefore, a necessary and sufficient condition that the two tangents expressed by (4′) and (5′) are the same is given by

$$(6)\qquad \frac{\dfrac{\partial g_1}{\partial \alpha}}{\dfrac{\partial g_2}{\partial \beta}} = \frac{\dfrac{\partial f_1}{\partial \alpha}}{\dfrac{\partial f_2}{\partial \beta}} = \frac{\dfrac{\partial f_1}{\partial \alpha}g_1 - \dfrac{\partial g_1}{\partial \alpha}f_1}{\dfrac{\partial f_2}{\partial \beta}g_2 - \dfrac{\partial g_2}{\partial \beta}f_2},$$

that is,

$$(7)\qquad \frac{\partial g_1}{\partial \alpha}\frac{\partial f_2}{\partial \beta} - \frac{\partial f_1}{\partial \alpha}\frac{\partial g_2}{\partial \beta} = 0,$$

$$(8)\qquad \frac{\partial g_1}{\partial \alpha}\left\{\frac{\partial f_2}{\partial \beta}g_2 - \frac{\partial g_2}{\partial \beta}f_2\right\} - \frac{\partial g_2}{\partial \beta}\left\{\frac{\partial f_1}{\partial \alpha}g_1 - \frac{\partial g_1}{\partial \alpha}f_1\right\} = 0.$$

Now considering (7) and (8) as a system of equations with respect to α and β, we have the following solutions:

$$(9)\qquad \alpha = \alpha(t_1, t_2),$$

$$(10)\qquad \beta = \beta(t_1, t_2).$$

In general, the equation of the straight line passing through two points $\{f_1(t_1, \alpha), g_1(t_1, \alpha)\}$ and $\{f_2(t_2, \beta), g_2(t_2, \beta)\}$ is expressed by

$$(11)\qquad Y - g_1(t_1, \alpha) = \frac{g_2(t_2, \beta) - g_1(t_1, \alpha)}{f_2(t_2, \beta) - f_1(t_1, \alpha)}\{X - f_1(t_1, \alpha)\}.$$

Hence, substituting expressions (9) and (10) into (11), we have the equation of the common tangent of a t_1-curve and t_2-curve; that is, we have an equation of the form

$$Y - Y_0(t_1, t_2) = \Phi(t_1, t_2)\{X - X_0(t_1, t_2)\}, \tag{12}$$

where Φ is a certain function of t_1 and t_2.

Next, solving (1) with respect to t_2 and putting $t_3 = t_3^{(0)} = \text{const.}$, we have

$$t_2 = f(t_1, t_3^{(0)}). \tag{13}$$

Substituting (13) into (12), we obtain the equation of the family of tangents, with a parameter t_1, of a t_3^0-curve:

$$Y - \overline{Y}_0(t_1, t_3^{(0)}) = \overline{\Phi}(t_1, t_3^{(0)})\{X - \overline{X}_0(t_1, t_3^{(0)})\}, \tag{14}$$

where $\overline{X}_0$, $\overline{Y}_0$ and $\overline{\Phi}$ are certain functions of t_1 and $t_3^{(0)}$, respectively.

Differentiating (14) partially with respect to t_1, we have

$$-\frac{\partial \overline{Y}_0}{\partial t_1} = \frac{\partial \overline{\Phi}}{\partial t_1}(X - \overline{X}_0) - \overline{\Phi}\frac{\partial \overline{X}_0}{\partial t_1}. \tag{15}$$

Hence, eliminating t_1 from (14) and (15), we have an equation of the form

$$G(X, Y, t_3^{(0)}) = 0, \tag{16}$$

which is an equation of a $t_3^{(0)}$-curve (Fig. 1).

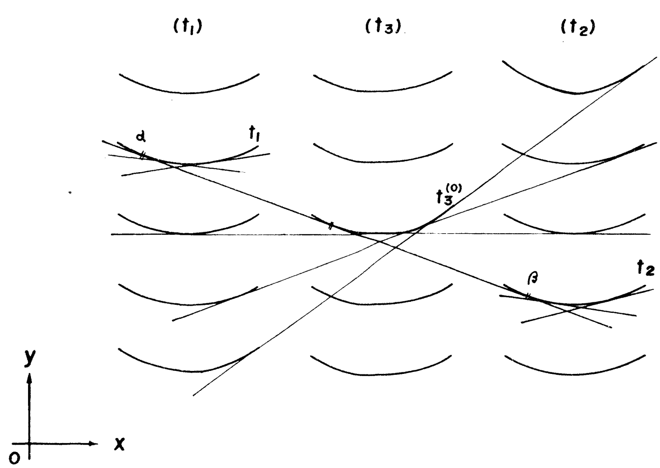

Fig. 1. Skeleton of the contact tangential chart by the enveloping method.

Writing x, y, t_3 for X, Y, $t_3^{(0)}$ in (16), we have the required equation of the family of (t_3)-curves:

$$G(x, y, t_3) = 0 \; ; \tag{17}$$

and finally we have obtained the three required equations, representing our contact tangential chart of the given functional equation (1), that is,

$$(t_1): \quad x = f_1(t_1, \alpha), \quad y = g_1(t_1, \alpha) \; ; \tag{2}$$

$$(t_2): \quad x = f_2(t_2, \beta), \quad y = g_2(t_2, \beta) \; ; \tag{3}$$

$$(t_3): \quad G(x, y, t_3) = 0 \, . \tag{17}$$

Or again we have three pairs of parametric equations,

$$(t_1): \quad x = f_1(t_1, \alpha), \quad y = g_1(t_1, \alpha) \; ; \tag{2}$$

$$(t_2): \quad x = f_2(t_2, \beta), \quad y = g_2(t_2, \beta) \; ; \tag{3}$$

$$(t_3): \quad x = f_3(t_3, \gamma), \quad y = g_3(t_3, \gamma) \, . \tag{18}$$

Example 1. We construct a contact tangential chart of the relation

$$t_1^2 + t_2^2 = t_3^2 \quad (t_1, t_2, t_3 > 0) \, . \tag{19}$$

Letting the parametric equations of (t_1)- and (t_2)-curves be

$$(t_1): \quad x = t_1^2 \cos\alpha - a \, , \quad y = t_1^2 \sin\alpha \quad (0 < \alpha < \pi) \; ; \tag{20}$$

$$(t_2): \quad x = t_2^2 \cos\beta + a \, , \quad y = t_2^2 \sin\beta \quad (0 < \beta < \pi) \, , \tag{21}$$

each of these equations represents a family of concentric circles (Fig. 2).

Calculating (7) and (8) by means of (20) and (21), we obtain, respectively,

$$\alpha = \beta \, , \tag{22}$$

$$\cos\alpha(t_2^2 + a\cos\beta) = \cos\beta(t_1^2 - a\cos\alpha) \, . \tag{23}$$

From these expressions, we have

$$\alpha = \cos^{-1}\left(\frac{t_1^2 - t_2^2}{2a}\right), \tag{24}$$

$$\beta = \cos^{-1}\left(\frac{t_1^2 - t_2^2}{2a}\right). \tag{25}$$

Hence according to (11), the equation of the straight line passing through the points $(t_1^2 \cos\alpha - a, \, t_1^2 \sin\alpha)$, $(t_2^2 \cos\beta + a, \, t_2^2 \sin\beta)$ is

$$Y - t_1^2 \sin\alpha = \frac{t_2^2 \sin\beta - t_1^2 \sin\alpha}{(t_2^2 \cos\beta + a) - (t_1^2 \cos\alpha - a)} \{X - (t_1^2 \cos\alpha - a)\} \, . \tag{26}$$

Substituting (22) into the above expression, we have

$$(27)\quad (t_1^2 - t_2^2)\sin\alpha \cdot X - \{(t_1^2 - t_2^2)\cos\alpha - 2a\}\, Y - (t_1^2 + t_2^2)\, a \sin\alpha = 0\,.$$

Again substituting (24) into (27), we obtain after some calculations

$$(28)\quad (t_1^2 - t_2^2)\sqrt{(4a^2 - (t_1^2 - t_2^2)^2)}\, X - \{(t_1^2 - t_2^2)^2 - 4a^2\}\, Y - \\ - (t_1^2 + t_2^2)\sqrt{(4a^2 - (t_1^2 - t_2^2)^2)}\, a = 0\,.$$

Putting $t_3 = t_3^{(0)} = \text{const.}$ in the given relation (19), we have

$$(29)\quad t_2^2 = t_3^{(0)^2} - t_1^2\,,$$

and substituting (29) into (28), we obtain

$$(30)\quad t\sqrt{(4a^2 - t^2)}\, X + (4a^2 - t^2)\, Y - a t_3^{(0)^2}\sqrt{(4a^2 - t^2)} = 0\,,$$

where

$$(31)\quad t \equiv 2t_1^2 - t_3^{(0)^2}\,.$$

Differentiating (30) partially with respect to t_1, we also obtain

$$(32)\quad 2(2a^2 - t^2)\, X - 2t\sqrt{(4a^2 - t^2)}\, Y + a t t_3^{(0)^2} = 0\,.$$

Hence, eliminating t from (30) and (32), and rewriting with $t_3^{(0)} = t_3$, we have the required equation of the (t_3)-curves in the xy-plane:

$$(33)\quad x^2 + y^2 = \left(\frac{t_3^2}{2}\right)^2,$$

and this is nothing but the equation of concentric circles (t_3) with the origin as their centers.

Fig. 2. shows our required contact tangential chart of the given relation (19).

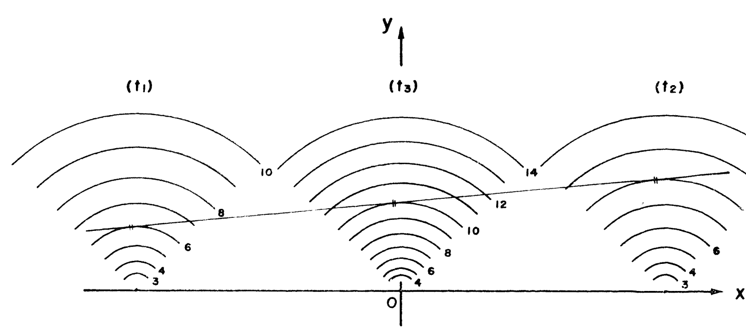

Fig. 2. Contact tangential chart of $t_1^2 + t_2^2 = t_3^2$. The figure shows that $t_1 = 6$, $t_2 = 8 \Rightarrow t_3 = 10$.

Remark 1. Our contact tangential charts of three variables being the dual of the concurrent charts consisting of three families of curves, we can construct the contact tangential charts automatically by the method of Adams' Scanner [4].

3. Contact tangential charts consisting of one curvilinear scale and two families of envelopes

Let the general functional relation of three variables (1) be given. We assume the following two pairs of equations:

$$(t_1): \quad x = f_1(t_1), \qquad y = g_1(t_1); \tag{34}$$

$$(t_2): \quad x = f_2(t_2, \alpha), \quad y = g_2(t_2, \alpha), \tag{35}$$

and the assumptions for f_i, g_i $(i = 1, 2)$ are similar to those for the equations (2) and (3).

Geometrically speaking, in this case the family of curves which is expressed by (2) degenerates into a curvilinear support, expressed by (34).

Firstly, the equation of the family of tangents of a t_2-curve of (t_2)-curves is expressed by

$$Y - g_2(t_2, \alpha) = \left(\frac{\partial g_2}{\partial \alpha}\Big/\frac{\partial f_2}{\partial \alpha}\right)\{X - f_2(t_2, \alpha)\}; \tag{36}$$

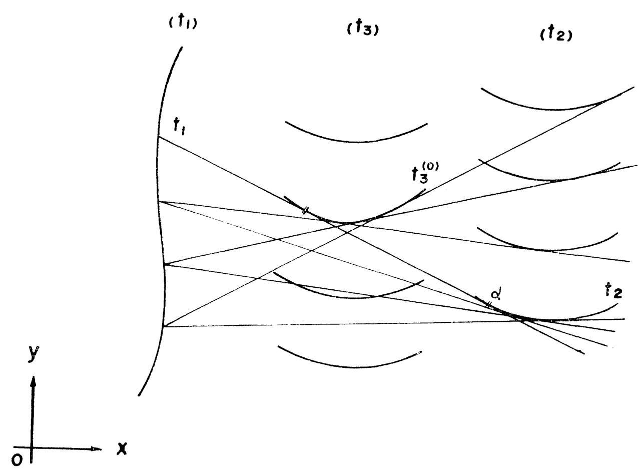

Fig. 3. Skeleton of the contact tangential chart consisting of one curvilinear scale and two families of envelopes.

and if one of these tangents passes through a scaled point t_1 of (t_1)-curvilinear scales, we have

$$g_1(t_1) - g_2(t_2, \alpha) = \left(\frac{\partial g_2}{\partial \alpha} \Big/ \frac{\partial f_2}{\partial \alpha}\right) \{f_1(t_1) - f_2(t_2, \alpha)\}. \tag{37}$$

Solving the above equation with respect to α, we may obtain the following functional form:

$$\alpha = \alpha(t_1, t_2) . \tag{38}$$

On the other hand, the equation of the straight line passing through two points $\{f_1(t_1), g_1(t_1)\}$ and $\{f_2(t_2, \alpha), g_2(t_2, \alpha)\}$ is

$$Y - g_1(t_1) = \frac{g_2(t_2, \alpha) - g_1(t_1)}{f_2(t_2, \alpha) - f_1(t_1)} \{X - f_1(t_1)\} . \tag{39}$$

Substituting (38) into (39), we have an equation which represents the straight line that passes through the point $\{f_1(t_1), g_1(t_1)\}$, and is tangential to a t_2-curve $\{f_2(t_2, \alpha), g_2(t_2, \alpha)\}$; that is,

$$Y - g_1(t_1) = \Phi(t_1, t_2) \{X - f_1(t_1)\} . \tag{40}$$

Again substituting (13) into the above expression, we have

$$Y - g_1(t_1) = \bar{\Phi}(t_1, t_3^{(0)}) \{X - f_1(t_1)\} . \tag{41}$$

Differentiating this expression partially with respect to t_1, and eliminating t_1 from the expression and (41), and, furthermore, writing x, y, t_3 for X, Y, $t_3^{(0)}$, we obtain the required equation of the family of (t_3)-curves:

$$G(x, y, t_3) = 0 , \tag{42}$$

or

$$(t_3) : \quad x = f_3(t_3, \beta) , \quad y = g_3(t_3, \beta) , \tag{43}$$

where β is a parameter of every t_3-curve.

Hence we have obtained the required three pairs of equations (34), (35) and (42) or (43), representing our contact tangential chart consisting of one curvilinear scale and two families of envelopes (Fig. 3).

Example 2. Construct a contact tangential chart of Ohm's law

$$\frac{E}{R} = I , \tag{44}$$

where $E \in \langle 1;100 \rangle$, [V]; $R \in \langle 1;5 \rangle$, [$\Omega$]; $I \in \langle 2;10 \rangle$, [A].

Let the parametric equations of (E)- and (R)-curves be, respectively,

$$(E): \quad x = -mE, \qquad y = 0; \tag{45}$$

$$(R): \quad x = R^2\alpha^2 + R^2, \quad y = 2R^2\alpha \quad (\alpha > 0), \tag{46}$$

where m is a scale modulus of rectilinear functional scale of (E), and α the parameter of parabolas (R).

The equation of the family of tangents of an R-parabola is obtained from (36) in the form

$$X - \alpha Y + R^2\alpha^2 - R^2 = 0; \tag{47}$$

and if one of these tangents passes through a scaled point $(-mE, 0)$, we have from (37)

$$-mE + R^2\alpha^2 - R^2 = 0. \tag{48}$$

Solving (48) with respect to $\alpha(\alpha > 0)$, we obtain

$$\alpha = \frac{1}{R}\sqrt{(R^2 + mE)}. \tag{49}$$

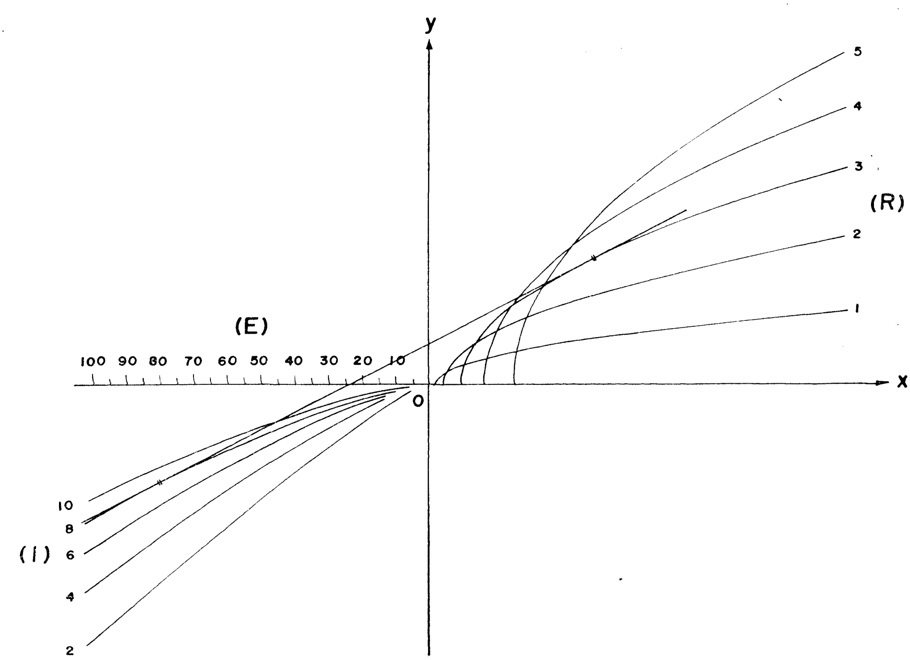

Fig. 4. Contact tangential chart of $E/R = I$. The figure shows that $E = 24\text{V}$, $R = 3\Omega \Rightarrow I = 8\text{A}$.

On the other hand, generally, the equation of the straight line passing through two points $(-mE, 0)$ and $(R^2\alpha^2 + R^2, 2R^2\alpha)$ is, from (39),

$$(50) \qquad 2R^2\alpha X - (R^2\alpha^2 + R^2 + mE)\, Y + 2R^2\alpha mE = 0\,.$$

Substituting (49) into (50), we obtain

$$(51) \qquad RX - \sqrt{(R^2 + mE)}\, Y + RmE = 0\,.$$

Eliminating R from the above expression and the given functional relation (44) we have

$$(52) \qquad EX - \sqrt{(E^2 + mEI^2)}\, Y + mE^2 = 0\,,$$

and this is nothing but the equation of straight lines enveloping an I-curve, with parameter E.

Then, differentiating (52) partially with respect to E, we have

$$(53) \qquad 2\sqrt{(E^2 + mEI^2)}\, X - (2E + mI^2)\, Y + 4mE\sqrt{(E^2 + mEI^2)} = 0\,.$$

From (52) and (53) we obtain, finally a parametric representation of (I)-curves, with parameter E:

$$(54) \qquad (I)\colon \quad x = -\frac{E(2E + 3mI^2)}{I^2}\,, \quad y = -\frac{2E\sqrt{(E^2 + mEI^2)}}{I^2}\,,$$

writing x, y for X, Y.

Or, eliminating E from the two expressions of (54), we have an algebraic equation of the fourth degree with respect to x and y, representing (I)-curves, that is,

$$(55) \qquad 4x^4 - 8x^2y^2 + 4y^4 - 4a^2x^3 + 36a^2xy^2 - 27a^4y^2 = 0\,,$$

where $a \equiv mI$.

Hence we have obtained three pairs of equations (45), (46) and (54), representing our required contact tangential chart, which is shown in Fig. 4.

Remark 2. Contact tangential charts consisting of two curvilinear scales and one family of envelopes were already studied by Edward Otto [1].

4. Contact tangential charts of four variables or more

Let the given functional relation of four variables be

$$(56) \qquad F_{1234}(t_1, t_2, t_3, t_4) = 0\,.$$

Assuming that the above expression is separable into the following two equations

$$(57) \qquad f(t_1, t_2, t_0) = 0 \,, \quad g(t_3, t_4, t_0) = 0 \,,$$

where t_0 is a parameter, we have two contact tangential charts from the theory of preceding sections, that is, one consisting of (t_1)-, (t_2)- and (t_0)-curves, the other of (t_3)-, (t_4)-, and (t_0)-curves.

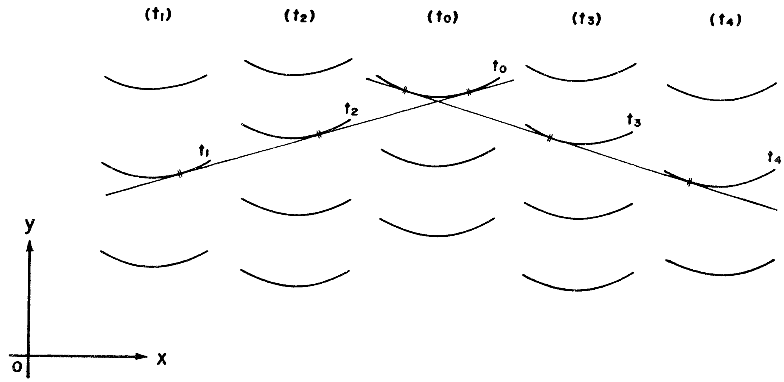

Fig. 5. Skeleton of the double contact tangential chart of four variables.

Thus obtained double contact tangential chart and the method of solution of the chart are shown in Fig. 5, where (t_0)-curves are those with no scale.

It is clear that the contact tangential charts of five variables or more can be constructed analogously.

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Souhrn

METODA KONSTRUKCE OBECNÝCH DOTYKOVÝCH NOMOGRAMŮ

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Budiž F_{123} reálná funkce tří reálných proměnných t_1, t_2 a t_3. V článku je uvedena metoda konstrukce dotykového nomogramu vztahu $F_{123} = 0$ metodou obálek. Jsou-li dány parametrické rovnice (t_1)- a (t_2)-křivek, je možno získat parametrickou rovnici (t_3)-křivek klasickou diferenciálně geometrickou metodou. Je uvedeno několik příkladů.

Dále je vyšetřován speciální případ obecných dotykových nomogramů, složených z jedné křivé stupnice a dvou soustav obálek.

V závěru se zkoumají dotykové nomogramy čtyř a více proměnných.

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