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Rhythmic maximal evenness: rhythm in voice-leading space

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RHYTHMIC MAXIMAL EVENNESS:

RHYTHM IN VOICE-LEADING SPACE

by

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ABSTRACT

Maximal evenness was first introduced in the music theory domain by John Clough and Jack Douthett. Later, the concept was explored by others such as Dmitri Tymoczko and Richard Cohn. Although maximal evenness was first explored in respect to pitch-classes, the concept can be understood in the rhythmic domain. An explanation of voice-leading space can be found here to create a conceptual foundation before departing to the implications of maximal evenness on rhythm. This thesis will then explore the concept further by exploring music from Steve Reich and György Ligeti to demonstrate applicability and deeper understand of the concept.

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LIST OF ABBREVIATIONS

ME	Maximal Evenness

1. Introduction

To this end, since duration is a measure of distance between time points, as interval is a measure of distance between pitch points, we begin by interpreting interval as duration.[1]
– Milton Babbitt

In 1962, Milton Babbitt proposed that pitch and rhythm can be viewed as analogous, in contrast to the belief that pitch and rhythm had to be treated separately. Babbitt's contribution to the field of music theory has opened up a wide variety of possibilities for future theorists to explore. By defining a direct relationship between pitch and rhythm, music becomes one coherent universe in which the two aspects of music are no longer separate. As stated in the epigraph above, interpreting durations as intervals allow an analogue to be defined between the behavior of pitch and rhythm. This means that we can now utilize theories about pitch and apply them to rhythm instead. With so many possibilities available, we are able to understand pitch concepts at a deeper level. This project continues to build on Babbitt's pitch-to-rhythm analogy by using a variety of theorists' contributions to determine the degree onto which voice leading is an inherent part of rhythm. This thesis contributes to the study of rhythmic voice leading by examining Steve Reich's "New York Counterpoint" and György Ligeti's "Étude 8: Fém."

In order to contribute to the theory of rhythmic voice leading, the discussion will draw upon a variety of sources that explore theories of voice-leading space. These theories will be applied to concepts of rhythm and the concept of rhythmic voice-leading space. The foremost pitch theory that will be used has been explored by John Clough, Jack Douthett,

[1] Milton Babbitt, "Twelve-Tone Rhythmic Structure and the Electric Medium." *Perspectives of New Music* 1, no. 1 (1962): 49–79.

and Dmitri Tymoczko.[2] They are among many theorists who have discussed maximal evenness (ME), a concept that relates intimately to voice leading and voice-leading space. The work that these authors have supplied can easily be expanded to aid in the description of rhythm as it exists in voice-leading space. This paper will argue that ME sets function in the background behind rhythmic patterns – much like how we can claim that there is a structure functioning in the background of tonal music. This thesis explores ME rhythmic patterns found in various pieces, which creates an argument for voice leadings' existence within the rhythmic domain.

1.1 Rhythmic Voice Leading Measures

The concept of voice leading has been explored throughout music history for many reasons including, but not limited to, part writing, counterpoint, and analysis. In a classroom, it is common to hear the phrases "tendency tones," "parallel motion," "contrary motion," and "voice doublings," all of which relate to voice leading but only in respect to pitch. Though many theorists have taken to explore the implications that voice leading has for music, it is even more interesting and revealing to take these theories and apply them to rhythm. Throughout this section we will explore voice-leading space and voice leading measures in regards to rhythm while using concepts created in the pitch domain for the point of departure.

Beyond simply writing four parts and being conscious of "good" voice leading, as

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[2] John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35 (1991): 93-173.

Dmitri Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011).

many musicians have been instructed to do through their theory education, it has been discovered that voice leading can be represented geometrically as a space. Once we think of a collection of rhythmic onsets (or chords in the case of pitch) as one dimensional points, then we can represent these collections in higher dimensional spaces. When these points exist in higher-dimensional spaces, we then are able to geometrically measure voice leading and use these "voice leading graphs" as analytical tools. Dmitri Tymoczko has explored voice leading space in his book *A Geometry of Music*, and most information to follow will be based on his findings, however, applying the concepts to rhythm instead of pitch.[3]

To begin, there should be a familiarity with labeling onsets within rhythm. A rhythmic onset can be labeled as the beat on which a rhythm occurs. This means that the two rhythms seen in Fig. 1 – 1 are equivalent in terms of onsets – the length of a held note does not matter, because onsets are only concerned with the moment of attack. The convention of labeling onsets starts with 0 and continues through $c - 1$, where c is the total number of beats in that pattern. Theorists have since explored the connection of pitch and

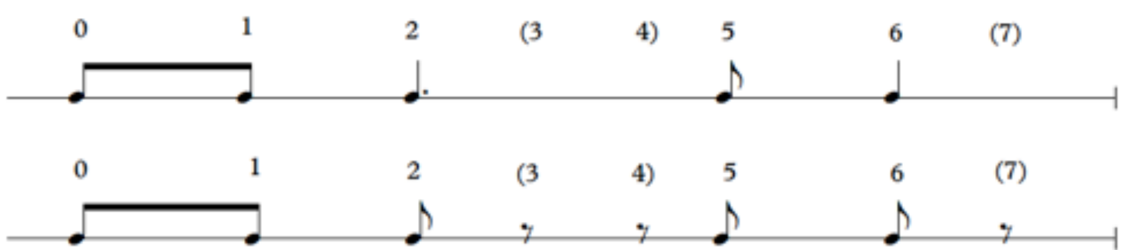

Fig. 1 – 1 Two rhythms representing the same onsets

[3] Dmitri Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011).

rhythm in this domain using the concept of beat-class as an analogue to pitch-class.[4]

By using the onset number labels, these rhythms can now be represented on a continuous line. This continuous line is an analogue to time, whereas the onset labels discretize the time depending on the rhythm in question. The onset numbers will represent different rhythmic values in relation to the rhythm. As seen in Fig. 1 – 2, an ♪ is used as the

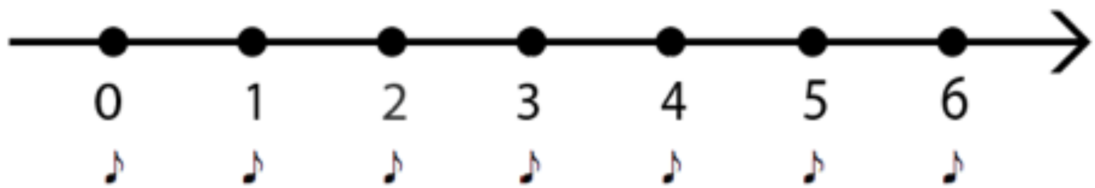

Fig. 1 – 2 Linear rhythmic space

unit of measure, meaning that the rhythm from Fig 1 – 1 can easily be represented on this line. Fig 1 – 3 shows multiple options for a unit of measure, displaying that time will always be represented as a continuous line but the values found on the line are flexible. Although subdivisions can be represented on these spaces (i.e. 𝅘𝅥𝅯 on the ♪ continuous line), any unit of rhythmic value is possible for representing real numbers as rhythmic values existing in time.

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[4] Richard Cohn, "Transpositional Combination of Beat -Class Sets in Steve Reich's Phase-Shifting Music," *Perspectives of New Music* 30, no. 2 (1992): 146–177.

Justin Colannino, Francisco Gómez and Godfried T. Toussaint, "Analysis of Emergent Beat-Class Sets in Steve Reich's "Clapping Music" and the Yoruba Bell Timeline," *Perspectives of New Music* 47, no. 1 (20099): 111-134.

John Roeder, "Beat-Class Modulation in Steve Reich's Music," *Music Theory Spectrum* 25, no. 2 (2003): 275–304.

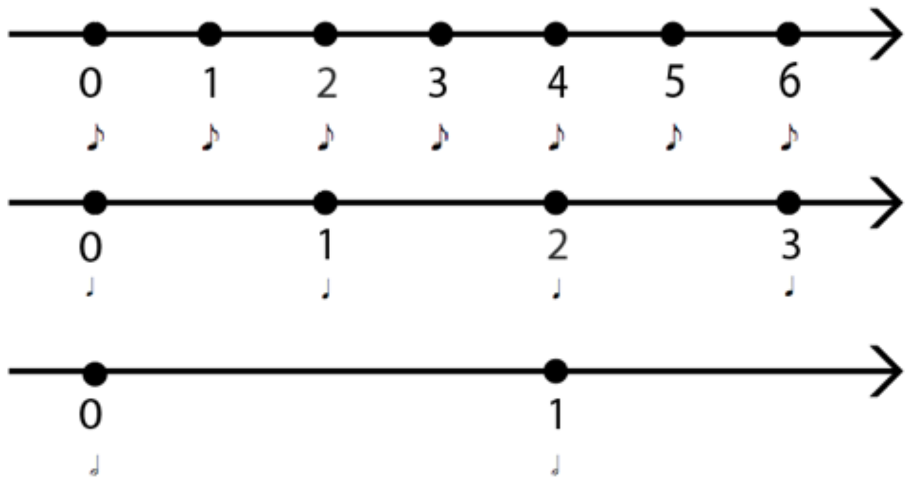

Fig. 1 – 3 Three possibilities of discretization of time

Metrical equivalence means that two onsets can be labeled as the same number if they are in the same metrical position. This is the analogous concept to "octave equivalence" in pitch-class terms. Let's say that the rhythm in Fig 1 – 1 is in a meter where downbeats or barlines occur at every whole-note span. Metrical equivalence will allow us to consider onsets in positions 0 and 8 to be equal. When metrical equivalence is applied, the space becomes circular, as seen in Fig. 1 – 4 for the rhythmic universe of 8. This circular representation can be applied to a variety of universes and will be labeled as 0 to $c - 1$, where c equals the cardinality of the universe. (This

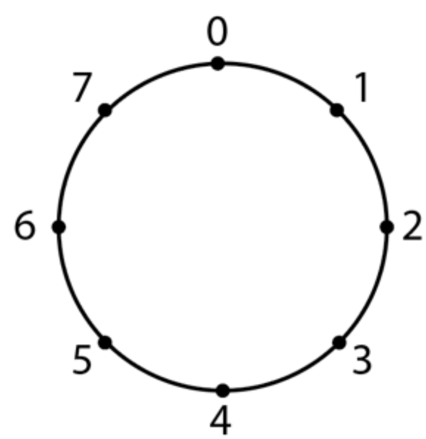

Fig. 1 – 4 Circular rhythmic space in modulus 8

is analogous to the onset labeling convention as described above.) Using this circular representation will allow us to understand and compare different rhythmic patterns within the same beat universe. Fig. 1 – 5 shows an example of a rhythm as seen on the circular rhythmic space. In a sense, metrical equivalence eliminates the continuity of time in which the rhythm exists and allows us to think of each rhythm as a collection.

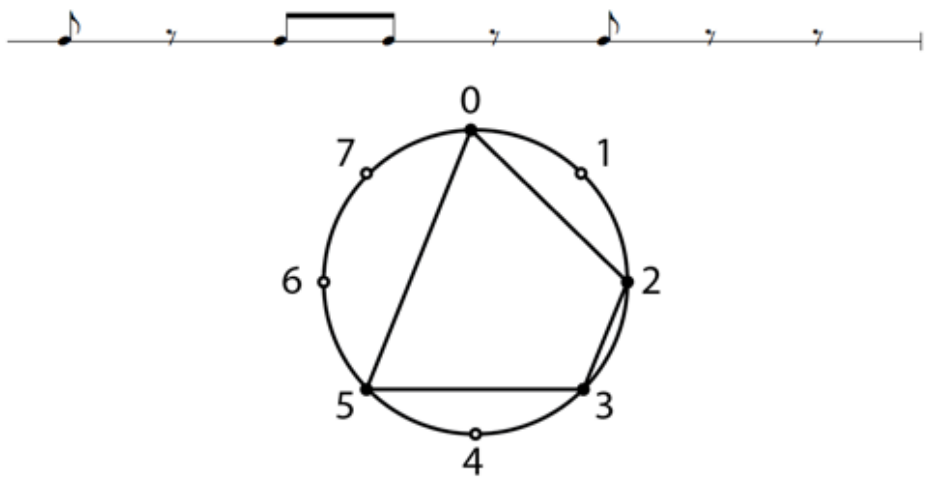

Fig. 1 – 5 An example rhythm shown on circular rhythmic space

A rhythm can be defined as a collection of time points, which can be labeled with onset numbers as described above. These collections of time points are analogous to a pitch-class set. To further our discussion, there needs to be a definition of the elements in the rhythm as "voices." In many theory classrooms, students start with SATB four-part writing in which the four voices are defined for the student, who is instructed to "lead" the voices to the next chord. This concept can be expanded to numerous voices which correspond to things that are musically real such as instrumentation and melodic lines. In terms of rhythm, defining a "voice" is not as straightforward, mainly because all rhythmic patterns exist temporally, whereas in the case of pitch the collection occurs vertically. In the vertical sense, we can trace the linear progression of notes over time in a specific voice. There is no concrete definition of what a rhythmic voice is, but our goal is to describe rhythmic voice leading in terms of a rhythmic voice moving as an abstract line from one rhythmic collection to another.[5] Each rhythmic collection can be thought of as a one-dimensional dot in which the orientation of the elements in the rhythm does not matter. Two symmetry operations will be discussed to follow in order to classify which rhythms belong in the same collection. Once the symmetry operations are defined, we then will be able to connect an abstract voice leading line between two collections classified as symmetrical under these operations.

In order to reach our conceptual goal, there are a few possibilities for defining a rhythmic voice.[6] The first suggestion here is to fix one onset in a rhythmic collection as being the starting point *x*. In doing this, the onsets will be specifically defined in relation to the onset labeled *x*, and the ordering of the onsets matters. In this particular case, each onset can be thought of as a voice, and the abstract line from one rhythmic collection to another can be traced as seen in Fig. 1 – 6. Two different voice leading paths are shown in this example from the same two rhythmic collections. Due to the fact that we must keep the ordering, the voice leading here is considered to be crossing-free. It should be noted that there are always two possibilities that have the smallest value of moves to consider with crossing-free voice leading, which are shown in Fig. 1 – 6. There are many other possibilities, but the voice leading values increase as *x* in the second collection moves to onsets farther from the *x* in the first collection. Of the two smaller voice leading possibilities, one possibility will usually account for a larger number of voice leading moves than the other. This definition of rhythmic voice is similar to how Babbitt's conception of

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[5] In other words, relating the notes as a bijective mapping, as described by Tymoczko 2008.

[6] These definitions are only a few examples to consider; other ways of defining rhythmic voice are possible.

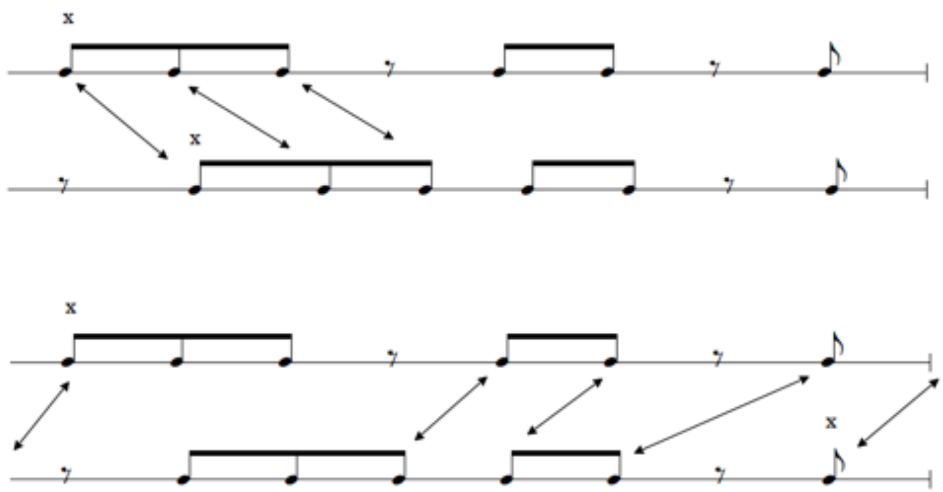

Fig. 1 – 6 Two possibilities of crossing free rhythmic voice leading

rhythms as a series whose elements we can define by onset numbers but also in terms of order numbers.[7]

The second suggestion for defining a rhythmic voice is to imagine a percussionist playing a drum using only three specific hits: 1) a single stroke; 2) a rim shot; and 3) a buzz. Each hit can then be defined as a voice respectively. In contrast to the way in which it was defined above, this version of voice does not privilege any voice as being the start; therefore, the ordering of the voices does not matter. This means that if the percussion rhythm was played in the order of: buzz, single stroke, then rim shot, followed by single stroke, rim shot, then buzz, there is a voice crossing in the voice leading. This concept can be seen in Fig. 1 – 7, which uses Fig. 1 – 6's rhythm to show the voice crossing move. Each individual onset can be related to the specific hit the percussionist uses. The move from one rhythmic pattern to the other is analogous to the change of ordering of the

[7] Milton Babbitt, "Twelve-Tone Rhythmic Structure and the Electric Medium." *Perspectives of New Music* 1, no. 1 (1962): 49–79.
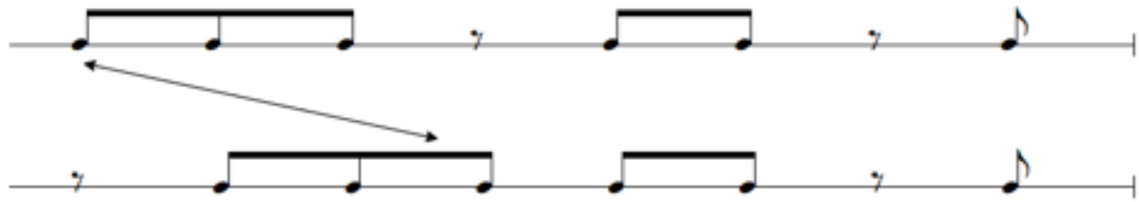
Fig. 1 – 7 Crossing voice leading

percussionists' hits, representing the concept of voice crossing in the onset collection. There are other ways to think about the concept of rhythmic voice, but it is important to note the distinction between crossing and crossing-free for the sake of this discussion. The most important take away is that when defining a rhythmic voice, the goal is to be able to trace the abstract line that the voice creates from one rhythm to another, either in succession, or comparatively from different rhythmic patterns in a piece.

By defining rhythm as a collection of time points, two rhythms can be equivalent under two symmetry operations. A symmetry operation will leave the essential identity of the rhythm unchanged. Taken from Tymoczko's definitions of symmetry operations, the first we are concerned with is "octave equivalence," which has been discussed above as

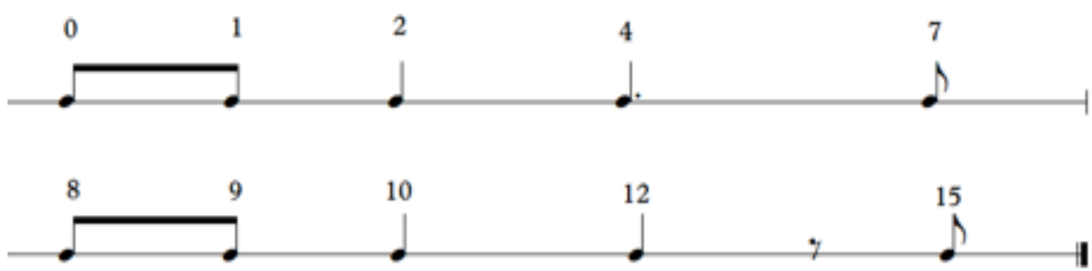

Fig. 1 – 8 Two equivalent rhythms because of the metrical equivalence property

metrical equivalence to show the circular representation of a rhythmic collection.[8] This means that in any cardinality universe of *c*, any onsets labeled as the same onsets mod*c* will be equal to each other. For example, in an 8-beat universe, any onset found at 0, 8, or 16 will be equal to one another under the metrical equivalence property. In moving between one rhythmic pattern to another, the rhythms will be considered equivalent if all of their onsets are in same metrically equivalent placement. This is seen in Fig. 1 – 8. All onsets of the second pattern are equivalent to the first mod*c*, and the onsets in position 4 and 12 are equivalent regardless of their rhythmic value.

The second symmetry operation that will leave the identity of the rhythm unchanged is the "permutation symmetry." Two rhythms symmetrical under permutation symmetry will simply reorder the voices in the rhythmic pattern. As a reminder, as described above, the concept of rhythmic voice can be conceptualized in a variety of ways. Regardless of using the crossing or crossing-free definition of the rhythmic voice, a permutation will reassign rhythmic voices to new voices. In terms of pitch, we can think about the chords D F♯ A and A D F♯ as being equal under permutational symmetry because the same notes were simply reordered. When thinking of a single chord in this way, this symmetry would be meaningless because the voices in one chord do not necessarily mean anything. The symmetry becomes important when we then think about the voice leading between two chords. For example, under the permutation symmetry, the voice leading from A C♯ E to D F♯ A and A C♯ E to A

[8] Dmitri Tymoczko, "Chapter 2: Harmony and Voice Leading," in *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011), 28-64.

D F♯ are both the same path between two points even though they will be considered different voice leadings. The rhythms used in Fig. 1 – 7 are an example of two rhythms that are permutations of each other because the onsets from one are reordered to the onsets of the other. Again, if we represent these collections as a point then there exists the same path between the two rhythms but they have different voice leadings. More about representing rhythmic collections as one-dimensional points in voice leading space will follow.

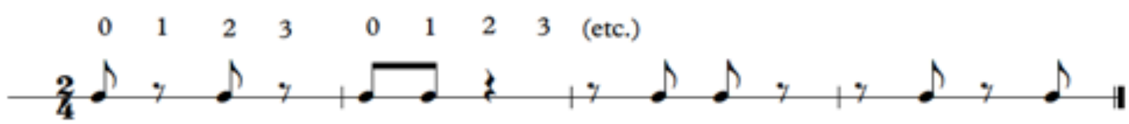

Fig. 1 – 9 Sample rhythmic pattern

In order to understand rhythmic voice-leading space and the path in which the voice leading travels, we will need to be able to map any rhythmic universe onto the appropriate geometric space. By associating a dimension with a rhythmic voice (any conception of voice), a *n*-dimensional space results from having *n* number of voices. Using the geometric

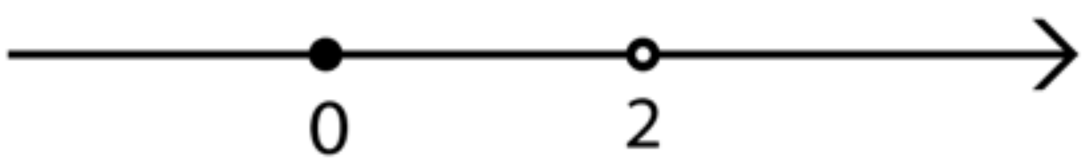

Fig. 1 – 10 Ordered pair of onsets in m. 1 of Fig. 1 – 9

spaces will allow us to use line segments to represent voice leading movement. Let's start with geometric space for two onset pairs. Fig. 1 – 9 shows a rhythmic pattern occurring in $\frac{2}{4}$ time where each measure contains only a total of two onsets. If we represent these onsets on a continuous line then we will have to order them from which they occur. The first measure is shown on a line in Fig. 1 – 10, where the first occurring onset is filled in black to represent that this an ordered onset pair and distinguish the difference between the two. To move even further, let us take this two-onset pair together and represent it as a one-dimensional point. This will now allow us to represent rhythms in a two-dimensional space. For example, Fig. 1 – 11 shows a graph in which onsets 0 through 4 are labeled on both axes and the two dots are representing a randomly chosen 0,2 onset pair and a 2,3 onset pair respectively. In Fig. 1 – 10 there are two separate one-dimensional points representing each onset, whereas in Fig. 1 – 11 there are single one dimensional points representing two onsets. These two-dimensional spaces can represent any two-onset pairs and the Cartesian grid can extend infinitely for any onset representation.

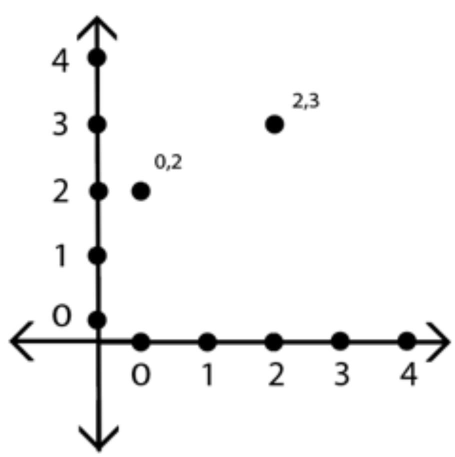

Fig. 1 – 11 Single points to represent an ordered onset pair

The graph used in Fig. 1 – 11 can be used to trace voice leading moves between rhythms. For example, using Fig. 1 – 9's rhythmic pattern, we can trace the onsets as points on the graph, as seen in Fig. 1 – 12. We can determine that: 1) any vertical move will result in

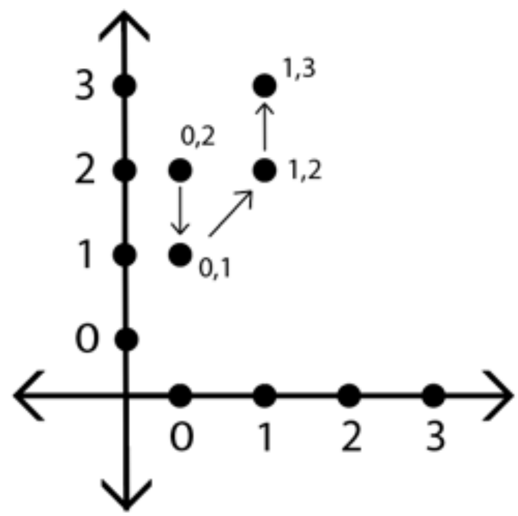

Fig. 1 – 12 Rhythmic pattern of Fig. 1 – 9 represented on two-dimensional space

the second onset to move one position while the first onset in order remains the same; 2) any horizontal move will result in the first onset to move one position while the second onset in order remains the same; and 3) any diagonal move will result in both onsets moving one position. This will allow us to graphically measure the onset moves and therefore quantify the voice leading value.

Currently, for this two-dimensional graph, the pitch space is ordered which means that we can extend this graph to be infinitely continuous in all directions. If this were the case, the metrical equivalence as mentioned earlier would not be present; instead, we will need to define which "octave" that the onset occurs within using subscripts. The grid in Fig. 1 – 13, where each point represents an ordered pair of onsets, can be extended indefinitely. Moves horizontally will change the value of both onsets in the same direction, moves vertically will change the value of both onsets in contrary motion, and diagonal moves will change only one value of the onset pair. Fig. 1 – 13 is a representation of Cartesian space in which we can plot ordered pairs of onsets. Once we apply metrical equivalence to Fig. 1 –

13, the Cartesian space then becomes a toroidal space. This is analogous to the familiar *Tonnetz* in which a panel of onset collections (or chords) connects to itself to create a doughnut-shaped continuous space. The doughnut shape occurs here because there is no distinction of the metrical position as before on the panel shown in Fig. 1 – 13, meaning that the right and left edges of that panel can be "glued" together to form a three-dimensional tube whose ends can be connected to create the doughnut. When mapping an onset collection on the toroidal space, the space is oriented, meaning that there is a distinction between up and down or left and right.

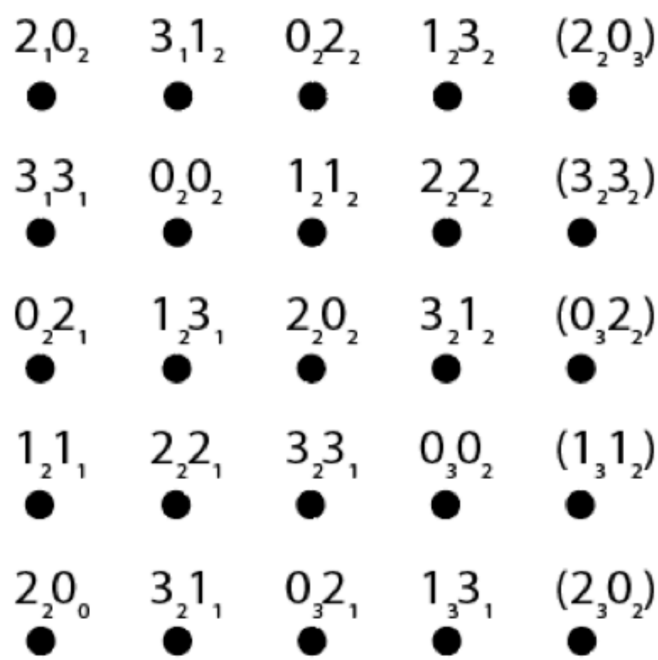

Fig. 1 – 13 Ordered pair of onsets grid. This grid can be expanded infinitely, the subscripts define which "octave" the onset exists in.

Once permutational symmetry is applied to the space it becomes non-orientable. This is because all orderings of a pair no longer matter, so there will be no consistent distinctions between which direction the path moves. Most surfaces that we encounter are orientable and have been described above as a plane and toroidal space. To define orientable verses non-orientable space, imagine someone sitting at their dinner table, on which a fork lies on the table next to their right hand. That person can travel anywhere in the world in any direction that they please, but if they return to their dinner table, the fork will remain on their right side. This person is in an orientable space in whose right and left sides can always

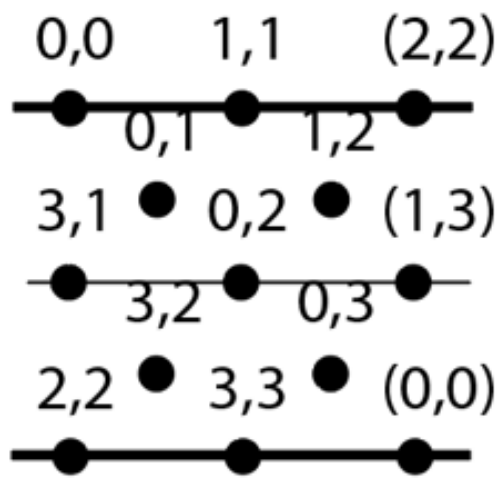

Fig. 1 – 14 One panel of ordered pair of onsets grid, metrical equivalence applied. Points on the right edge at (2,2), (1,3) and (0,0) will flip and "glue" to left edge, creating a Möbius strip.

be defined. In a non-orientable space, that same person can travel around and return to their dinner table and their fork would be on their left side. In a non-orientable space, there is no definition of left and right and up and down. A single panel as seen in Fig. 1 – 14 can represent a non-orientable space. The upper right-hand corner is now equal to the lower left-hand corner, and the upper left-hand corner is equal to the lower right-hand corner. We can then "glue" these corners together to create one continuous Möbius strip, which we can utilize to quantify any movement between two two-onset pairs. The edges of the Möbius strip will represent doublings, or unisons which can be identified as 0,0 or 2,2 and are identified by the bold line on Fig. 1 – 14. The center of the Möbius strip will contain perfectly even collections as seen as all transpositions running down the middle of the 0,2 collections. Two onsets can divide the 4-beat universe perfectly evenly down the middle and are shown by the line running across the middle of the panel. The four points surrounding the 0,2 collection are *nearly* even, meaning that these collections do not split the universe perfectly but are still close to even. Whenever the cardinality of the set does not divide the cardinality of the universe, collections on the middle of the grid will not exist; instead, the closest to the middle we can get are called *maximally even* collections, which will be explored in greater detail in section 1.2.

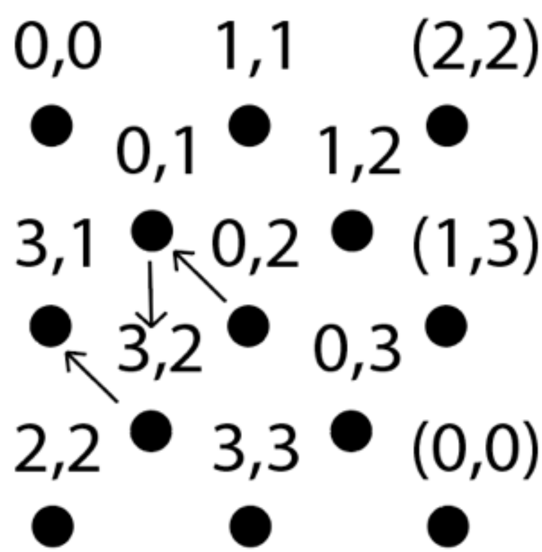

Fig. 1 – 15 Tracing the voice leading moves of Fig. 1 – 9

Using this panel as a grid, we can now trace the voice leading moves in the rhythm found in Fig. 1 – 9, as seen in Fig. 1 – 15. The same results are found as seen in Fig. 1 – 12, with one, two, and then one voice leading moves accounted for between each onset pair. The horizontal outer edges, which represent doublings, will act as mirror boundaries, as shown on Fig. 1 – 16 as the arrow between onset pairs 3,1 and 1,3. Fig. 1 – 16 also has a corresponding rhythm in which a high and a low voice have been indicated. Fig 1 – 16a shows the first measure at 3,1 followed by the second measure showing the voice crossing to 1,3. Fig. 1 – 16b shows each intermediate stage from 3,1 → 0,1 → 1,1 → 1,2 → 1,3.[9] Once the two-onset pair hits the "mirror," the result is a voice crossing. The vertical outer

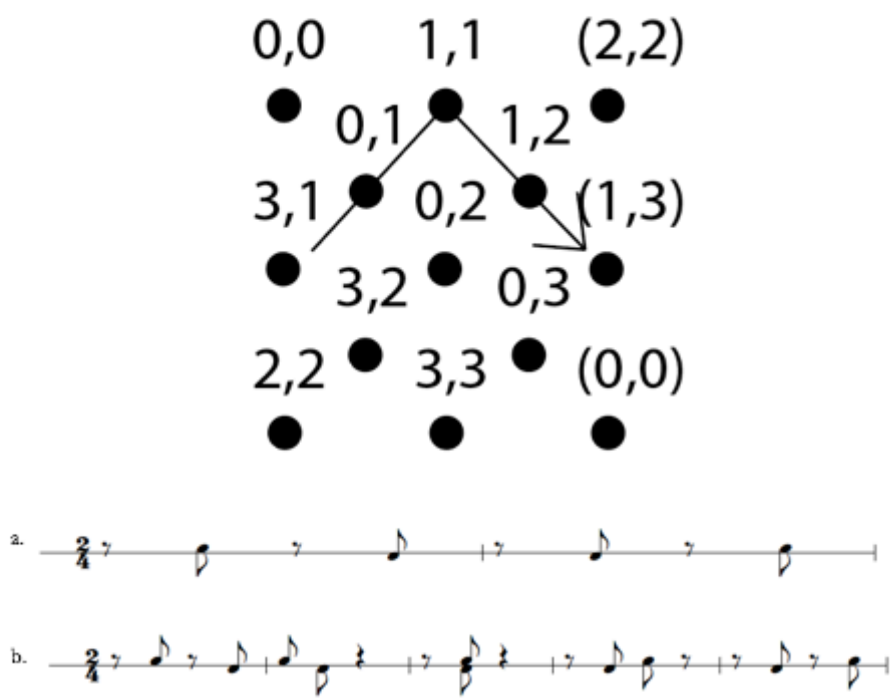

Fig. 1 – 16 Any voice leading bouncing off the horizontal boundary will result in a voice crossing. a. The rhythmic example represents a high and low voice with this voice crossing. b. Represents each intermediate move through the grid

[9] It should be noted that there is another way to trace this voice crossing as well from 3,1 → 3,2 → 3,3 → 0,3 → 1,3.

boundaries are what is "glued" together on the Möbius strip, meaning that as we follow paths from one onset pair to another they will exit on the right of the panel and return on the left side to continue the transposition relation. This can be traced as the arrow in Fig. 1 – 17, which traces the transposition of 0,1 to 1,2 to 3,2, maintaining the same distance between the two onsets. This transposition is also shown as a rhythm in which the first onset in each pair is a high voice and the second onset is a low voice. Although all of these examples are shown in a 4-beat universe, a similar panel can be selected to represent any size universe characterized by two-onset pairs in that space. On the panels shown above the ♪ determined the onset value. The same panel can be created to represent any note value depending on the

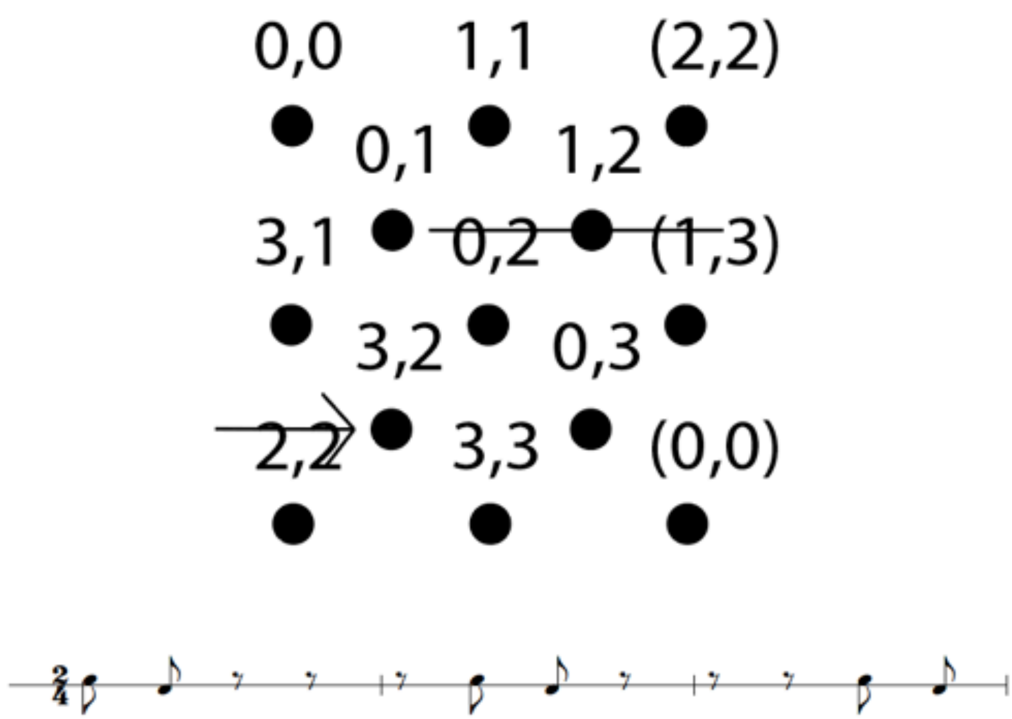

Fig. 1 – 17 Any transpositionally related voice leading move will continue off the right side of the panel and appear on the left side of the panel. The rhythmic example represents this transposition over the course of three measures.

rhythm in question. All rhythmic values exist in any selection of lattice space, but this specific lattice discretizes ♪.

Obviously, we are not only concerned with two-onset rhythms, but in order to understand voice leading space it is best to start there. To move even further, we will now explore three-onset rhythm collections. Fig. 1 – 18 shows a rhythm in which each measure

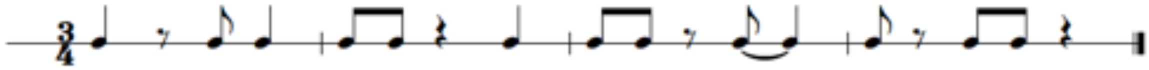

Fig. 1 – 18 Example rhythm where each measure only has three onsets

has 3 onsets. Just as we started before in the two-onset pairs, we can simply imagine a one-dimensional point representing a three-onset collection. We can no longer represent these collections on a two-dimensional graph, though, and must include a z-axis that will create the three-dimensional space. The z-axis is imagined as "popping out" from the x- and y-axes, in order to correctly account for the three distinct onsets in a collection. When applying metrical equivalence and permutational symmetry to the space we no longer get a Möbius strip, but now a non-orientable three-dimensional twisted triangular prism. Within this twisted triangular prism, we are able to represent any three-onset collections anywhere in the space. One horizontal slice of this three-dimensional triangular prism existing in a 12-beat universe is seen in Fig. 1 – 19.[10] The corners of the triangle represent the 3 possible three-onset unisons; the edges have two unisons and one other onset, and at the center of the slice

[10] This diagram is taken from: Dmitri Tymoczko, "Chapter 3: A Geometry of Chords," in *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011), 65-114.

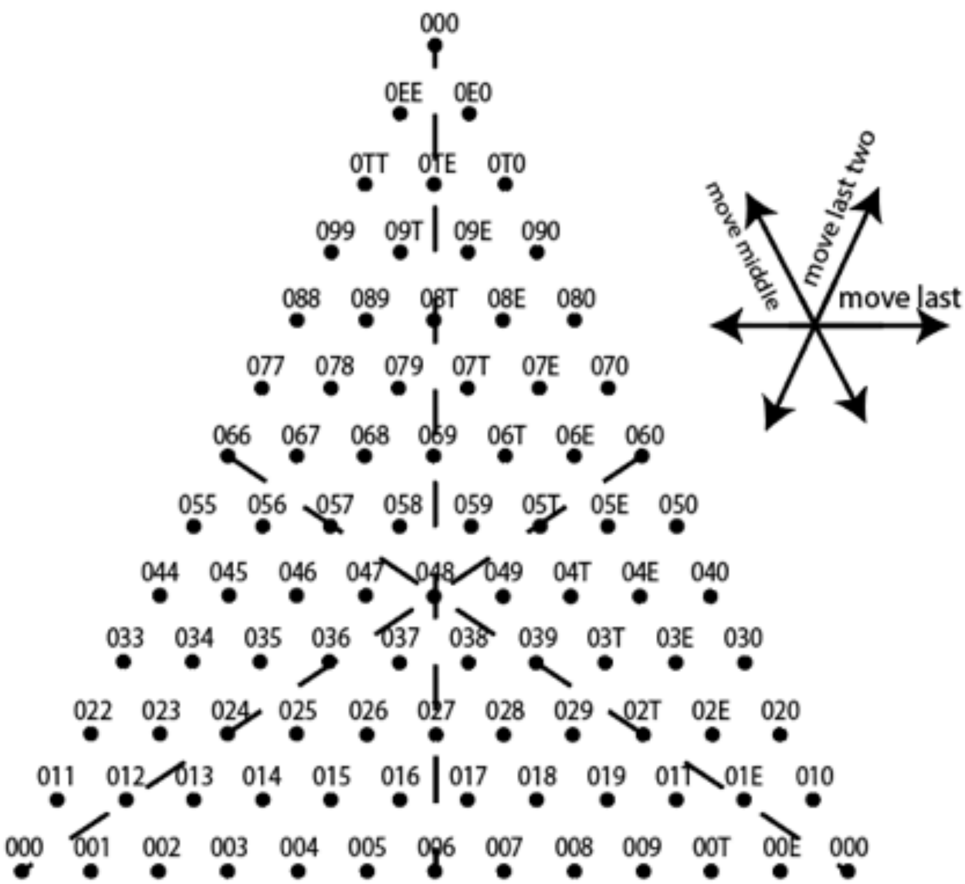

Fig. 1 – 19 Horizontal slice of three-dimensional three-onset collection space

is the perfectly even three-onset collection of (0, 4, 8). In terms of voice leading line segment representations on this space, there are a total of 4 types of moves as seen on the guide to the right of the horizontal slice on Fig. 1 – 19. A NW/SE move will move the middle onset a value of one; a NE/SW move will move the last two onsets one value, equal to two voice leading moves; and a horizontal move will move the last onset one value. The last voice leading moves are shown on Fig. 1 – 19 as dotted lines; which, they can be thought of as originating on the edge, representing only two distinct onsets (one doubled) as far apart as possible in this 12-beat universe. With each move, the line will move farther apart until it reaches the perfectly even collection; then, the onsets will move closer together until it ends in the all three-onset unison corner. Geometrically on this three-dimensional triangular prism, the three corners will connect to one another along its line of transposition. This means that the top of the triangle in Fig. 1 – 19 will transpositionally continue down to the lower left corner and then to the lower right corner, wrapping around the prism a total of three times before returning to the same place. If the rhythm in Fig. 1 – 18 is considered to be on a 𝅘𝅥𝅯 grid, we use a 12-onset universe space to map the onset collections as seen in Fig. 1 – 20. The line segments represent the voice leading moves between each three-onset collection.

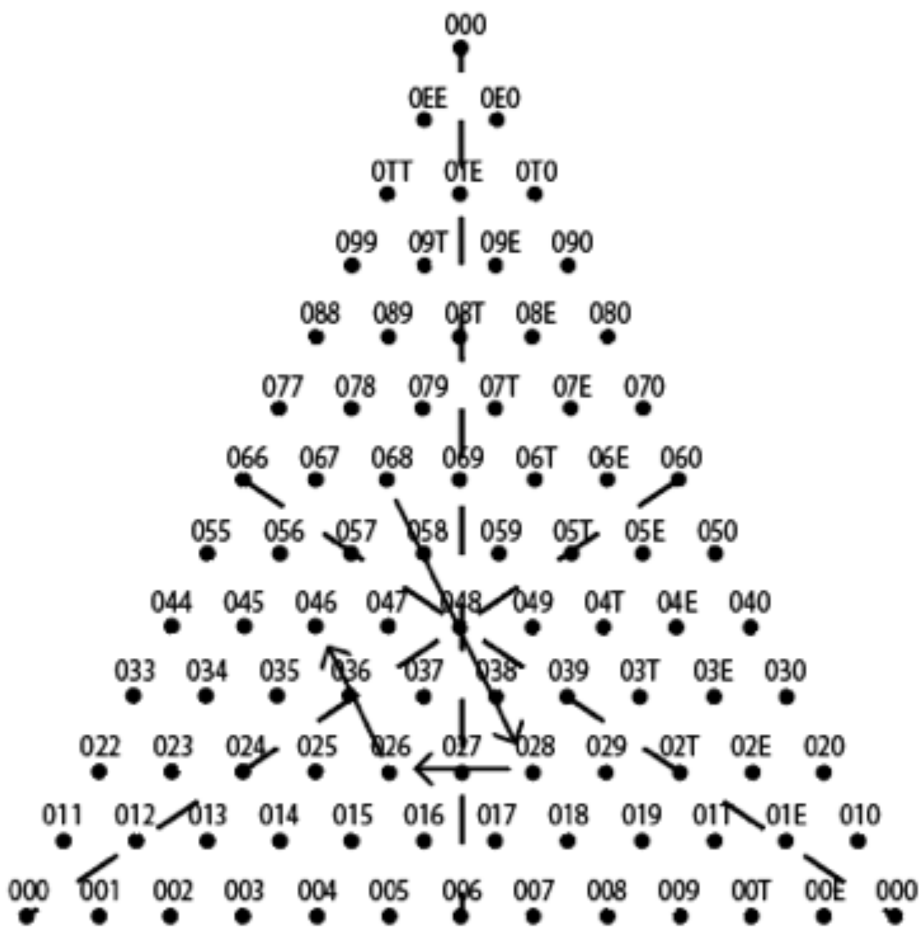

Fig. 1 – 20 Voice leading representation of Fig. 1 – 18 rhythm

Not all rhythms exist in patterns of only three onsets; there is an infinite amount of beat size universes (*c*) with an infinite number of values of onsets within that universe. The two- and three-onset collections and their spaces can be expanded to fit any of these sizes. A collection of *n* number of onsets will exist in that *n*-dimensional space. So, for two-onset pairs, the rhythms exist in a two-dimensional space and a three-onset collection exist in a three-dimensional space, etc. It is clear that at this point the visual representations begin to get harder to imagine as we move past three-dimensional spaces. The take away from the two of these spaces defined should be that all universes and onset collections can be mapped geometrically such that we can quantify the voice leading values by using line segments between the onset collection points on the respective *n*-dimensional space. Especially in regards to the following sections, it is important to note that the perfectly even collections exist at the center of the space, nearly even collections are near the center, and unisons and uneven collections are found at the outer edges of the space.

1.2 Maximal Evenness

The theory of ME is a concept first introduced to musical set theory by theorist John Clough and mathematician Jack Douthett.[11] Specifically in relation to voice-leading space, ME has since been explored by a variety of other music theorists.[12] In order to understand

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[11] John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35 (1991): 93-173.

[12] Emmanuel Amiot, "David Lewin and Maximally Even Sets," *Journal of Mathematics and Music* 1, no. 3 (2007): 157-172.

Clifton Callender, Ian Quinn, and Dmitri Tymoczko, "Generalized Voice-Leading Spaces," *Science* 320, no. 5874 (2008): 346-48

Richard Cohn, *Audacious Euphony* (New York: Oxford University Press, 2012).

how ME sets behave in voice leading space we should use perfectly even sets as a point of departure. Perfectly even sets exist in the middle of voice leading space as seen in Fig. 1 – 14 and Fig. 1- 19. In order to imagine this, we can think of a collection of notes as represented by one point. This means that any rhythmic pattern can be represented as a one-dimensional point, as described above. That one-dimensional point exists on a continuous line, which would hold all transpositions of that collection as the dot moves along the line. This continuous line of perfectly even collections in the three-onset universe lies in the middle of a three-dimensional prism. Fig. 1 – 21 uses the augmented triad as an example wherein a) the three-note collection is represented as a point; b) is shown on the continuous line of all transpositions of the augmented triad; and c) is running through the center of a prism. We can compare this example to Fig. 1 – 19, in which the central perfectly even augmented triad

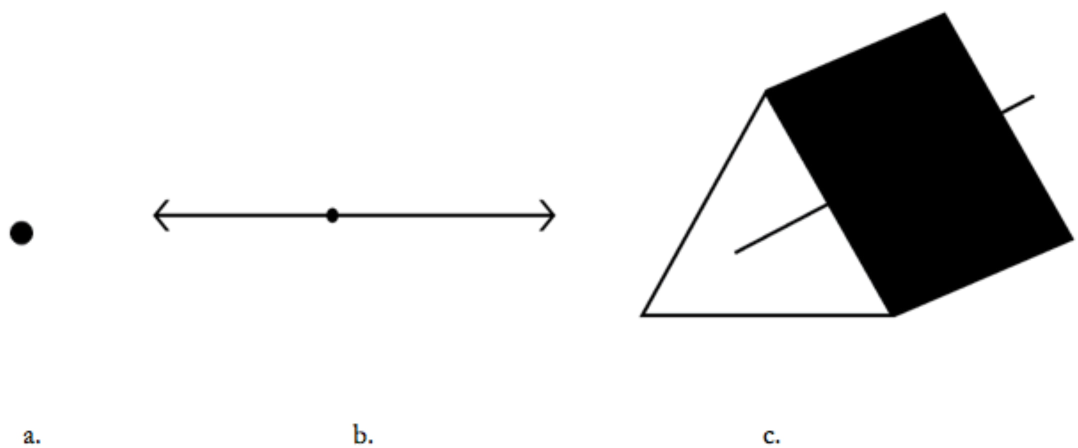

Fig. 1 – 21 a. Augmented triad as one point. b. Augmented triads all related by transposition on one line. c. Augmented triad continuous line in center of continuous three-dimensional prism

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Dmitri Tymoczko, "Geometry and the Quest for Theoretical Generality," *Journal of Mathematics and Music* 7 no. 2 (2013): 127-144.

Dmitri Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011).

is found in the center of the space at 048. The six points found around the central augmented triad spell out major and minor triads via only one voice leading move from the perfectly even collection. As we move further form the center of the space, the more uneven the collections become. This will hold true for any collection existing in this *n*-dimensional universe space.

ME sets behave in a similar but different way in comparison to perfectly even sets. Due to the fact that the elements are coprime, there is no way to have the set exist in the center of the space because it cannot be divided evenly. Therefore, while there is a central "augmented triad" line that runs through the middle of the *n*-dimensional space, there is no actual collection that can be seen realized on that equal-tempered grid. The coprime ME sets are close to the center of the space, so they still have strong connections to voice leading but cannot move as efficiently as the perfectly even sets do. These ME collections will be the central topic to be discussed in the section to follow.

Clough and Douthett define ME to be a quality of a collection in which every basic interval has either one or two adjacent specific intervals – meaning that the collection is as spread out as possible in the universe that it exists within.[13] The mathematics behind ME will be explored more in section 1.3. To define a ME set, d = cardinality of the set and c = the size of the universe that d exists within. ME rhythmic patterns will be the main focus of this discussion, but using the chromatic pitch-class analogy as a place for departure. For a familiar example, let $c = 12$ and $d = 7$. Fig. 1 – 22 represents one of the rotations that will yield the ME result of $7 \rightarrow 12$ represented both on a circle and as a rhythmic pattern. We

[13] John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35 (1991): 93-173.

have spaced *d* as evenly as possible over *c*, and the results here are at the elements: 0, 1, 3, 5, 6, 8, and 10, which happens to yield a particular mode of the diatonic scale. This ME 7 → 12

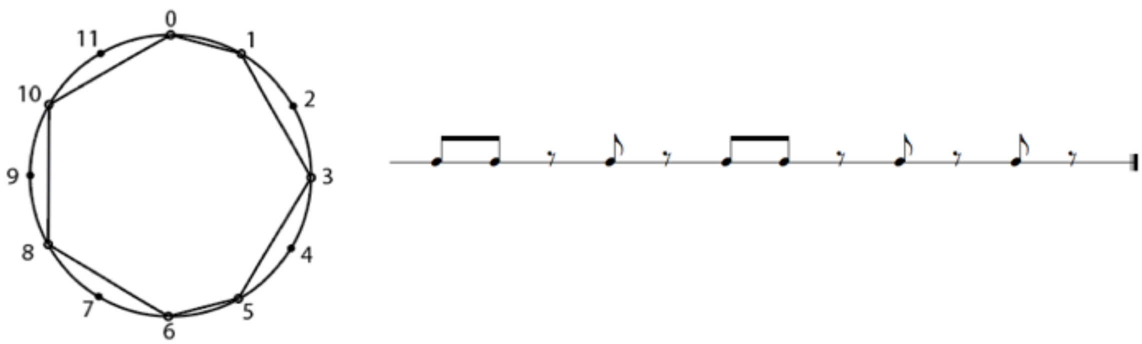

Fig. 1 – 22 ME 7 → 12 set represented on a circle and as a rhythmic pattern

set can be realized at any rotation of this pattern around the circle or within the beat universe. When translating these numbers into pitches, each rotation of the pattern would change the key of the scale. To explain these relationships further, more information in regards to rotation, the generator cycle, and the J-function will follow in section 1.3.

ME sets can be classified into three different categories to further understand the basic structure of each set and its implications in relation to voice-leading space. The first category will include examples of perfectly even sets, which have been discussed earlier in this section. Perfectly even sets result when *d* divides *c* evenly. For example, let *d* = 3 and *c* = 12, which will result in 3 elements spaced 4 units apart at numbers 0, 4, and 8, as seen in Fig. 1 – 23 on a circle and as a rhythmic pattern. In relation to the pitch analogy, this configuration will result in an augmented triad, which divides the octave *perfectly* evenly. These sets are special cases of ME sets and exist within the center of voice-leading space, as mentioned above.

The second category of ME, which the analyses to follow are most concerned with,

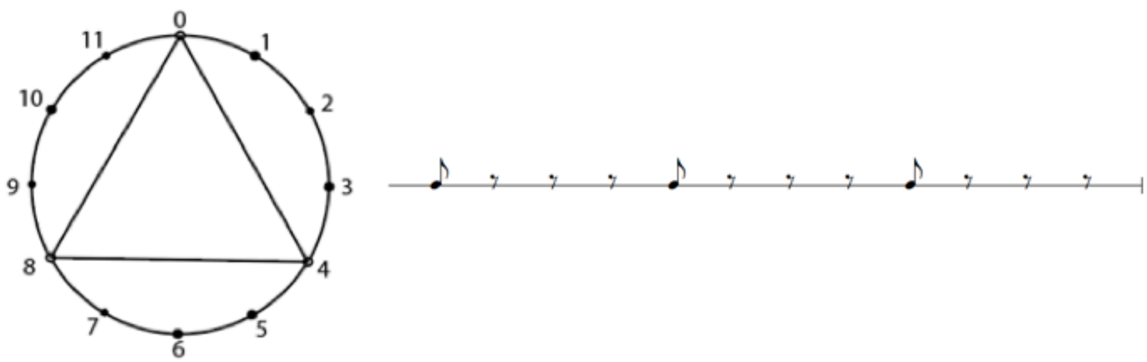

Fig. 1 – 23 Perfectly even 3 → 12 set represented on a circle and as a rhythm

will be defined when *c* and *d* are coprime, meaning that they do not share any common factors. We have already discussed an example of this second category in which *c* = 12 and *d* = 7. (See Fig. 1 – 22 for reference). These ME sets are *generated*, meaning there is a specific interval that can be repeatedly added to itself to form the entire universe. The circle of fifths shows that the chromatic universe can be generated by one interval added to itself repeatedly: the perfect fourth (or fifth). Any sets belonging to this ME category are also *well-formed,* meaning that they contain up to two distinct intervals between consecutive elements in the set. All ME collections are generated and well-formed, but not all generated collections are well-formed. The deep scale property also belongs in this category, though not all ME sets here are deep, which will be discussed in section 1.4.[14] The following analyses will use ME sets as a basis for the quality, function, and voice leading between rhythms in order to understand the choices the composers have made in their music.

The third category of ME sets exists when *c* and *d* share one common factor. When dealing with these ME sets an interesting aspect arises, which can be seen in Fig. 1 – 24. In

[14] Except for the set: (0124) mod 6.

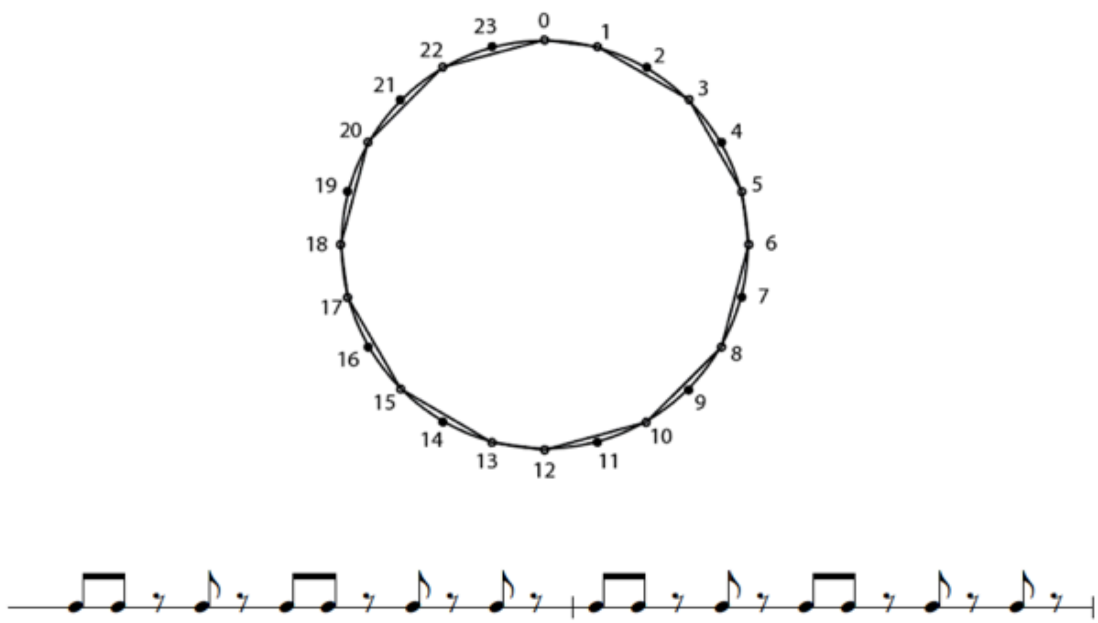

Fig. 1 – 24 ME set when c = 24 and d = 14

this example, c = 24 and d = 14, so the common factor that they share is 2. In comparison to Fig. 1 – 22, the same 7 → 12 ME set is present in the 14 → 24 ME set and is repeated a total of two times. Any set that belongs in this category can be broken perfectly into as many sub-periods as the common factor they share. In this specific case, we have two identical sub-periods of the 7 → 12 ME set. Comparing and contrasting the sub-periods in these circumstances creates another layer of understanding in regards to the components of the entire universe. This category of ME sets will be important in both analyses of pieces by Ligeti and Reich later in this study.

All three of these categories will be used as ways of describing and classifying rhythmic patterns in relation to ME. The first category of classification shows that the perfectly even set exists in the middle of voice leading space, meaning that the movement between different sets in the universe can be related to one another via minimal distance moved, which has been explored in section 1.1 and earlier in this section. The second category exists close to the center of voice leading space but also presents more properties such as well formedness, connections to generated collections, and deep properties.[15] The last category can be thought of as a mixture of the first two categories because it can be reduced into perfectly divided smaller co-prime sets. In addition to the universe as a whole, the periodic nature of these sets will allow a new layer of comparisons within the larger universe.

ME sets have a strong connection to voice-leading space and have been explored by many others in relation to how these sets exist within tonal music.[16] Using these discoveries as a basis, this discussion will expand to rhythms to further explain the way in which ME sets behave within music and therefore bridge connections between pitch space and rhythmic space. ME is a very flexible category and will allow for a wide variety of beat universe sizes, in contrast to the common c = 12 universe that many others have explored when restricted to pitch space.[17] ME sets exist closest to the center of voice-leading space, which will allow connections to the concept of rhythmic voice leading in addition to many other properties that these rhythms contain.

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[15]Norman Carey and David Clampitt, "Aspects of Well-formed Scales," *Music Theory Spectrum* 11, no. 2 (1989): 187-206.

Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11, no. 1 (1967): 32-59.

[16] See footnote 5.

[17] Of course, other universes have been explored within pitch space in regards to alternate tuning systems.

1.3 The J-Function, the Generator, and Well-Formedness

ME sets can be thought of as placing elements as spread out as possible over the course of the universe they exist within. A theorem proved by Clough and Douthett states that all ME sets can be derived from the J-function, in which we can mathematically track the placement of each element.[18] For this equation, let c = cardinality of the universe, d = size of the set, m = mode index and k = an integer between 0 and $d - 1$:

$$J_{c,d}^{m}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor$$

The equation uses the floor function, which rounds down to the nearest integer. For example: $\lfloor 2.3 \rfloor$, $\lfloor 2.6 \rfloor$, and $\lfloor 2.9 \rfloor$ all equal 2. To define a complete ME set, the J-function would be used as follows:

$$J_{c,d}^{m} = \{\, J_{c,d}^{m}(\mathrm{k}) \,\}_{k=0}^{d-1} = \{J_{c,d}^{m}(0),\ J_{c,d}^{m}(1),\ J_{c,d}^{m}(2),\ \ldots,\ J_{c,d}^{m}(d-1)\}$$

As an example, let c = 12, d = 7, and m = 0 in the J-function:

$$J_{12,7}^{0}(0) = \left\{\left\lfloor \frac{12(0)+0}{7} \right\rfloor, \left\lfloor \frac{12(1)+0}{7} \right\rfloor, \left\lfloor \frac{12(2)+0}{7} \right\rfloor, \left\lfloor \frac{12(3)+0}{7} \right\rfloor, \left\lfloor \frac{12(4)+0}{7} \right\rfloor, \left\lfloor \frac{12(5)+0}{7} \right\rfloor, \left\lfloor \frac{12(6)+0}{7} \right\rfloor\right\}$$

These variables yield the diatonic scale at the numbers: 0, 1, 3, 5, 6, 8, 10, a D♭ Major scale, and can be seen in Fig. 1 – 22. Before we discuss further the meaning of the variable m, it is

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[18] John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35 (1991): 93-173.

useful to see how the J-function works visually. Imagine two concentric circles, the smaller containing 7 points equally spaced around the circle, and the larger containing 12 points equally spaced around the circle. Jack Douthett refers to the inner circle as the *beacon* and the outer as the *filter*.[19] Now, picture each point of the beacon shining a light perpendicular to the circumference of the beacon. For each light it will either (1) line up with a hole on the filter's circle and continue to shine through, or (2) hit the inside wall of the filter's circle and travel counterclockwise and shine through the first hole it confronts. The visualization of this for the 7 → 12 ME set can be seen in Fig. 1 – 25.

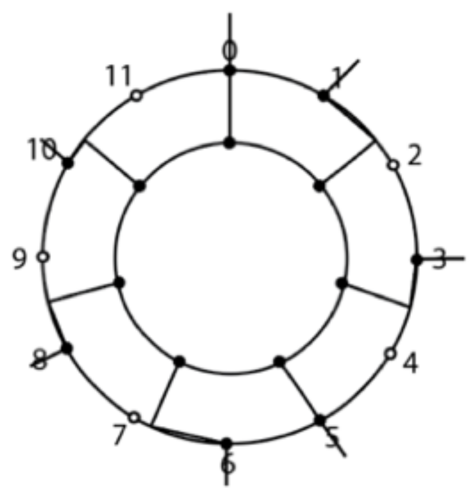

Fig. 1 – 25 Visualization of the J-Function for the 7 → 12 ME set

The *m* variable determines what Clough and Douthett label as the "mode index" and defines the rotation of the beacon. Let's use the same previous example but instead with $m = 1$. With this mode index, the J-Function will yield: 0, 1, 3, 5, 7, 8, 10, which translates to an

[19] Douthett, Jack, "Filtered Point-Symmetry and Dynamical Voice-Leading," in *Music Theory and Mathematics: Chords, Collections, and Transformations*, ed. Jack Douthett, Martha M. Hyde, and Charles J. Smith (Rochester, NY: University of Rochester Press, Boydell & Brewer, 2008): 72–106.

A♭ Major scale in pitch class terms. When $m = 2$, the set will translate to a E♭ Major scale. As the m value increases, the scalar result will continue in this fashion – moving around the familiar circle of fifths. In all cases of the second category of ME sets, there will be a value that generates the full universe. For the diatonic case, this value is equal to a perfect fifth. The m variable determines which mode the set belongs to, in relation to the set's generator cycle. This means that the m variable for any given set creates a generalized circle of fifths.

All ME sets in the second category are generated, meaning that one value can be added repeatedly to itself to create the full set. In the third category of our ME sets, c and d share a common factor, and the set can be derived from a generated collection in another universe and broken up into as many periods of the common factor to form the full set. To find the generator we can solve the equation:

$$gd = \pm 1 \pmod{c}$$

For example:

$$5 \times 7 = -1 \pmod{12}$$

$$7 \times 7 = 1 \pmod{12}$$

Now that the generator has been determined for the values of 5 or 7, we can now create the generator cycle for this specific ME set:

0 5 10 3 8 1 6 11 4 9 2 7

0 7 2 9 4 11 6 1 8 3 10 5

The two solutions of g and $-g$ are complements of each other, meaning that the generator cycles will be the same in reverse order. If we now translate the generator of 7 into pitches, we get the result in the succession of the circle of fifths and accounting for every note within the universe:

C G D A E B F♯ C♯ A♭ E♭ B♭ F

In relation to the J-function, the *m* variable will determine in which "key" that the ME set is existing within. As *m* increases by 1, the universe changes its *mode index* one spot in the generator cycle for that set.

ME sets that share a common factor cannot be generated, but instead are derived from generated collections that have a smaller value of *c*. For example, the ME 14 → 24 set will repeat the ME 7 → 12 set two times to complete the universe, which we have seen in Fig 1 – 24. We must first determine the generator of the smallest set, which is 7. The smaller set will need to be repeated twice to create the larger set, so we can simply add 12 to every number in the smaller set, yielding the results:

0 7 2 9 4 11 6 1 8 3 10 5 | 12 19 14 21 16 23 18 13 20 15 22 17

The dividing line shows where the pattern is repeated; everything to the right of the dividing line is equal to everything on the left in mod12 but the generator cycle will need to be repeated to account for the larger universe of 24.

Each ME set in the second category has a generator cycle that completes the universe the set exists within. Once the generator is defined using the equation above, any ME set will be present when the number of elements equal to value of *d* is found consecutively within the generator cycle. For example, in the ME 7 → 12 set, because *d* = 7, a ME set will be completed when 7 elements along the cycle are present, as seen here:

$$\begin{array}{l} \underline{0\ \ 7\ \ 2\ \ 9\ \ 4\ \ 11\ \ 6\ \ 1\ \ 8\ \ 3\ \ 10\ \ 5} \\ 0\ \ 7\ \ 2\ \ 9\ \ 4\ \ 11\ \ 6 \end{array}$$

The 7 elements present in this case are identical to the results that we gathered when putting these variables into the J-function when $m = 0$. When the m variable is equal to 1, the results are as follows in relation to the generator cycle:

0	7	2	9	4	11	6	1	8	3	10	5
0	7	2	9	4	11						5

There still are 7 elements in a row on the generator cycle, meaning that the ME pattern is still accounted for. The element at number 6 simply moves number 5 from the change of one m value to the other. Spelled as pitches, the result of the change in the m value is as follows:

$m = 0$: C, D, E, F♯, G, A, B

$m = 1$: C, D, E, F, G, A, B

This relationship can be represented as a minimal voice leading move – the difference of these results is only one-half step between F♯ and F-natural, which effects the key change of the diatonic set from G major to C major.[20]

Carey and Clampitt discuss the concept of well-formedness and its implication for voice leading in the diatonic space.[21] They define "A generated scale is *well-formed* if its generator always spans the same number of step intervals."[22] Using this definition for

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[20] Hook, Julian, "Signature Transformations," in *Music Theory and Mathematics: Chords, Collections, and Transformations*, ed. Jack Douthett, Martha M. Hyde, and Charles J. Smith (Rochester, NY: University of Rochester Press, Boydell & Brewer, 2008): 137-60.

Dmitri Tymoczko, "Geometry and the Quest for Theoretical Generality," *Journal of Mathematics and Music* 7 no. 2 (2013): 127-144.

[21] Norman Carey and David Clampitt, "Aspects of Well-formed Scales," *Music Theory Spectrum* 11, no. 2 (1989): 187-206.

[22] Norman Carey and David Clampitt, "Self-Similar Pitch Structures and their Duals, and Rhythmic Analogues," *Persepctives of New Music* 34, no. 2 (1996): 62-87.

generating a collection by consecutive fifths, the well-formed collections are: the dyad of the perfect fifth, three successive perfect fifths, pentatonic, diatonic, and chromatic scales. This explanation is not limited to collections generated by the perfect fifth; the well-formed property can be applied to any collection that has a generator, which will be useful in the following analyses. There is another property of these well-formed collections that was previously mentioned – all of them only contain up to two distinctive intervals when put into scalar order. For example, the chromatic scale only contains minor seconds; the diatonic contains minor and major seconds; the pentatonic scale contains only major seconds and minor thirds; and so on. We can derive other collections using the perfect fifth as a generator but these will not have the well-formed quality with only two distinct intervals. This means that three successive perfect fifths, let's say F, C, G, D, arranged in scalar order: F, G, C, D, will create three different intervals, the major second, perfect fourth, and minor third. Carey and Clampitt organize these findings onto circles in order to show that the well-formed scales are symmetrical, but not all generated collections will yield this symmetry, as seen in Fig. 1 – 26. This layout is not symmetrical in the sense that we are normally used to – the scales themselves are not symmetrical. Instead, they make a symmetrical pattern when there is a generator cycle plus the extra interval that is not in the cycle. Once equating the generator to this extra interval, the scales will show symmetrical geometric patterns and maintain only two distinct intervals in the entire collection. The lines shown through the center of the circles connect the notes in scalar order which determines the number of intervals between consecutive notes. Not all generated collections are well-formed, but all well-formed collections are generated.

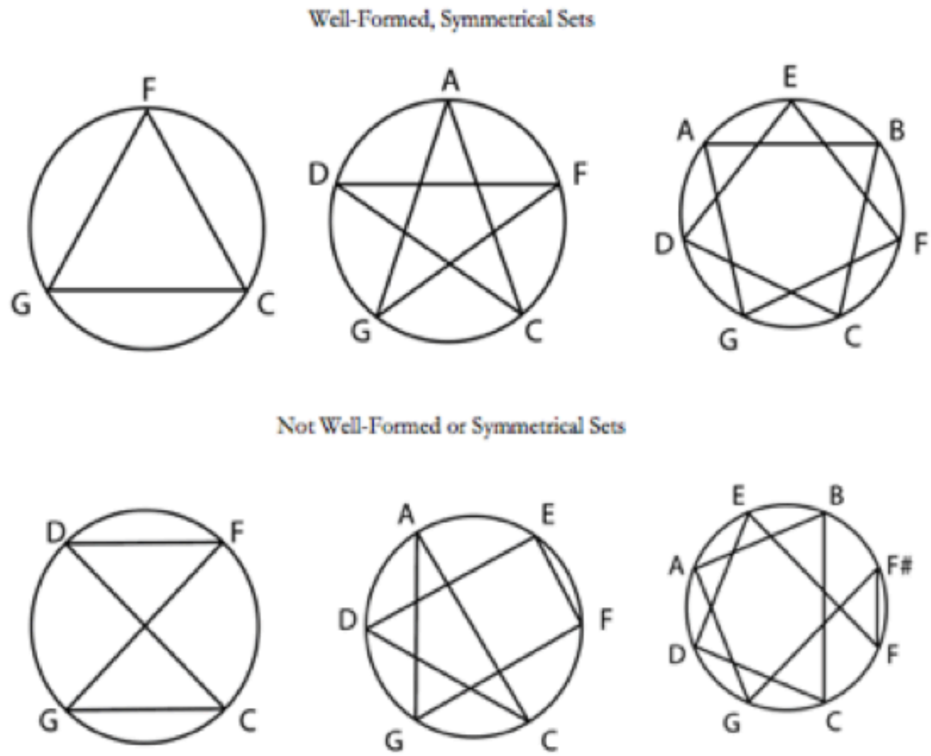
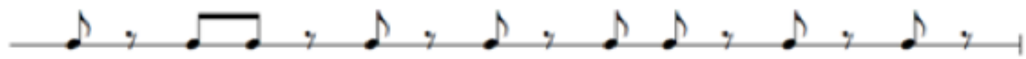


Fig. 1 – 26 Well-formed and not well-formed sets generated from a succession of perfect fifths

These properties of well-formedness and ME can be applied to rhythm as well as pitch. For example, the rhythm seen in Fig. 1 – 27a can be described as a ME 9 → 16 set with m = 2. Fig. 1 – 27b shows the generator cycle for this ME set along with the onsets of the rhythm highlighted below it. This ME set falls into our second category, in which c

a.



b.

<u>0 7 14 5 12 3 10 1 8 15 6 13 4 11 2 9</u>

0 7 14 5 12 3 10 2 9

Fig. 1 – 27 a. Rhythmic example of ME 9 → 16. b. The rhythm's onsets in relation to its generator cycle

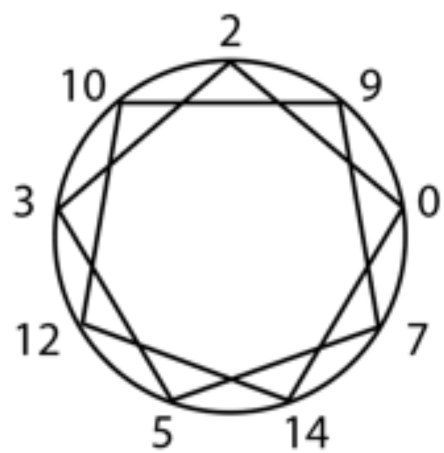


Fig. 1 – 28 Rhythmic example of ME 9 → 16 in generator order with lines connecting scalar order

and *d* are coprime. Using the same format at Carey and Clampitt, Fig. 1 – 28 shows this rhythm on a circle in which the numbers around the edge are in generator order and the lines connecting them represents scalar order. Between each number in scalar order we only find two values of either 1 or 2 (mod16). The scalar order here reflects that of the rhythmic example as seen in Fig. 1 – 27a. We can clearly see that this rhythm is a well-formed scale, because it is generated by a single value, contains only two distinct values in scalar order, and preserves symmetry.

Our third category of ME, in which *c* and *d* share a common factor, still preserves the two-step-type property but not the well-formedness property, because the whole set is not generated. For example, Fig. 1 – 29a shows a rhythmic ME 10 → 16 set, while Fig. 1 – 29b shows rhythm in relation the generator cycle for this set. *C* and *d* in this case share a common factor of 2, meaning that the generator cycle will be broken into two sub-periods of ME 5 → 8, as seen by the dividing line in Fig. 1 – 29b. Fig. 1 – 30 shows the ME 5 → 8 rhythm in generator order with the inner lines connecting the scalar order. This visual

a.

b.

0 5 2 7 4 1 6 3 | 8 13 10 15 12 9 14 11

0 4 1 6 3 | 8 12 9 14 11

Fig. 1 –29 a. ME 10 → 16 rhythm b. The rhythm in relation to the twice repeated generator cycle

represents the well-formedness property because it contains only two distinct values between onsets in scalar order, is generated by one value, and is symmetrical. The generator cycle for a ME set is of great importance in determining the function of the *m* variable in the J-

Fig. 1 – 30 ME 5 → 8 in generator order with lines connecting scalar order

function in addition to defining the well-formedness property. In the analyses to follow for both Ligeti and Reich, the concept of ME as applied to the rhythms will aid in understanding the construction and choices these composers made.

1.4 Deep Scale Property and Subset of ME sets

Carlton Gamer explains the deep scale property, in which all intervals formed between every note in a scale has their own unique multiplicity.[23] One well known example of this property can be identified within the diatonic scale; whose interval vector is <254361>. For future analyses in this paper, we will be using interval histograms as used by Godfried Toussaint to show the number of intervals within a set, in which each box represents one occurrence of that interval, and the x-axis of these graphs shows ½ the size of the universe.[24] The interval histogram for the diatonic scale found in the chromatic universe is seen in Fig. 1 – 31. It is clear from this histogram that each interval occurs a unique number of times, meaning that this scale has the deep property. When looking at these intervals, we can rank the number of occurrences of each interval from most to least occurrences: 5 → 2 → 3 → 4 → 1 → 6. This ranking is identical to the generator cycle for this set, in which anything greater than *c*/2 uses its complementary value mod *c*.

Fig. 1 – 31 Interval histogram of the diatonic scale

[23] Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11, no. 1 (1967): 32-59.

[24] Toussaint, Gottfried, *The Geometry of Musical Rhythm: What Makes a Good Rhythm Good?* (CRC Press, 2013).

The deep scale property only exists in coprime ME sets when $d = \frac{c}{2}$ or $\frac{c+1}{2}$.[25] This is because all interval classes, or all but one interval class, must be present to create the deep scale property. The $\frac{c}{2}$ case would yield all but one interval represented, and the $\frac{c+1}{2}$ case would result in the smallest interval with value of 1. A simple counting explanation for these limits can be defined; a value of *d* less than $\frac{c}{2}$ or $\frac{c+1}{2}$, will not contain enough total intervals to give a distinct multiplicity to each interval. On the other hand, as the value of as *d* increases past either $\frac{c}{2}$ or $\frac{c+1}{2}$ a "floor" occurs in relation to the interval histogram, meaning that there are too many intervals to account for in order for every interval to have its own

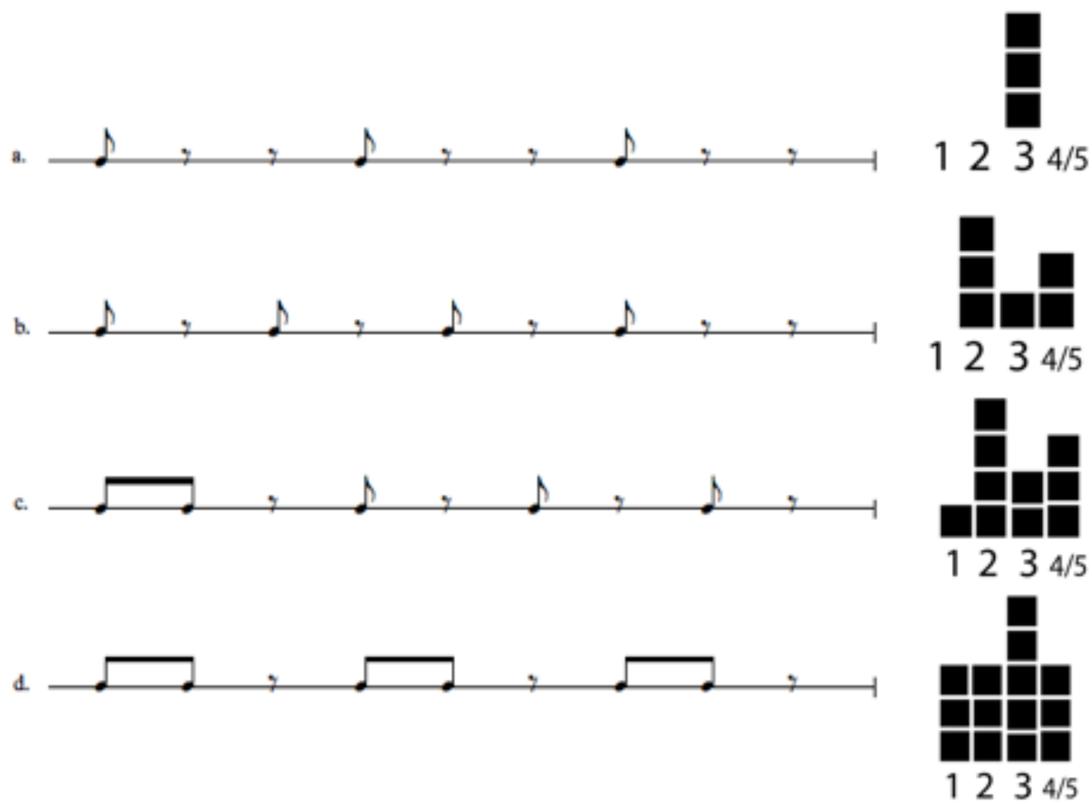


Fig. 1 – 32 Interval histograms of a. 3 → 9 b. 4 → 9 c. 5 → 9 and d. 6 → 9

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[25] Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11, no. 1 (1967): 32-59.

distinct multiplicity. The "floor" means that the lowest multiplicity will be shared by other intervals. In these cases, the intervals occurring most frequently will remain in generator order, and the rest of the intervals will flatten out and share their values with other intervals. Fig. 1 – 32 shows an example of a. less than $\frac{c}{2}$; b. $\frac{c}{2}$; c. $\frac{c+1}{2}$; and d. greater than $\frac{c+1}{2}$ cases for a universe of 9.[26]

A fair amount of rhythms that are of interest in the following analyses are subsets of ME sets. There are a few qualities of their intervals in these subsets in relation to the generator cycle that needs to be defined. For example, let us consider a subset of the ME 7 → 12 set in which an onset on the periphery of the generator cycle is missing. Fig. 1 – 33 shows an example of this rhythm, the onsets in relation to the generator cycle, and the interval histogram for this set. The interval ranking for this set is as follows: 5 → 2 → 3 → 4

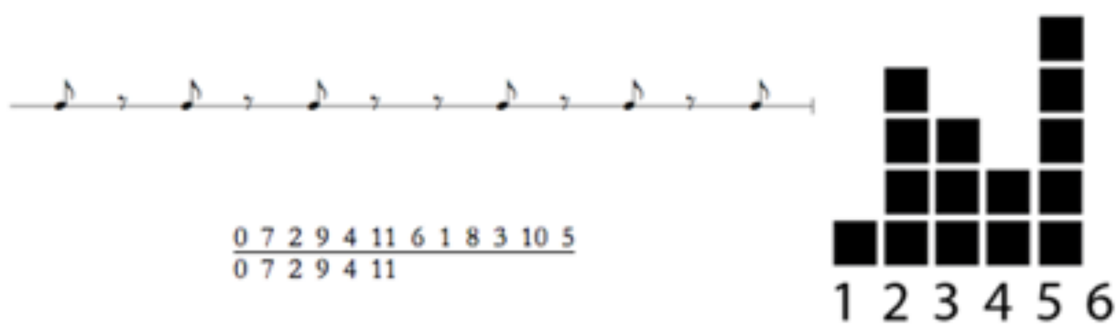


Fig. 1 – 33 6 note subset of ME 7 → 12 set with corresponding generator cycle and interval histogram

→ 1 → 6. With the peripheral note missing, the interval histogram still reflects the deep property and maintains the same interval ranking as the full ME 7 → 12 set. In the complete

[26] The generator cycle does not remain in this interval histogram because it is a perfectly even division.

set, the occurrences of interval 6 are of importance because it is the least frequently occurring of all the intervals. The absence of interval 6 affects the interval quality because it is the rarest interval in the rhythm. Regardless of missing the rarest interval from the subset as a result of a missing peripheral onset, this interval histogram is not affected to a great degree because the ranking remains the same, meaning that the overall rhythmic quality is not greatly affected.

For comparison, Fig. 1 – 34 shows a rhythm subset of ME 7 → 12 missing an onset from the middle of the generator cycle, the onsets in relation to the generator cycle, and the corresponding interval histogram. By looking only at the interval histogram, we see that the interval values are “leveling out” and the interval ranking is now: 5 → 2/3 → 1/4 → 6. The

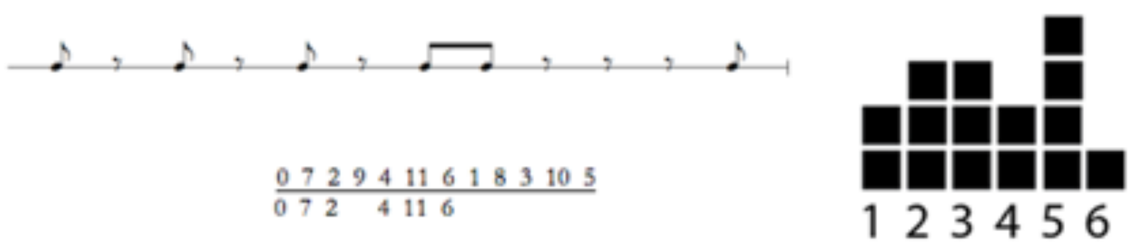


Fig. 1 – 34 6 note subset of ME 7 → 12 set with corresponding generator cycle and interval histogram

deep property no longer applies to this particular rhythm, and the generator cycle can no longer be found in the interval ranking pattern. It is important that the rarest interval of 6 is still present, but overall, the interval quality is affected to a greater degree than that of a missing onset on the edge of the generator cycle. In the analyses to follow the distinction between ME subsets missing the outer or inner onsets will play a role in the rhythmic structure, interval quality, and will aid in understanding further the results of the choices that the composers have made in their compositions.

2.1 György Ligeti's "Étude 8: Fém"

Between the years 1985 – 2001 the Hungarian composer György Ligeti wrote his cycle of 18 Études for piano, divided between three different books. The focus of this analysis will be on his "Étude 8: Fém," which, translated from Hungarian, means "metal." The majority of this piece is composed in open fifth chords, and Ligeti advises the performer to play "Always hard and metallic." Between the right and left-hand parts Ligeti constructs a polyrhythm in which the r.h. contains cycles of 18 beats and the l.h. contains cycles of 12 beats, which means that the down beats of the polyrhythm line up every 12 measures or 144 ♪. As a convention throughout this analysis, to determine where each rhythmic cycle starts, the label will be as follows: measure#, ♪#. The majority of this analysis will focus on how the r.h. and l.h. rhythms can be described in terms of ME and rhythmic voice leading. The discussion will then move to how the two hands can be described as interacting with one another, which will lead to an overall concept regarding the form.

This first portion of this analysis will concentrate on the r.h. rhythmic pattern and its connection to ME and rhythmic voice leading. The first rhythmic pattern of the r.h. lasts for a total of 18 ♪ and is rhythmically reproduced in Fig. 2 – 1. Although the score indicates a

Fig. 2 – 1 R.h. rhythmic pattern for mm. 1 – 33.

meter of $\frac{12}{8}$, Ligeti informs the performer that the measure markings are only there to help with the synchronization of the hands. Therefore, these examples will not include any time signature markings or barlines, as can be seen in Fig. 2 – 1. This opening 18-beat rhythmic pattern of the r.h. is repeated for a large portion of the piece – all the way through measure 33.

There are a few points of interest to be explained here in relation to ME, which will aid in a deeper understanding of the function of this rhythmic pattern in relation to the whole piece. This rhythm can be described as a subset of a 13 → 18 ME, which can be seen in Fig. 2 – 2, where $m = 0$. This ME set can be defined in our second category, in which c and d are co-prime, meaning that 13 → 18 does not contain any common factors, so this is

Fig. 2 – 2 ME 13 → 18 set

the most reduced this set can be. The second category also means that this pattern contains a generator, which will be important further along in this analysis. When comparing the r.h. pattern with the ME set, it is noticed that the rhythmic pattern only contains 11 onsets, and not 13. This means that we need to be able to explain and understand the impact that the

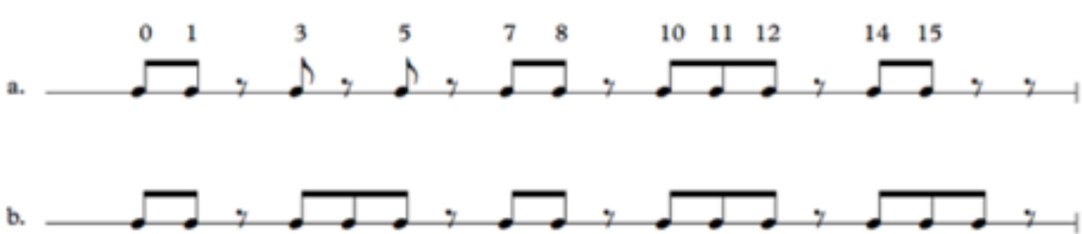


Fig. 2 – 3 a. The r.h. rhythmic pattern b. 13 → 18 ME set where $m = 4$

two missing onsets create. There are two different possibilities to describe the r.h. pattern in relation to the ME set and to do so we must use different *m* qualities within the J-function. The first possibility is where $m = 4$. Fig. 2 – 3 shows the comparison of the r.h. pattern to that of the ME set. We can easily determine from this visual that if we fill in onsets in positions 4 and 16 of the rhythm, it would result in the ME set. Fig. 2 – 4 shows the second possibility in which the r.h. pattern can relate to the ME set, which is when $m = 5$. The r.h. pattern now would be complete when onsets at position 4 and 17 are added. Thus the two different possibilities are either $m = 4$ or 5, meaning that the ME sets that the rhythm could be related to are adjacent to each other on the generator cycle. The absence of onset in position 4 is common to both possibilities in addition to the absence of either of the two last beats of the cycle.

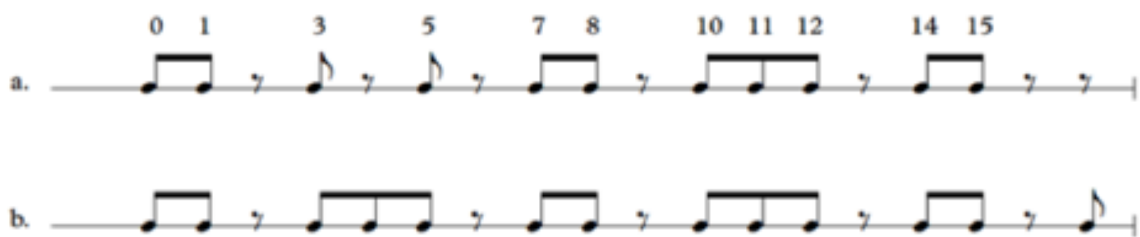



Fig. 2 – 4 a. The r.h. rhythmic pattern b. 13 → 18 ME set where $m = 5$

Using the formula as explained above in section 1.2, (gd = ± 1mod*c*), we can determine that the generator for this set is 7. As a reminder, the ME set is realized when 13 of these onsets are listed consecutively, without any spaces between them. Fig. 2 – 5 shows

0	7	14	3	10	17	6	13	2	9	16	5	12	1	8	15	4	11
0	7	14	3	10	_					_	5	12	1	8	15		11

Fig. 2 – 5 Generator cycle for 13 → 18 ME set compared to r.h. pattern

the generator pattern across the top and the r.h. rhythmic onsets for measures 1-33 below it. The underscored spaces in the r.h. rhythm represent the two different possibilities of the missing onsets in position 16 or 17, as seen in Figs. 2 – 3 and 2 – 4, while the absence of onset in position 4 is common to both possibilities. There is a large difference between a missing onset in the middle of the ME set missing versus one in the outer parts of the set, such that this rhythm is affected by both small and large impacts from the absent onsets. The 4th onset of the r.h. pattern happens roughly in the middle of the ME set (either onset number 6 or 7 of the generator pattern, depending on the interpretation we are considering). As described in section 1.4, a missing onset in the middle of the generator cycle affects the interval content to a greater degree than that of a peripheral onset. However, in this specific case, there are a large number of onsets in the set, so the effect of a missing onset will not impact the interval content to as great a degree as would be the case with a smaller set. It is beneficial to analyze the interval content using the interval histograms that were introduced in section 1.2 because they visually represent information about how the rhythm is constructed. Fig. 2 – 6 shows the interval graph of the 13 →18 ME set, with the number of intervals at 9 doubled, because it is half of the universe of the set and should be counted twice to more accurately represent the interval count. This rhythm can be classified as being *partially deep*, meaning that there are too many

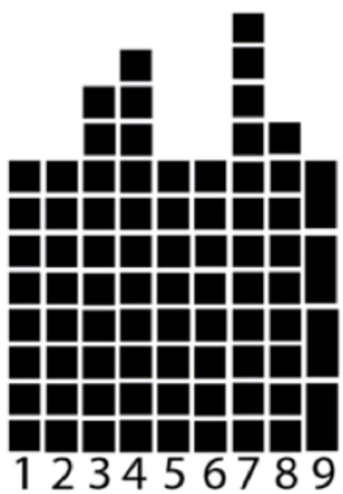


Fig. 2 – 6 Interval histogram of 13 → 18 ME set (*c*/2 interval doubled)

onsets to allow the interval graph to be deep, as described in section 1.2. The results here are distinct interval frequencies for as many intervals as possible, not including the "floor" in which the other intervals are present. It will be beneficial to rank the occurrence of the intervals; these rankings will be directly linked to the generator cycle. As a reminder, if this were a deep rhythm then the interval occurring the most frequently would be the generator, followed by the next interval in the cycle etc. Therefore, in the case of ME 13→18 the ranking pattern is 7 → 4 → 3 → 8 → everything else, where we use the complements within this universe of all numbers greater than 9 in the generator cycle.[27] Depending on how large the set is, there will be a smaller or larger number of frequencies that are unique to that interval before the other intervals all retain the same frequency, as described in section 1.2.

First, let us trace the impact that the removing either onset 16 or 17 has on this ME set, which happens to yield the identical interval histograms seen in Fig. 2 – 7. Something really interesting occurs here: even though there is a decreased number of occurrences for each interval, the ranking pattern for this graph is 7 → 4 → 3 → 8 → 1→ and then

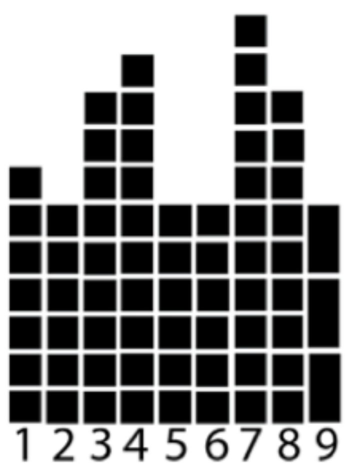


Fig. 2 – 7 Interval histogram of ME 13 → 18 and missing onset 16 or 17

[27] This is identical to the familiar interval vector analysis in the 12-note universe.

everything else. With the removal of this peripheral onset, the ranking pattern includes one more distinct interval, and which also happens to be the next interval in the generator pattern. Using this as a basis for interpretation, it is possible that Ligeti has chosen this rhythm because he wanted to be able to include more unique intervals between the onsets than that of the ME pattern while still retaining the basic interval content.

Fig. 2 – 8 shows the interval graph of the r.h. rhythm for measures 1 – 33 of the piece. For obvious reasons, the more onsets we take away from the set, the smaller number of intervals will be present, but this graph now reveals a new ranking of 7→ 4 → 3/8 → 2/9 → 1 → 6. The removal of this middle onset affects the interval graph to a larger degree than the removal of the peripheral onsets seen in Fig. 2 – 7. The first two intervals in the ranking fall in line with the generator cycle of the ME set, but the order begins to get jumbled after that. In comparison to Fig. 2 – 6, this set contains a total of 6 different interval multiplicities, and most of the intervals lose two of their values. The generator cycle ranking stays intact for intervals 7 → 4, and intervals 3 and 8 are now combined on the same level. Interval 6 now occurs the fewest amount of times in this set, which makes it more distinct from that of Fig. 2 – 7. With the other subsets and even in the ME set itself, there was a certain point in the ranking that would be labeled as "everything else", whereas here there is a full ranking pattern, with two interval frequencies shared by two intervals.

Fig. 2 – 8 Interval histogram of r.h. rhythm

<u>Intervals in Relation to Rhythmic Values (In Relation to Generator Cycle)</u>

7/11 = 𝅗𝅥.‿♪ 14/4 = 𝅗𝅥 3/15 = ♩. 10/8 = 𝅝‿𝅝 17/1 = ♪

6/12 = 𝅗𝅥. 13/5 = 𝅗𝅥‿♪ 2/16 = ♩ 9 = 𝅝‿♪

Table 2 -1

In interpreting this graph even further, it might be useful to think about the intervals in relation to rhythmic values, as seen in Table 2 – 1. By relating the interval frequencies to different rhythmic values, we can then determine the prominent values that help to define the rhythmic profile of the piece. The most prominent rhythmic value present in this rhythm is equal to 𝅗𝅥.‿♪, which does not easily translate to a common meter. This rhythmic value is roughly 2 ½ the length of the 18-beat pattern, which highlights even more the oddity that this rhythm implies; the prominence of 𝅗𝅥.‿♪ will give a feeling of instability to the listener. On the other hand, the next most frequent rhythmic value occurrence of the 𝅗𝅥 might be easier for the listener to hear. This means that many of the onset intervals are 2 beats away from each other. For reference, an interval histogram with a high prominence of interval 1 would represent a rhythm that contains many eighth note onsets in a row, whereas an interval histogram with a low frequency of interval 1 would have onsets not adjacent to one another. At this point, there is no need to go too deep into the meaning behind these prominent rhythmic values; instead, it should be noted that the 𝅗𝅥.‿♪ and 𝅗𝅥 rhythmic values are the most common to the first 33 measures of the r.h. in the piece and fall in line with the generator cycle. The choice that Ligeti made for the r.h. includes two different layers in regard to the onsets removed – one of which is the removal of a peripheral note on the ME

13 → 18 generator cycle, and the other a removal that changes the interval content drastically away from the ME sets to include more unique interval frequencies and stray away from the generator cycle ranked occurrences.

	0	7	14	3	10	17	6	13	2	9	16	5	12	1	8	15	4	11
m. 1, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 34, ♪1	0	7	14	3	10		6					5	12	1	8	15		11
m. 35, ♪7	0	7	14		10	17	6	13	2	9		5					4	11
m. 37, ♪1	0	7	14		10	17	6	13	2	9					8		4	11
m. 38, ♪7				3	10		6	13		9	16	5	12	1	8		4	
m. 40, ♪1	0	7	14	3	10	17	6		2	9	16	5	12	1	8	15		11
m. 41, ♪7	0	7	14	3		17		13			16	5	12				4	11
m. 43, ♪1			14	3	10	17	6	13		9	16			1	8		4	
m. 44, ♪7	0	7	14	3	10	17	6	13	2	9	16	5	12	1	8		4	11
m. 46, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 47, ♪7	0	7	14	3	10							5	12	1	8	15		11
m. 49, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 50, ♪7	0	7	14	3	10							5	12	1	8	15		11
m. 52, ♪1	0	7	14	3	10	17						5	12	1	8	15		11
m. 53, ♪7	0	7	14	3	10	17						5	12	1	8	15		11
m. 55, ♪1	0	7	14	3	10	17						5	12	1	8	15		11
m. 56, ♪7	0	7		3	10							5	12	1	8	15		11

Table 2 – 2 All r.h. rhythms in relation to generator of ME 13 → 18

There are a total of ten different transformations of this r.h. pattern from mm. 34 – 57, all resulting in a variety of different relationships to the ME set as well as to the first rhythmic pattern. All of these patterns, excluding mm. 58 – end, are shown in Table 2 – 2 in relation to the ME 13 →18 generator pattern. Some general observations can be gathered from the information presented on this table, the first of which is that there is a clear process in this section. The rhythms from mm. 1 – 33 reflect strong similarities to those of the rhythms found in mm. 46 – 56. Therefore, it can be concluded that the rhythmic process starts in an established stable area, moves to a middle section which contains transformations of this rhythm in forms far from the stable area, and then finishes its process by returning to the beginning material.

	0	7	14	3	10	17	6	13	2	9	16	5	12	1	8	15	4	11
m. 1, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 34, ♪1	0	7	14	3	10		6					5	12	1	8	15		11
m. 35, ♪7	17	6	13	2	9		5					4	11	0	7	14		10
m. 37, ♪1	17	6	13	2	9					8		4	11	0	7	14		10
m. 38, ♪7	16	5	12	1	8		4					3	10		6	13		9
m. 40, ♪1	0	7	14	3	10	17	6		2	9	16	5	12	1	8		4	
m. 41, ♪7	0	7	14	3		17		13			16	5	12	1	8	15		11
m. 43, ♪1	16			1	8		4				14	3	10	17	6	13		9
m. 44, ♪7	11	0	7	14	3	10	17	6	13	2	9	16	5	12	1	8		4
m. 46, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 47, ♪7	0	7	14	3	10							5	12	1	8	15		11
m. 49, ♪1	0	7	14	3	10							5	12	1	8	15		11
m. 50, ♪7	0	7	14	3	10							5	12	1	8	15		11
m. 52, ♪1	0	7	14	3	10	17						5	12	1	8	15		11
m. 53, ♪7	0	7	14	3	10	17						5	12	1	8	15		11
m. 55, ♪1	0	7	14	3	10	17						5	12	1	8	15		11
m. 56, ♪7	0	7		3	10							5	12	1	8	15		11

Table 2 – 3 All r.h. rhythms rotated

Table 2 – 3 rotates all of the r.h. patterns to more clearly reflect their relationship to m. 1 while also retaining the missing onset 4 for all rhythms. For example, m. 37, ♪1 found on Table 2 – 2 now has the onset at 4 rotated to be in line with 5 so that we can clearly

determine the relationship that each of these rhythms has with the opening r.h. pattern. There are a total of 3 different rotations used here: 1) back one ♪ so that onset in position 0 lines up with the onset in position 17; 2) back two ♪ so that the onset in position 0 lines up with the onset in position 16; and 3) forward one move along the generator cycle so that the onset in position 0 lines up with the onset in position 7. These 3 different rotations show that all rhythms can be closely related to one another by one simple move, regardless of how unclear the relationship between the onsets becomes. By keeping the onset numbers the same as found in the piece, there can be some clear relationships can be found. An example

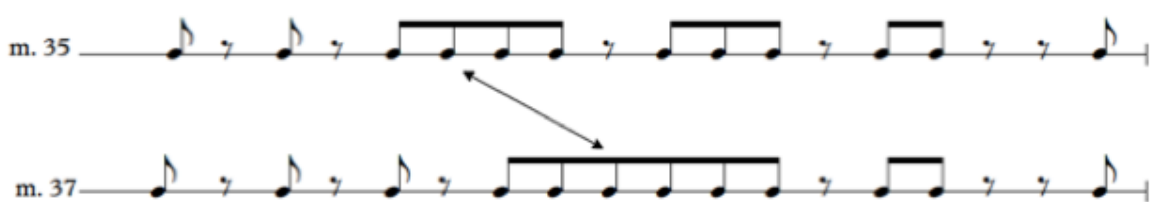


Fig. 2 – 9 The two rhythms of m. 35 and m. 37 with an arrow to indicate VL move

of this can be seen in the transformation between m. 34, ♪1 and m. 35, ♪7 – they are actually revealed here to be the same rhythmic pattern, but rotated by one eighth note. If m. 35, ♪7 is rotated by another eighth note and the onset at position 0 is removed, the rhythm in m. 38, ♪7 is the result.

The relationship between mm. 35, ♪7 and 37, ♪1 is the first case of a rhythmic voice leading move used in this piece. The two measures are rhythmically reproduced in Fig. 2 – 9, with an arrow to identify exactly where this voice leading move occurs, totaling a value of 3 ♪ moves. As mentioned in section 1.1, there is a distinction between crossing and crossing free voice leading. In these examples, we will be using crossing voice leading to more simply trace the transformations of the rhythmic patterns from the movement of only one onset. Fig. 2 – 10 shows both interval histograms for each rhythm, which visualize the impact on

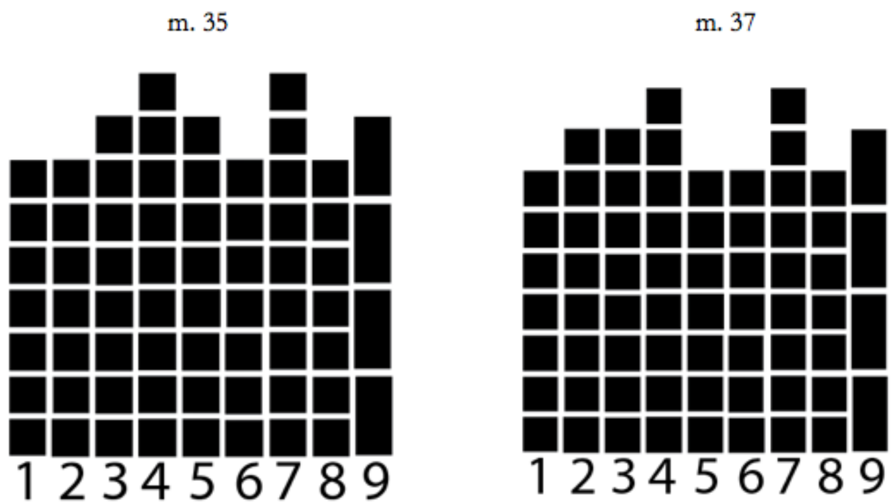


Fig. 2 – 10 Interval histograms of m. 35 and m. 37 r.h. rhythms.
(*c*/2 interval doubled).

the interval content that the voice leading move creates. In regards to the histograms, a simple move can be seen between one of the occurrences of interval 5 in m. 35, ♪7 with a change in position to interval 2 in m. 37, ♪1; every other interval stays at the same frequency of occurrence. The interval ranking for these graphs also stays the same: 7/4 → 2/5/9 → everything else, meaning that the voice leading move did not impact the ranking at all. A voice leading of this nature is a way in which Ligeti can add variety to his rhythmic pattern while still retaining the similar interval qualities and overall feeling of the rhythm.

Table 2 – 3 reveals that the section of the piece from mm. 38 – 44 diverges greatly from the two sections flanking it. Within these measures of the piece, onsets begin to get jumbled together and no longer can be clearly related to that of m. 1. These rhythms build to m. 44, ♪7 in which the onsets are saturated, missing only one to complete the set to in its

entirety. One way to reconcile the transformation of these rhythms to this point is to relate all of the rhythms of the same cardinality by their voice leading moves. Other than the example shown between mm. 35, ♪7 and 37, ♪1, the cardinality that is common between multiple rhythms happens to be the same as the beginning – 11 onsets. These occur at mm. 1, ♪1, 38, ♪7 and 43, ♪1. These three measures are rhythmically reproduced in Fig. 2 – 11 along with their corresponding interval histograms and arrows to indicate voice leading moves between each rhythm. These rhythms are the rotated rhythms used in Table 2 – 3 in order to show the smallest voice leading moves between the set. This means that these voice leading values exist within these sets but are not necessarily present in this form on the music's surface. The voice leading move from m. 1, ♪1 to m. 38, ♪7 totals 5, with the move from onset in position 1 to onset in position 6. The voice leading move from m. 38, ♪7 to m. 43, ♪1 totals 8, with the move from onset in position 7 to onset in position 1 and onset in position 14 to onset in position 16. The interval histograms do not show as simple a relationship as seen between mm. 35, ♪7 and 37, ♪1, because these voice leading moves are much larger and therefore have a greater impact on the occurrences of the intervals.

It is revealing to see how these intervals are ranked between these three different rhythms:

m. 1, ♪1: 7 → 4 → 8/3 → 9/2 → 1 → 6,

m. 38, ♪7: 3 → 7/4 → everything else,

m. 43, ♪1: 5 → 3/7/8 →2/4/6/7 → 1.

Clearly, as the piece moves through these measures, the interval histograms begin to level out quite a bit, so that there are only three distinct interval frequencies in m. 38, ♪7 and four in m. 43, ♪1. What is more is the fact that both m. 38, ♪7 and m. 43, ♪1 stray rather far away from the interval ranking of the ME set that is functioning in the background. When comparing these rhythms to the generator cycle, as seen in Table 2 – 3, these rhythms span a total of 14 and 15 elements, meaning that within them are 3 or 4 different possibilities for the ME 11 → 18 set. The distance these are from the ME set is reflected in the results we found in the interval histograms, which are highlighted by the out of order generator cycle influenced interval graphs. The last two rhythms that haven't been considered to this point

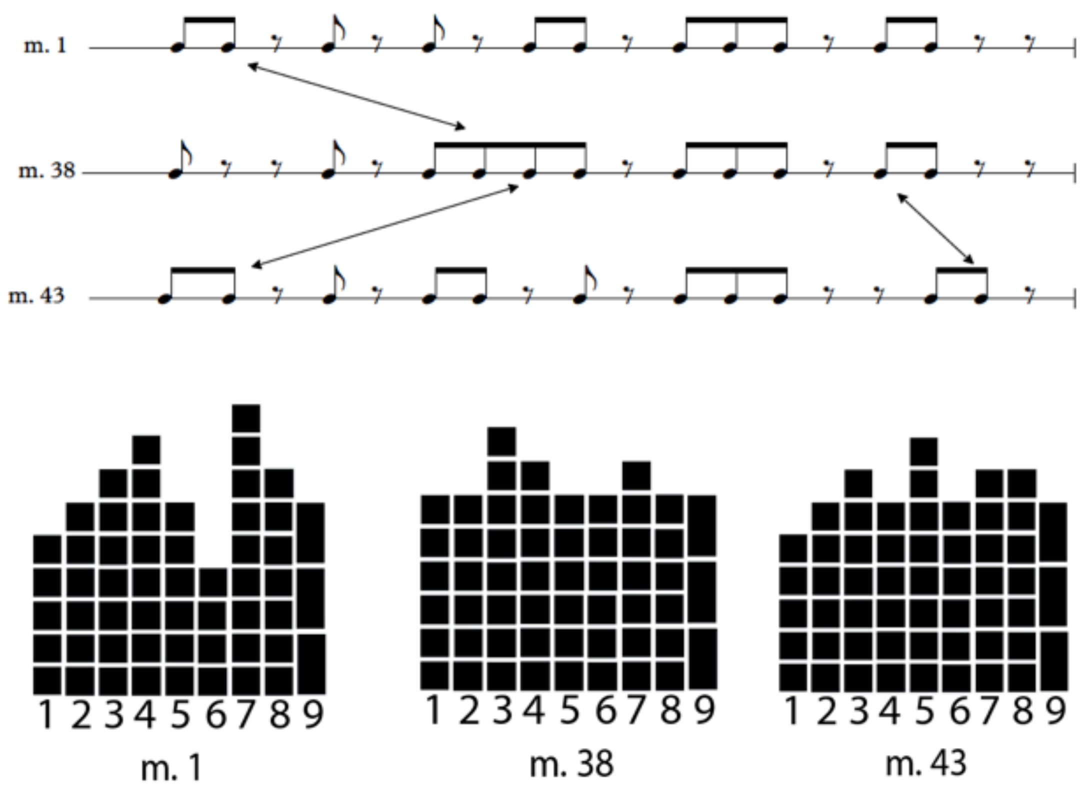


Fig. 2 – 11 Rhythms of m. 1, 38 and 43 and their corresponding interval histograms

are those found in mm. 40, ♪1 and 41, ♪7, which are rhythmically reproduced with their corresponding interval histograms in Fig. 2 – 12. Due to the number of onsets present in the 18-beat universe, these interval graphs contain many more occurrences of intervals overall. We can observe that within these measures, the interval histograms are leveling out, only showing three distinct interval frequencies for both measures respectively.

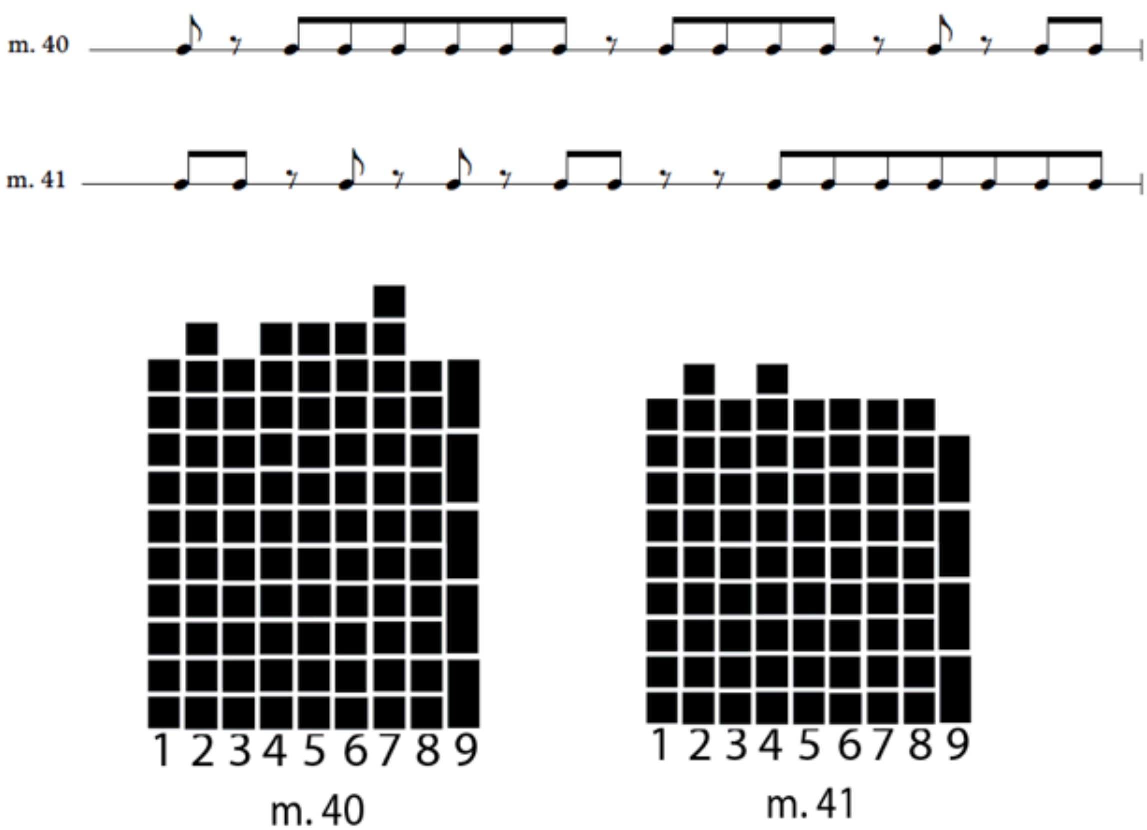


Fig. 2 – 12 R.h. rhythms of mm. 40 and 41, and corresponding interval histograms.

Measure 52, ♪1 clears up some ambiguity remaining from the beginning of the piece – one of the peripheral onsets that was missing from the ME set is now present. Fig. 2 – 13 shows the rhythm and the corresponding interval histogram of this rhythm. As we observed earlier in Fig. 2 – 7, the removal of a peripheral onset creates an interval histogram that reflects that of the generator pattern, with more distinct interval frequencies present than that of the ME set we are comparing it to. Here, we do have the peripheral onset, but now the onset at position 4 is missing, which does not cleanly fit into a relationship with the ME set. The interval ranking is as follows: 7 → 4 → 2/3/6/9 → 5/8 → 1. The intervals of 7 and 4 follow those of the generator cycle, but the next interval that should be the

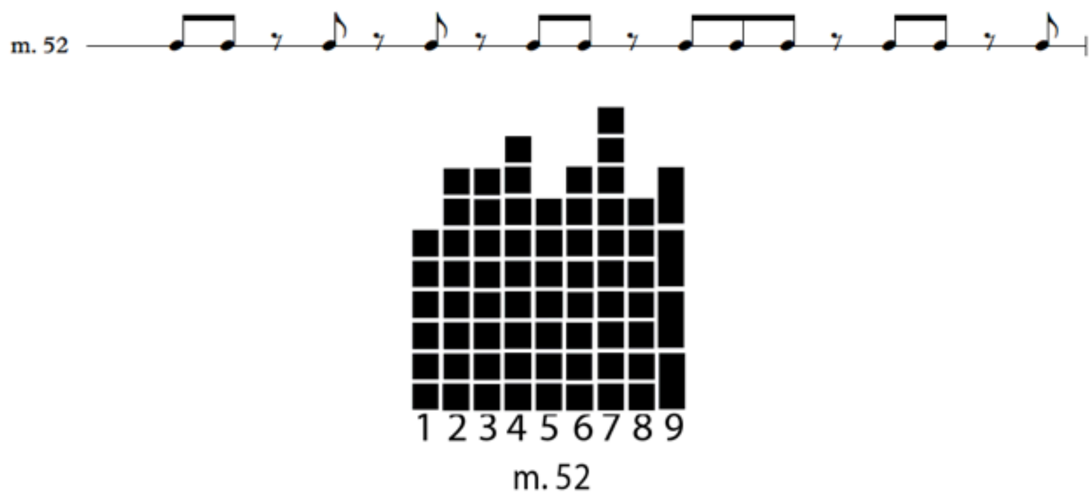


Fig. 2 – 13 m. 52 rhythm and corresponding interval histogram

third most common, 1, actually has the lowest occurrence. We see here a rhythmic pattern that is as close to the ME set as the piece will realize, but with interval content that is actually quite distant from it. Soon after this 12-note subset of the ME 13 → 18 set is completed and repeated two more times, an even smaller 10-note subset is written in m. 56, ♪7 as the last 18-beat pattern of the r.h. for this section. This rhythm and interval histogram can be seen in Fig. 2 – 14. The interval histogram only follows the fact that 7 is the most prominent interval, while again, everything becomes jumbled from there.

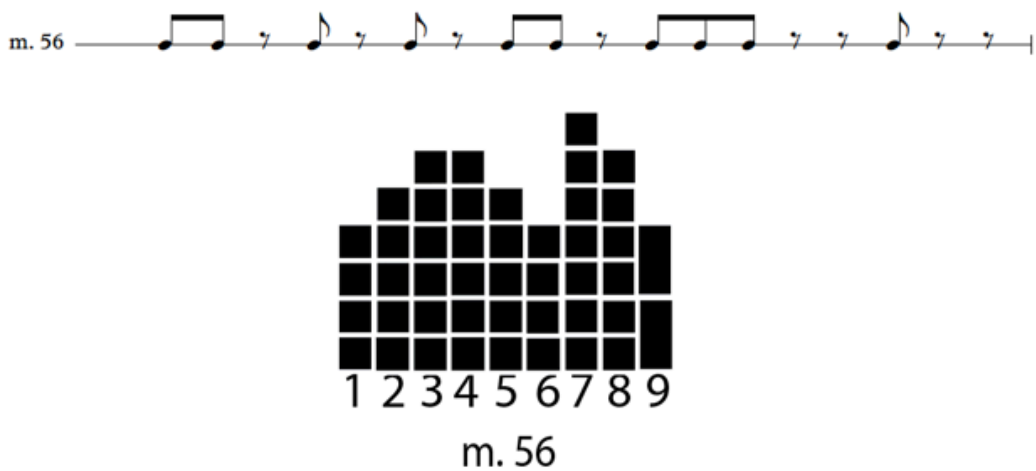


Fig. 2 – 14 m. 56 rhythm and interval histogram

Now that the r.h. rhythms of this section of the piece have been accounted for via relation to the ME set or voice leading moves, it is time to look at the l.h. rhythms and draw connections between the two and their overall relationship to the piece as a whole. While the

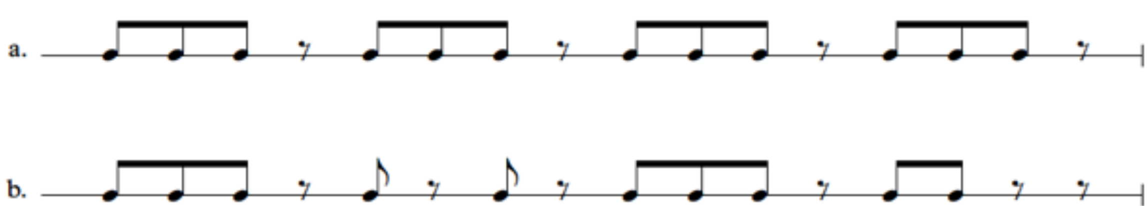



Fig. 2 – 15 a. ME 12 → 16 b. l.h. rhythm

r.h.'s pattern follows that of an 18-beat cycle, the l.h. rhythms follows a 16-beat cycle, meaning that every 144 ♪ the polyrhythm will line up. Just as seen in the r.h., the 16-beat rhythm of the l.h. is repeated until m. 34, ♪5, defining the stable area of reference for the entire section. A similar aspect occurs in the l.h. as in the r.h.: this is also a subset of a ME set and, even more, there is also two missing onsets in this rhythm as well. The functioning ME set in the background is 12 → 16, meaning that the l.h. rhythm contains a total of 10 onsets. Fig. 2 – 15 shows the l.h. rhythm rhythmically reproduced along with the ME 12 → 16 set for comparison. It is clear to see that with the addition of onsets in position 5 and 14 in the l.h. pattern, the result would be the ME 12 → 16 set. This ME set that is functioning in the background of the l.h. rhythm fits into our third category as explained in section 1.2, in which *c* and *d* share a common factor: 4. This means that we can actually split this rhythm into 4 groups of ME 3 → 4 sets. Fig. 2 – 15 then reveals that only two out of the four groups of four beats do not match the ME 3 → 4 subperiod sets.

It is beneficial to look at this rhythm in relation to the generator cycle, which looks different from that of the r.h. because of the shared common factor. Here, the generator happens to be 1, because the two factors' cardinalities are only one apart from one another.

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

Fig. 2 – 16 The four subperiods of the generator cycle for ME 12 → 16

0	1	2	3
0	**1**	**2**	
4	5	6	7
4		**6**	
8	9	10	11
8	**9**	**10**	
12	13	14	15
12	**13**		

Fig. 2 – 17 The four components of the generator cycle for ME 12 → 16 with l.h. rhythm shown underneath.

The generator cycle used in the case will actually be stacked to show the four different sections of the rhythm, as seen in Fig. 2 – 16. This means that we will need three numbers highlighted in a row under each group of four to create a ME set; therefore, as seen in Fig. 2 – 17, sections 1 and 3 of the l.h. rhythm are part of the ME set, while the other two are missing one onset respectively. The last onset of each group of four is the onset missing from each section of the rhythm, while the onsets two and three of the second and fourth groups are missing.

Instead of using a table like that used for the r.h. to visualize how these rhythms move through the piece, here each 16-beat rhythm is stacked into 4 groupings of 4 to make the transformation from one rhythm to another appear quite obvious. All l.h. rhythmic patterns are reproduced in this stacking fashion in Table 2 – 4. Measure numbers that are written in italics have the same rhythm as the first l.h. pattern of the piece. The rhythm outlined for m. 57, ♪1 only contains 12 beats, because that is where the section break occurs in the piece, and the 16-beat pattern is not fully realized. Just as with the r.h., Table 2 – 4 allows us to draw connections between different l.h rhythm transformations throughout the piece. What is more, the transformations parallel those transformations that we saw in the r.h. rhythms. In regards to the r.h, m. 40, ♪1 is roughly where the pattern from m. 1 could no longer be used as relating the new measures easily. This is similar to the l.h., where in m. 39, ♪9 the rhythms begin to diverge from that of the m. 1, ♪1 pattern.

m. 1, ♪1			
0	1	2	_
4	_	6	_
8	9	10	_
12	13	_	_

m. 34, ♪5			
0	1	2	_
4	_	6	_
8	9	10	11
12	13	_	_

m. 35, ♪9			
0	1	2	_
4	_	6	_
8	9	10	_
12	13	_	_

m. 37, ♪1			
0	1	2	_
4	_	6	_
8	9	10	11
12	13	_	_

m. 38, ♪5			
0	1	2	_
4	_	6	_
8	9	10	_
12	13	_	_

m. 39, ♪9			
0	1	2	_
4	5	_	7
8	9	10	11
12	13	_	_

m. 41, ♪1			
0	1	_	3
_	5	6	7
8	9	_	11
12	_	14	_

m. 42, ♪5			
_	1	2	3
4	5	6	7
_	9	_	11
12	_	14	_

m. 43, ♪9			
_	1	2	3
4	_	6	7
_	9	10	11
12	13	14	15

m. 45, ♪1			
0	1	2	3
4	5	6	7
_	9	10	11
_	13	14	_

m. 46, ♪5			
0	1	2	_
4	_	6	_
8	9	10	_
12	13	_	_

m. 47, ♪9			
0	1	2	_
4	_	6	_
8	9	10	_
12	13	_	_

m. 49, ♪1			
0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

m. 50, ♪5			
0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

m. 51, ♪9			
0	1	2	3
4	_	6	_
8	9	10	_
12	13	_	_

m. 53, ♪1			
0	1	_	3
_	5	_	7
8	9	10	11
_	13	14	_

m. 54, ♪5			
0	_	2	_
4	5	_	7
8	9	10	11
12	13	_	15

m. 55, ♪9			
0	1	2	3
4	5	6	7
8	9	10	11
_	13	_	15

m. 57, ♪1			
_	1	2	_
4	5	6	_
_	9	_	_

Table 2 - 4 All l.h. rhythms stacked in 4 groups of 4 to represent repeated generator cycle. *Italics* represent the same rhythm as m. 1.

The first 38 measures of the l.h. rhythm utilize only two different rhythms – the one seen in m. 1, ♪1 and the other seen in m. 34, ♪5. The difference between the two rhythms is not too drastic because m. 34, ♪5 only adds one onset in position 11 to that of the m. 1, ♪1 rhythm. For reference as well, the interval histograms of both rhythms respectively are shown in Fig. 2 – 18. For obvious reasons, the histogram for m. 34, ♪5 will contain more

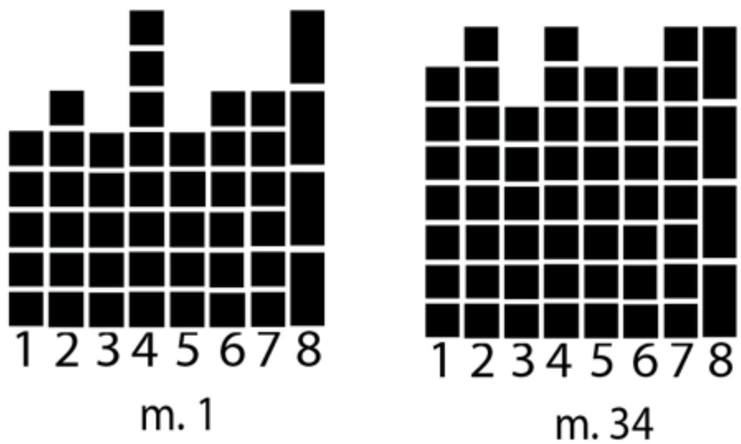


Fig. 2 – 18 Interval histograms of m. 1 and m. 34 l.h. rhythms

interval occurrences than that of m. 1, ♪1 because it has an additional onset. The transformation from m. 1, ♪1 to m. 34, ♪5 impacts the histograms by showing that the number of interval occurrences are flattening out; the flat aspect of this histogram is related to that of the ME set functioning in the background. Although both rhythms contain only 3 distinct multiplicities, the histogram in m. 34, ♪5 has a much more level surface. For even further understanding, Fig. 2 – 19 provides the interval histogram for the ME 12 → 16 set, so that we can now compare how these transformations can be related to the set. The histogram for the ME 12 → 16 set has two spikes in its graph; one at interval 4 and the other at 8, which is expected because of the information explored in section 1.4, but

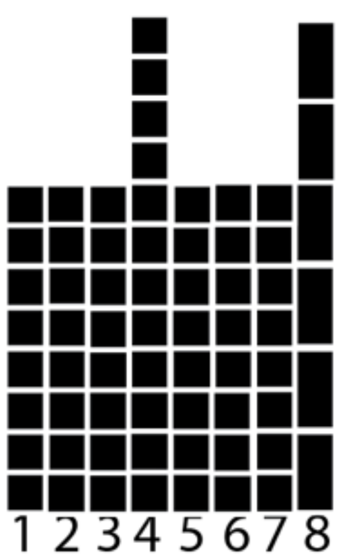


Fig. 2 – 19 Interval histogram of ME 12 →16

the rest of the interval levels remain at the same level. The rhythm of m. 1, ♪1 retains the spikes at these two intervals but lessens the number of occurrences and retains a relatively flat number of other occurrences. On the other hand, m. 34, ♪5's intervals are very similar in their number of occurrences and therefore flatten out the graph to a larger degree, including that of the intervals of 4 and 8.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
m. 1, ♪1	0	1	2		4		6		8	9	10		12	13		
m. 34, ♪5	0	1	2		4		6		8	9	10	11	12	13		
m. 35, ♪9	0	1	2		4		6		8	9	10		12	13		
m. 37, ♪1	0	1	2		4		6		8	9	10	11	12	13		
m. 38, ♪5	0	1	2		4		6		8	9	10		12	13		
m. 39, ♪9	0	1	2		4	5		7	8	9	10	11	12	13		
m. 41, ♪1	0	1		3		5	6	7	8	9		11	12		14	
m. 42, ♪5		1	2	3	4	5	6	7		9		11	12		14	
m. 43, ♪9		1	2	3	4		6	7		9	10	11	12	13	14	15
m. 45, ♪1	0	1	2	3	4	5	6	7		9	10	11		13	14	
m. 46, ♪5	0	1	2		4		6		8	9	10		12	13		
m. 47, ♪9	0	1	2		4		6		8	9	10		12	13		
m. 49, ♪1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
m. 50, ♪5	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
m. 51, ♪9	0	1	2	3	4		6		8	9	10		12	13		
m. 53, ♪1	0	1		3		5		7	8	9	10	11		13	14	
m. 54, ♪5	0		2		4	5		7	8	9	10	11	12	13		15
m. 55, ♪9	0	1	2	3	4	5	6	7	8	9	10	11		13		15
m. 57, ♪1		1	2		4	5	6			9			(			)

Table 2 – 5 All l.h. rhythms for mm. 1 – 57. The parentheses indicate the beats that do not exist because of the double bar section division.

What might also help in our analysis is to look at the rhythms in a similarly to how we looked at the r.h. patterns, as seen in Table 2 – 5. Due to the fact that the generator is value 1 in this case, all of the rhythms will not be rotated and are represented as they naturally occur in the piece. Although it might not be completely obvious, this section of the piece can be divided into four different sections: Section 1 from mm. 1 – 38; Section 2 from mm. 39 – 45; Section 3 from mm. 46 – 50; and Section 4 from mm. 51 – 57. Section 1 has been described above in which only two different rhythms are utilized, and their respective difference is only the addition of one onset. In looking ahead to Section 2, the rhythm of m. 34, ♪5 will actually play a role in the relationship of new rhythms to come to to voice leading moves. The bridge between Section 1 and Section 2 (m. 38, ♪5 to m. 39, ♪9) adds two additional onsets – the difference being onsets in positions 5, 7 and 11, found in the new rhythm at m. 39, ♪9.

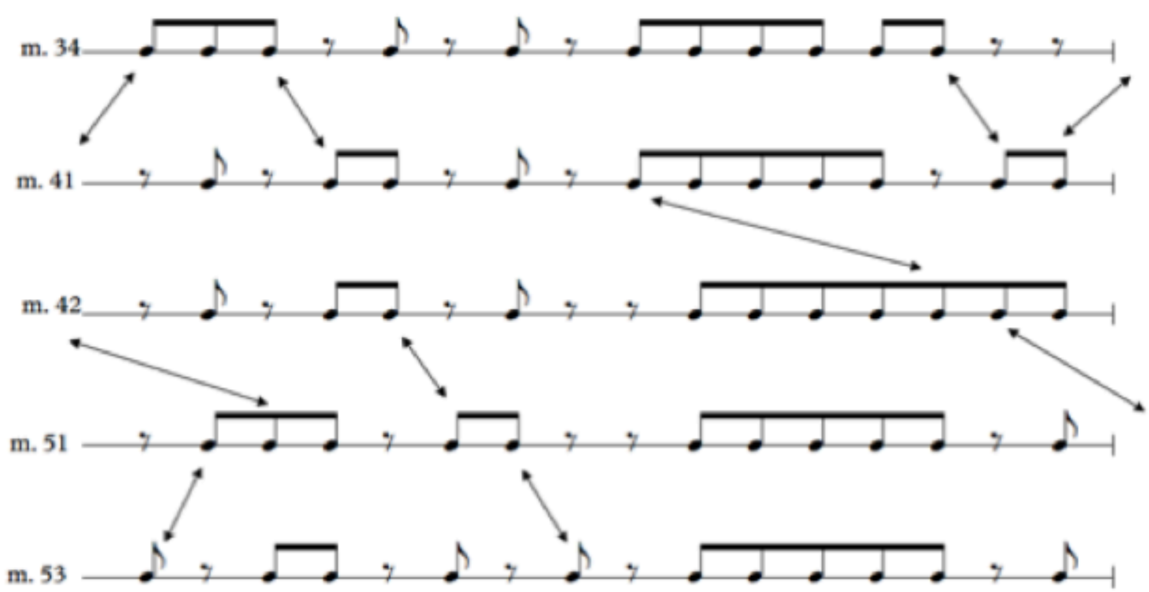



Fig. 2 – 20 VL moves between 11 onset sets

There are voice leading connections that can be made throughout all sections to show how coherent these rhythms actually are. In order to see this cohesion, it is beneficial to compare all rhythms of the same cardinality and assess the voice leading values between them all. Fig. 2 – 20 shows all rhythms used in the l.h. of cardinality of 11 onsets, with arrows to indicate the voice leading moves between the rhythms. Some of these rhythms are rotated in order to achieve the smallest voice leading moves between the sets. Between m. 34, ♪5 and m. 41, ♪1, there is a total value of three voice leading moves, with three distinct eighth notes moving their position between the two. M. 41, ♪1 and m. 42, ♪5 have a total

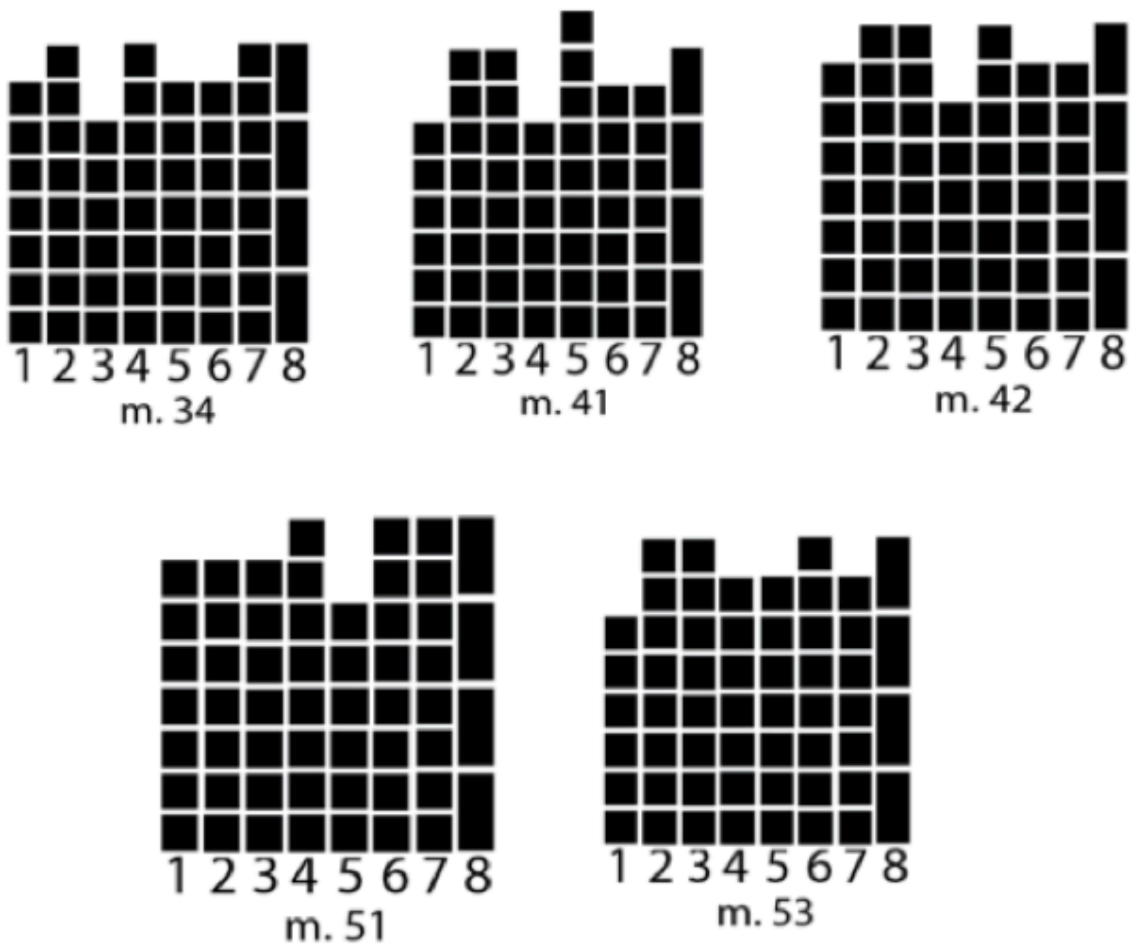


Fig. 2 – 21 All interval histogram of 11 onset rhythms in the l.h.

value of five voice leading moves only from the movement of one eighth note. In moving further through the piece, we see that between m. 42, ♪5 and m. 51, ♪9 there is again a total of, five voice leading moves, now from the movement of 2 eighth notes. The last instance of a transformation of an 11-onset set is found between m. 51, ♪9 and m. 53, ♪, with a total of only two voice leading moves from the movement of two of the eighth notes. These 11-onset patterns can be thought of as a thread throughout all the sections because of the voice leading moves between these measures. For reference as well, Fig. 2 – 21 shows all the interval histograms of these 11-onset rhythmic patterns. All of these rhythms contain the same number of interval occurrences with same basic shape retained between all of them. The relatively flat surface of them all retains the same feeling of all these rhythms. M. 41, ♪1 is quite interesting because it is the only one that has a spike at one of its values, where the other histograms have four intervals at their highest occurrences.

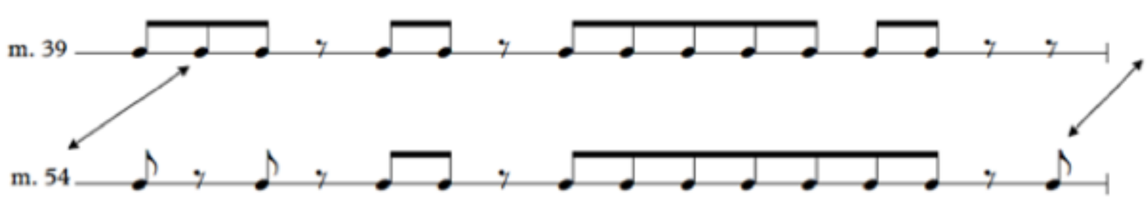



Fig. 2 – 22 VL move between m. 39 and m. 54

A few other voice leading moves should be noted between cardinalities of 12 and 13. There are two rhythms found with cardinality 12 in m. 39, ♪9 and m. 54, ♪5, which also happen to link sections 2 and 4 together. Fig. 2 – 22 shows the relationship between these two rhythms, totaling a voice leading value of only two with the movement of one eighth note. As opposed to the histograms seen for l.h. rhythms of cardinality 11, the histograms for these two rhythms show that this voice leading actually impacts the occurrences to a much greater degree. Fig. 2 – 23 shows these results because m. 54, ♪5 is not as flat a histogram, while also showing that intervals 3 and 5 are the most prominent. This is much different from that of m. 39, ♪9 and even from the ME set that is functioning in the background of the piece.

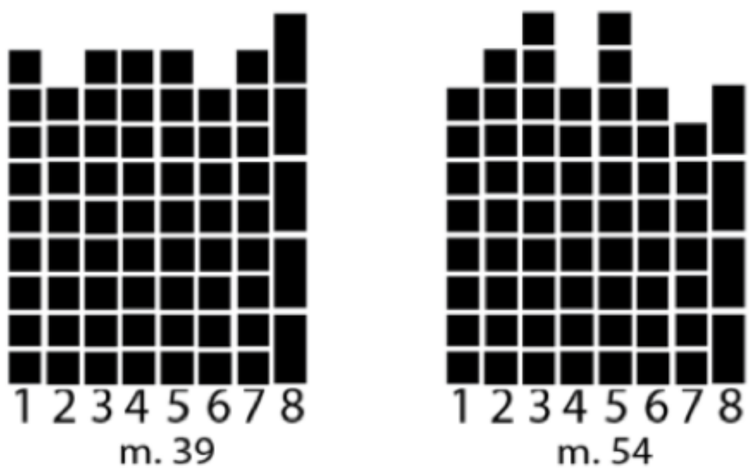


Fig. 2 – 23 Interval histograms of m. 39 and m. 54

The smallest voice leading move between sets of the same cardinality is found between m. 43, ♪9 and m. 45, ♪1 which presented consecutively in the piece. This is an unsurprising fact because the cardinality of these sets is 13, which is only missing 3 onsets to complete the entire set. Therefore, there are not many possibilities among the eighth notes that could be moved between these sets. Fig. 2 – 24 shows the voice leading move between the two rhythms, which is equal to the value of one. The interval histograms also show information that we should not be surprised by – the more onsets there are within the universe, the higher the floor will be, as described in section 1.4. There are only two distinct values in each of these histograms; m. 45, ♪1 contains one extra interval in the highest occurrence, but overall the rhythmic quality is unaffected.

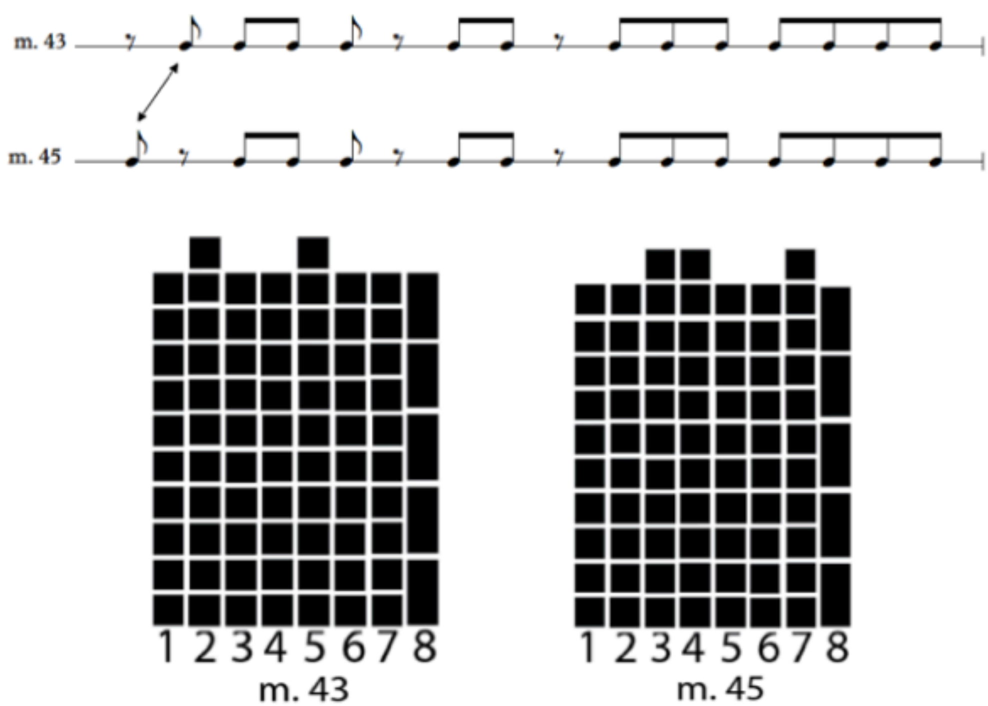


Fig. 2 – 24 VL move between rhythms of m. 43 and m. 45 and corresponding histograms

Now that rhythms in both hands have been explored, we can create a chart of the overall form that this piece creates in measures 1 – 57, as seen in Table 2 – 6. The transformations and qualities found in each hand complement each other in the process of the piece. The form can be thought of as starting in a stable area, moving towards unstable material, and then returning to the beginning stable material. There should be some

	Measure Numbers	Quality
Section 1		
R. H.	1 – 39	Utilization of two rhythms connected by addition of one onset or rotation.
L. H.	1 – 38	Use of only two rhythms with the difference of only 1 onset
Section 2		
R. H.	40 – 44	Drastically strays away from beginning pattern.
L. H.	39 – 45	Drastically strays away from beginning pattern.
Section 3		
R. H.	46 – 50	Return to rhythm of m. 1.
L. H.	46 – 50	Return to rhythm of m. 1 and then saturation of all onsets
Section 4		
R. H.	52 – 56	Addition of peripheral onset to rhythm of m. 1.
L. H.	51 – 56	VL transformations from m. 1.

Table 2 – 6 Form of mm. 1 – 57

mention of the section from m. 58 – end of the piece. After ending the first section in a high register and the loudest dynamics of the piece, Ligeti plunges the listener into a "simple but afar" contrasting section. Here, Ligeti introduces the ♩. to be the basic rhythmic value, in contrast to the ♪ in the beginning section. In this section, there is no polyrhythm present, and the onset strikes between the hands coincide for most of this ending material. The open fifths motive from the beginning of the piece prevails in this section and creates a feeling of a faraway echo after experiencing the fast-paced hard and metallic section from earlier. This section can be interpreted as a place for the listener to reflect on the fast and well-structured material previously stated. The echo-like material allows the listener to be immersed in the ambient feeling of open fifths that could not be as easily experienced in the first section. The effect of this last section is stronger when listening to the piece from the beginning – the feeling of the open fifths translates to the listener more strongly after exposure to the stark resonance that they present at the end.

Overall, analyzing this piece in terms of ME and rhythmic voice leading allowed for a deeper understanding of the structure in the choices of rhythms in which Ligeti made. By relating the r.h. and the l.h. rhythms to their respective ME sets that are functioning in the background, a better understanding of the form of the piece arises. The polyrhythm allows there to be variety through consistency – the beat cycles remain and the relationship between them changes. The beginning section utilizes this variety and consistency to create a piece that is pulled through interesting relationships between the two streams. The ME sets and rhythmic voice leadings through the progression of the piece also reflect this variety through consistency because the ME set in the background stays consistent while the surface level onsets change to structure the form of the first section. The last section acts as a place for reflection, leaving the audience to wonder about the intricate rhythmic structure the first section presented.

3.1 Steve Reich's New York Counterpoint, Mvt. 3

The effect, by change of accent, is to vary the perception of that which in fact is not changing.
- Steve Reich

Steve Reich's *New York Counterpoint* is scored for eight clarinets, two bass clarinets, and one live clarinet. The composer instructs the performer to record all of the parts his or herself first, and then play the live clarinet part with the recording. The piece as a whole contains three movements written in $\frac{3}{2} = \frac{6}{4}$ or $\frac{12}{8}$ meter, which Reich states will create ambiguity in regards to perceiving the piece as 3 groups of 4 ♪ or 4 groups of 3 ♪. The epigraph above is found in the program notes to the third movement, in which Reich highlights this metrical ambiguity as the bass clarinets switch from one accented meter to the other. This metrical ambiguity, in addition to connections to ME and rhythmic voice leading, will be the basis of this analysis of the third movement of this work.

Before diving in too deep into the nuts and bolts of the rhythmic patterns and interactions of all the parts, it is important to define the basic form and rhythmic canon that is the foundation of the structure of this movement. After introducing the basic rhythm at the beginning of the piece, Reich uses the rhythm to create a canon by shifting the pattern a ♩, or ♩. while the basic rhythm continues in a different part. This creates variety while still retaining the rhythmic qualities found in this basic rhythm. Once the complete canon is present, the live clarinet part takes a special role in the piece by highlighting subsets of the resultant rhythm, which is the complete rhythm of the combined parts in canon. Between the multiple clarinet parts' accompaniment and the live clarinet, the rhythmic canon creates intriguing resultant rhythms that the live clarinet articulates or builds up to through the piece. The bass clarinet parts, as Reich also points out, switch their articulation of beats depending on the time "change" ($\frac{6}{4} = \frac{3}{2}$ to $\frac{12}{8}$) to highlight either 4 groupings of 3 ♪ notes or 3 groupings of 4 ♪ notes, with ambiguity in regards to registral choices. All the material of the piece derives from three rhythms: 1) the opening rhythm, which is used as a canon or will be built up to as a goal; 2) the subset of the resultant rhythm of the canon, highlighted by the live clarinet part; and 3) the two bass clarinets' rhythms, in which Reich utilizes to emphasize the ambiguity of the meter of the piece. The overall form of this movement is shown in Table 3 – 1.

Rehearsal Number	Description	Canon Delay	Time Signature
61 – 62	Introduction of basic rhythm	–	$\frac{6}{4} = \frac{3}{2}$
62 – 65	Live Clarinet and Clarinet 6 build to canon	♩	$\frac{6}{4} = \frac{3}{2}$
65 – 67	Addition of Clarinet 2, 3, and 5 to canon	♩	$\frac{6}{4} = \frac{3}{2}$
67 – 70	Bass Clarinets added and build to their completed pattern	♩ (only in Cl. parts, not B. Cl.)	$\frac{6}{4} = \frac{3}{2}$
70 – 73	Live clarinet plays rhythm derived from resultant of the canon	♩	$\frac{6}{4} = \frac{3}{2}$
73	Live clarinet plays rhythm derived from resultant of the canon B. Cl. meter changes	♩.	Cl: $\frac{6}{4} = \frac{3}{2}$ B. Cl: $\frac{12}{8}$
74	Same as 71 (same resultant pattern) B. Cl. notes change register from 71	♩	$\frac{6}{4} = \frac{3}{2}$

75	Same as 73 (same resultant pattern)	♩.	Cl: $\frac{6}{4} = \frac{3}{2}$ B. Cl: $\frac{12}{8}$
76	Same as 71 (same resultant pattern) B. Cl. notes do NOT change register from 71	♩	$\frac{6}{4} = \frac{3}{2}$
77	Same as 73	♩.	Cl: $\frac{6}{4} = \frac{3}{2}$ B. Cl: $\frac{12}{8}$
78 – 80	Same as 74	♩	$\frac{6}{4} = \frac{3}{2}$
80 – 82	Same as 73	♩.	Cl: $\frac{6}{4} = \frac{3}{2}$ B. Cl: $\frac{12}{8}$
82 – 84	Same as 74	♩	$\frac{6}{4} = \frac{3}{2}$
84 – 86	Same as 73, no B. Cl. starting at 85	♩.	Cl: $\frac{6}{4} = \frac{3}{2}$ B. Cl: $\frac{12}{8}$
86 – 88	Same as 71, minus Cl. 3 and 6 starting at 87	♩	$\frac{6}{4} = \frac{3}{2}$
88 – end	Live clarinet plays basic rhythm, not resultant	♩	$\frac{6}{4} = \frac{3}{2}$

Table 3 – 1 Basic form and rhythmic process of *New York Counterpoint* Mvt. 3

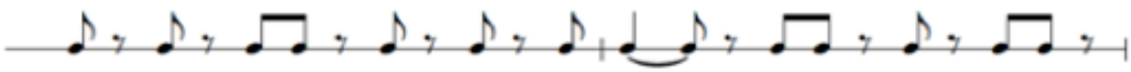


Fig. 3 – 1 Basic rhythm

It is important to analyze the basic rhythm used for the canon and determine the qualities that the rhythm has in relation to ME. The rhythm used as a basis for the rhythmic process of this movement is seen in Fig. 3 – 1, which is first played by Cl. 7 and 8 as the introduction to the movement. This rhythm has a total of 13 onsets in a 24-beat universe, and in comparison to a 14 → 24 ME set, the rhythm is missing one onset and is one voice leading move away from the ME set. Fig. 3 – 2 reproduces the basic rhythm in relation to the 14 → 24 ME set. By looking at this comparison, we can see that the onset in position 14

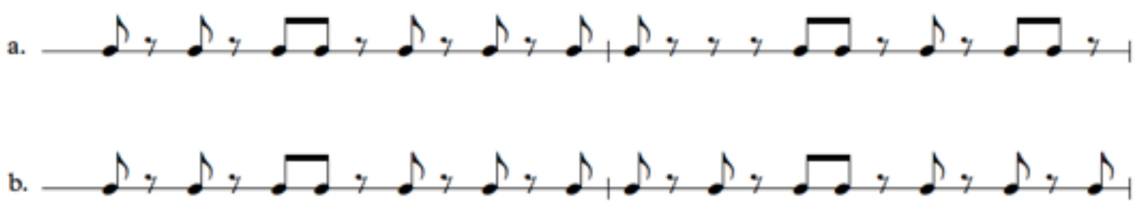


Fig. 3 – 2 a. Basic rhythm with tied note eliminated to show only onsets
b. ME 14 → 24 where *m* = 8

is missing and the onset in position 22 can be moved to an onset in position 23 in order to relate the two rhythms. A 14 → 24 ME set falls into our third category as described in section 1.2. Here, the 7 → 12 ME set is repeated a total of two times to complete the 14 → 24 ME set. The basic rhythm shares the same first measure with this 14 → 24 ME set, meaning that the first measure is a complete 7 → 12 ME set.

Due to the fact that it is a rotationally symmetrical rhythm, a ME pattern with a common factor x, the $\frac{c}{x}$ interval will have the maximum value for that cardinality. In these cases, there is no way obtain the deep scale property as described in section 1.4. Instead, the result is a deep-scale histogram pattern for the $\frac{d}{x} \rightarrow \frac{c}{x}$ ME set, but with each frequency represented x times (according to equivalences mod $\frac{c}{x}$). Intervals equal to 0 mod $\frac{c}{x}$ get the value of interval 0, or the cardinality of the pattern. In this example, that means that the deep scale histogram from 7 →12 ME set will have each interval frequency represented two times and interval 12 will be the maximum frequency occurrence (look ahead to Fig. 3 – 4 for reference). The ME 7 → 12 interval histograms can be seen in Fig. 3 – 3 with corresponding complementary intervals seen below all intervals. When this is expanded to the 14 → 24 universe, the complementary intervals are combined with one another in the interval ranking and do not include the special cases of 6 and 12, which divide the universes in half. This means that the generator pattern functions in the same way from one universe to another if they are related by a common factor.

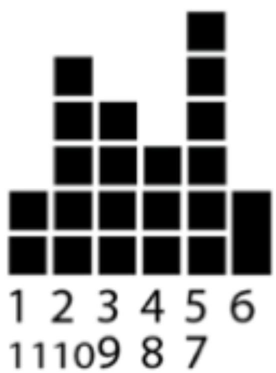


Fig. 3 – 3 ME 7 → 12 set with complimentary intervals

For even further comparison between the rhythm used in the movement and the ME set that it is related to, we can look at the interval histograms that they create, as seen in Fig. 3 – 4. The generator for this ME set can either be 5 or 7, which explains the high occurrences of these intervals in the histograms. In comparing these two rhythms in Fig. 3 – 4, there will be fewer interval occurrences overall in the basic rhythm, but the spikes at interval 5 and 7 are common to both. Each interval is not affected to a great degree, with the exception of 12 losing two occurrences between the ME set and the basic rhythm used. The

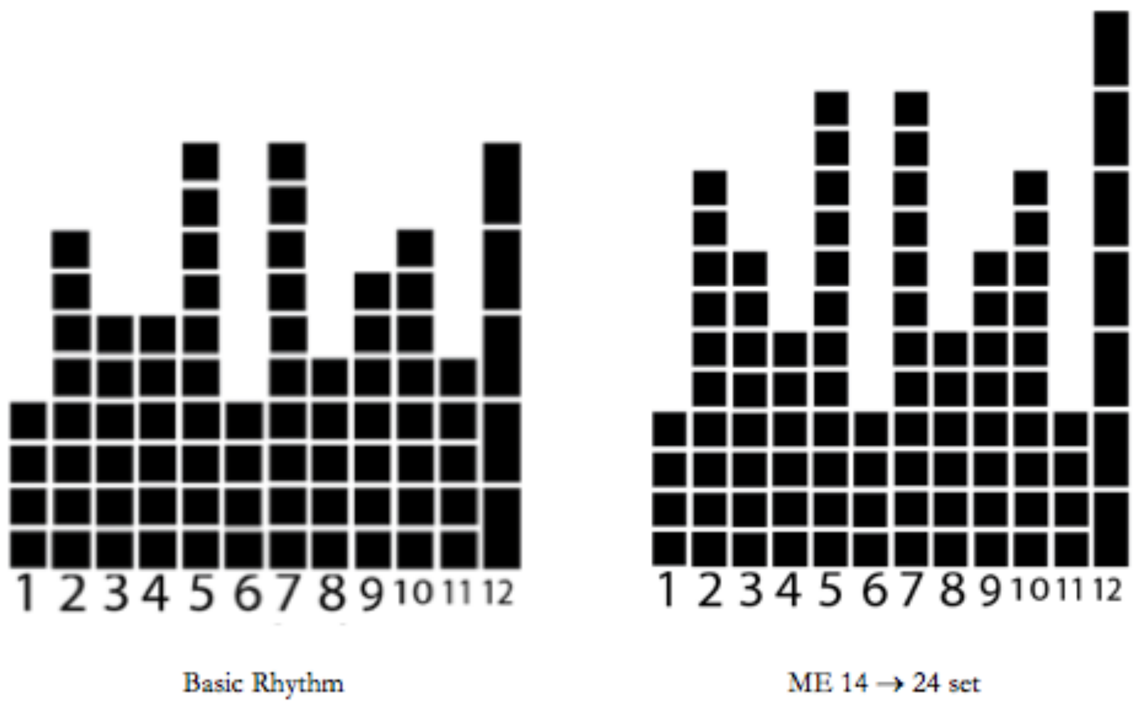


Fig. 3 – 4 Interval histograms of basic rhythm and ME 14 → 24 set

interval ranking for the ME 14 → 24 set is: 12 → 5/7 → 2/10 → 3/9 → 4/ 8 → 1/6/11, and for the basic rhythm is: 5/7/12 → 2/10 → 9 → 3/4 → 8/11 → 1/6. The basic shape of the ranking remains the same between each rhythm, with the exception of 12 no longer representing an interval of symmetry (because the full ME set is not present), and interval 3 and 4 are now equally represented instead of 3 being more frequent. In interpreting these differences, it is particularly significant that 3 and 4 are now shared in frequency. As stated above, Reich intends there to be ambiguity between 3 groupings of 4 or 4 groupings of 3. If he were to use the ME set, then interval 3 would be the most prominent, but instead, the choice to make these intervals the same frequency even further highlights the duple and triple meter ambiguity that he is concerned with.

The ME set functioning in the background shares a common factor, therefore, it is useful to look at the rhythm in the generated collection as stacked, just as we did for the l.h. part of the previous Ligeti analysis. This way, we can clearly visualize the differences between the two measures found in one ME-related pattern. Fig. 3 – 5 shows the basic rhythm in this

0	7	2	9	4	11	6	1	8	3	10	5
0	7	2	9	4	11						5
12	19		21	16						22	17

Fig. 3 – 5 Basic rhythm stacked in two parts in relation to ME 7 → 12 generator cycle

fashion; running above the rhythm is the generator cycle for ME 7 → 12, and each row shows the first and second measures respectively. Just as stated above, it is clear to see that the first measure of the pattern is ME, because there are 7 onsets existing consecutively in respect to the generator pattern. The second half of the basic rhythm is a different, though, in that contains only 6 onsets and represents a relationship to the first half of the rhythm. If the onset in position 11 moves to position 22, and the onset in position 2 is removed from the first measure rhythm, the result is that of the second measure rhythm. When thinking about this in comparison to two sets of ME 7 → 12, the first measure is comparable to when *m* = 5 and the second measure is comparable to when *m* = 4. The result we can conclude from looking at the rhythm in this way is that it is comprised of two ME sets that have *m* values adjacent to one another on the generator cycle in addition to alterations between the two halves. This allows Reich to have consistency between the ME sets used but variety in terms of the *m* values and changes made.

After Reich introduces this movement with the basic rhythm that will be used as a canon and the basis of the movement, he composes a build in the live clarinet and Cl. 6 parts to a canon at the ♩ with the completion found at R65. Fig. 3 – 6 shows the rhythms as they build to the completed basic rhythm, shifted a ♩ in relation to the generator cycle, while Fig.

0	7	2	9	4	11	6	1	8	3	10	5
	7		9			6					
	19	14				18					

R. 62

0	7	2	9	4	11	6	1	8	3	10	5
	7	2	9	4		6					
	19	14	21			18	13				

R. 63

0	7	2	9	4	11	6	1	8	3	10	5
	7	2	9	4		6					
	19	14	21		23	18	13				

R. 64

0	7	2	9	4	11	6	1	8	3	10	5
0	7	2	9	4	11	6					
	19	14	21		23	18	13				

R. 65

Fig. 3 – 6 Rhythm building to canon at the ♩ in relation to generator cycle

3 – 7 shows the interval histograms of the same rhythms. In regards to the generator cycle, we can see that in R62 and R63 there is only one onset changed between each half of the pattern, meaning that each half is closely related to the other. The largest transformation from one rehearsal number to the other is also found between these two because there is an

addition of four onsets. Between R63 and R64 there is only one added onset, and between R64 and R65 there are two added onsets.

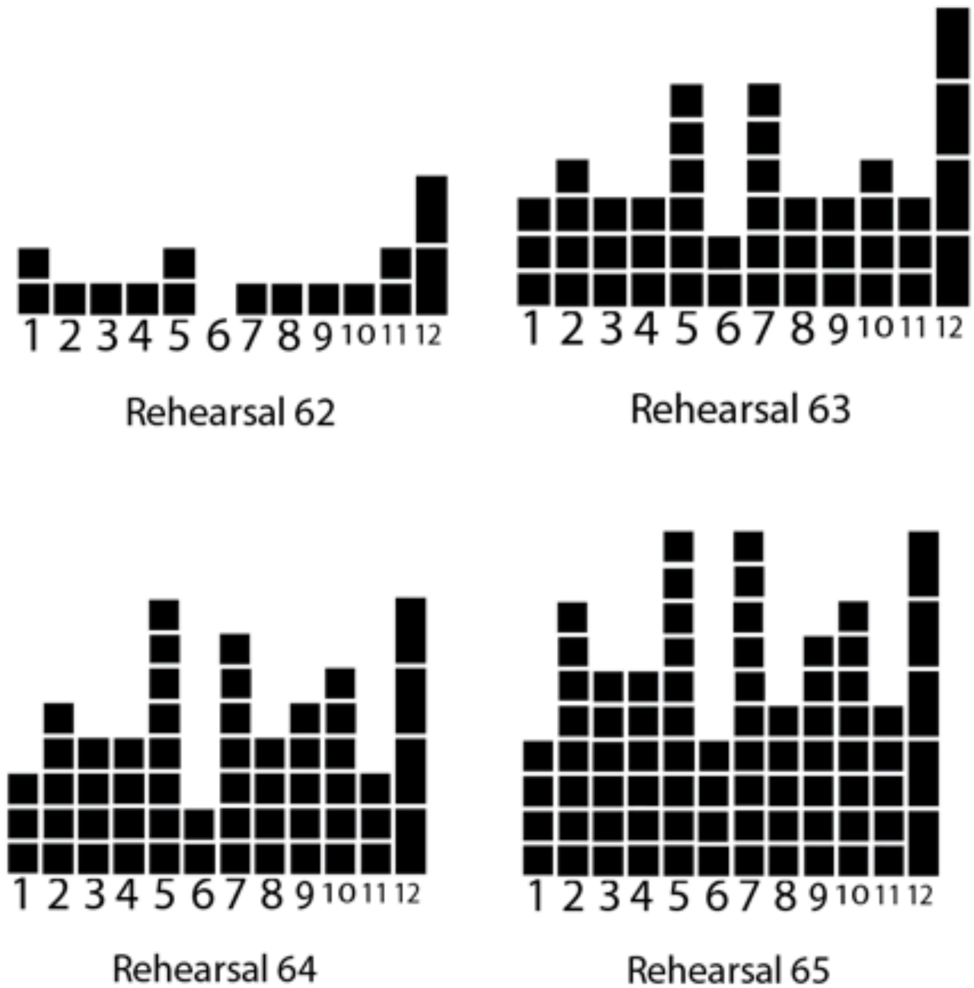


Fig. 3 – 7 Interval histograms of rhythm building to canon at the ♩

The interval histograms reflect what we have seen so far in relation to the generator cycle, but with more information in regards to interval ranking. The ranking is as follows for each respective rehearsal number:

62: 12 → 1/5/11 → 2/3/4/7/8/9/10 → 6

63: 12 → 5/7 → 2/10 → 1/3/4/8/9/11 → 6

$$64: 5/12 \rightarrow 7 \rightarrow 10 \rightarrow 2/9 \rightarrow 3/4/8 \rightarrow 1/11 \rightarrow 6$$

$$65: 5/7/12 \rightarrow 2/10 \rightarrow 9 \rightarrow 3/4 \rightarrow 8/11 \rightarrow 1/6$$

As stated above, some points of interest in the completed rhythm are that 1) the overall ranking does not change that much in comparison to the ME pattern; 2) interval 12 is no longer an interval of symmetry; and 3) intervals 3 and 4 are now equally ranked. Keeping these concepts in mind, it is revealing to compare the build-up histograms to these qualities. Obviously, it is hard to compare R62's to that of the ME set or the completed pattern simply because there are not many onsets. A few points of interest do arise though: 1) if another frequency at interval 7 is added then the result is a symmetrical histogram, just like the ME set; and 2) the intervals 3 and 4 are equally represented, maintaining the ambiguity stated above. R63 is symmetrical, making it closely related to the ME set, while intervals 3 and 4 still remain equally represented. R64 seems like a mixture between R63 and R65 – it no longer symmetrical like R63 and overall has a similar histogram to R65, but with a small number of frequencies per interval. The most important aspect that continues throughout this buildup is that intervals 3 and 4 remain equally represented through the entire process – highlighting the metrical ambiguity that Reich is concerned with. In addition to this ambiguity, there is a mixture of representation between the completed rhythm and the symmetrical nature of the ME rhythm, revealing that the ME pattern is functioning in the background. The interval ranking for the remaining rehearsal numbers reveals exactly what we would expect – a buildup, and a strong relationship to the interval ranking of R65. R64 has a total of 7 distinct interval frequencies, which happens to the largest number out of all four of these rhythms. We see here that intervals 5 and 7 have different values, which is surprising in comparison to the goal rhythm at R65, and even in comparison to R63.

After Reich builds the live clarinet and Cl. 6 to a canon at the ♩, he emphasizes the canon by adding in three more clarinet parts. Between R66 and R70, Reich moves the two different rhythms that create the pattern to different Cl. parts to add variety and retain similarity. The bass clarinets are also added in this section, which will be explored later in this discussion. The section of R70 – R73 begins a process in which the live clarinet plays a subset of the resultant rhythm of the basic rhythm and the canon rhythm at the ♩. Fig. 3 – 8

0	7	2	9	4	11	6	1	8	3	10	5
0	7	2	9	4	11	6					5
12	19		21	16	23	18				22	17

R. 71

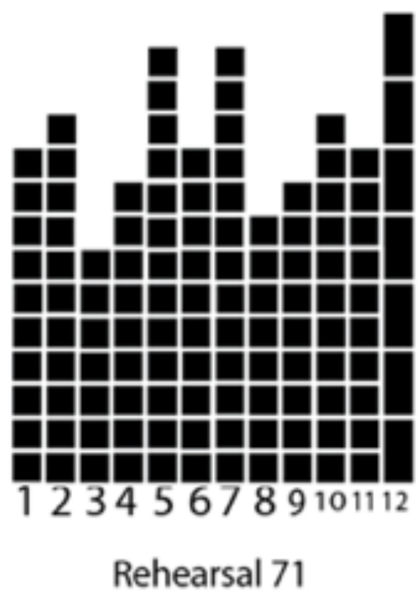


Fig. 3 – 8 Rhythm of R71 in relation to generator cycle and corresponding interval histogram

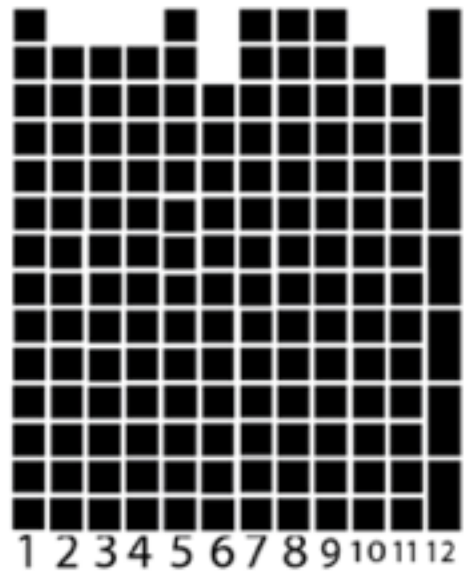
shows this subset of the resultant rhythm in relation to the generator cycle in addition to its corresponding interval histogram. Similar to R62 and R63, the two halves of the rhythm only have the difference of an onset in position 2 and then in position 10. The onsets in the universe are more saturated because there is a total of 16 onsets in the 24-beat universe. The interval histogram yields the interval ranking of: 12 → 5/7 → 2/10 → 1/6/11 → 4/9 → 8

→ 3. The main difference between this ranking and that of the basic rhythm are the occurrences of intervals 1 and 6. Even in tracing the buildup to the basic rhythm, intervals 1 and 6 were towards the bottom of the ranking, but in this example, they are much higher in the ranking. When looking at the ranking and disregarding the placements of interval 1 and 6, it is not far from the basic rhythm at all, which is expected because it is based upon the basic rhythm and its ♩ canon.

Instead of the previous canon at the ♩, R73 introduces a new pattern in which Reich creates a canon at the ♩.. In contrast to how he constructed this before, there is no buildup to this rhythm or establishment of it prior to adding in the live clarinet. Instead, the presentation of the canon and the subset of the resultant rhythm in the live clarinet happens all at once at R73. Fig. 3 – 9 shows the subset of the resultant rhythm of this canon and the corresponding interval histogram. The interval histogram has a rather flat surface, with no more relation to the interval rankings that we have explored prior to this. It is important to note that intervals 3 and 4 are still at the same frequency, which has been noted to be an important property of the rhythms throughout this piece in relation to the duple and triple ambiguity of meter. Although the interval histograms can be interpreted to represent rhythmic values, they might not align with one another in the sense of meter, so we cannot

0	7	2	9	4	11	6	1	8	3	10	5
0	7	2	9	4	11		1	8	3	10	5
12	19		21	16				20		22	17

R. 73



Rehearsal 73

Fig. 3 – 9 Subset of resultant rhythm of R73 in relation to generator cycle and corresponding interval histogram

solidly determine what values relate to an implied meter. With this number of onsets, the maximum interval height is 17 or 18, with the "floor" being 12. The subset of the resultant rhythm interval histogram has no interval values that rise above 14, and two intervals are represented at 12. In addition to the interval 3 and 4 ambiguity this rhythm creates, for the first time in the piece this rhythm represents all intervals roughly the same number of times. This rhythmic pattern, along with the one seen in Fig. 3 – 8, will switch back and forth throughout the second half of the piece. (See Table 3 – 1 for reference). Reich takes the listener through alternating rhythmic patterns that have a large amount of interval variety and others that do not. In addition to the change of canon at the ♩ and ♩., the interval content of these rhythms creates a feeling of rocking back and forth through material, which might aid in the change in perception that Reich is after.

There is one small difference in this subset of the resultant rhythm in regard to the ♩‿♪ from both patterns. The canon at the ♩. and subset of the resultant rhythm is seen in Fig. 3 – 10 for comparison. The true subsets of the resultant rhythm would have the live clarinet

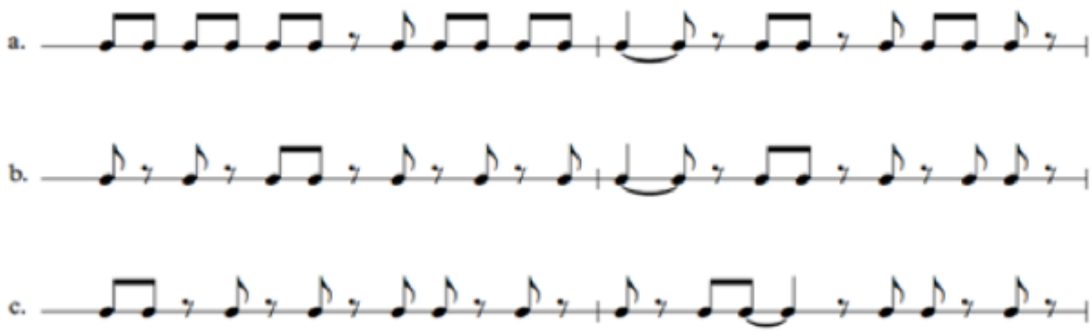



Fig. 3 – 10 a. Resultant rhythm b. Basic rhythm and c. Rhythm at the ♩.

playing an onset at position 14 in addition to everything else that Reich has written. Instead, Reich puts precedence on mimicking the basic rhythm and holds an onset through the moment of onset 14. The main reason for this is most likely because he wanted to retain the signature ♩‿♪ that helps the listener to discern where the canon is.

While all of the cited clarinet parts and live clarinet part are completing the process as described above, the two bass clarinet parts are highlighting an ambiguity found within the meter that Reich chose. The epigraph found at the beginning of this section is a testament to how Reich wants his audience to change perspective as this movement plays out. The two bass clarinet parts do not enter this movement until R67 and do not realize their full rhythmic pattern until R70. The buildup to the rhythmic pattern is not as dramatic as we have seen earlier in the live clarinet part, but instead, this rhythmic pattern is only 12 ♪ long and fits into one measure of the previously established 24-beat universe. The introduction of this rhythm is a complete 12 ♪ cycle but starts on beat 4. Fig. 3 – 11

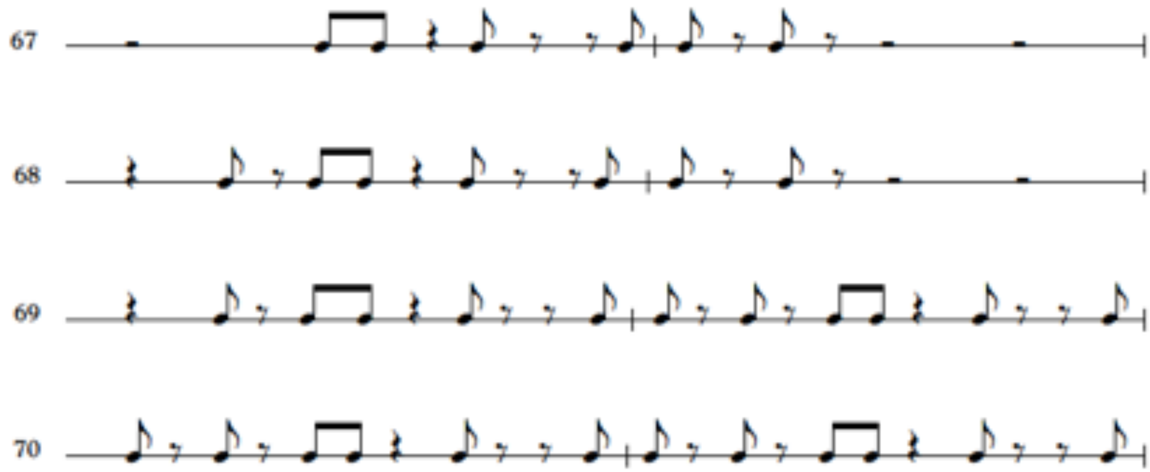



Fig. 3 – 11 Process of adding onsets to complete B. Cl. 9 rhythm at R70

shows the rhythm of the bass clarinet 9 part as it builds from R67 to 70. The first presentation of the rhythm at R67 spans the length of the pattern of 12 ♪ and proceeds to add onsets until there are two full presentation of the rhythm within the 24-beat universe that it exists within. Before discussing the metrical ambiguity that Reich alludes to in his program notes, the B. Cl. 9 part creates ambiguity regarding where the downbeat in this rhythmic pattern may start – with two onsets in a row like in R67, or does the pattern actually start like that of R70? In comparison to the basic rhythm that the other parts play, it is possible that the downbeat of the pattern begins at the actual down beatbeat of the measure, because their first four onsets are in line with each other, but there is no way to determine this fact concretely.

While the B. Cl. 9 part is building up to the rhythm at R70, the B. Cl. 10 part is doing the same, but its final rhythm is shifted by a 𝅗𝅥 in relation to the B. Cl. 9's rhythm. Fig. 3 – 12 shows this buildup to the shifted rhythm in relation to the B. Cl. 10 part. Due to the fact that there is no establishment of this rhythmic pattern prior to both B. Cls. starting

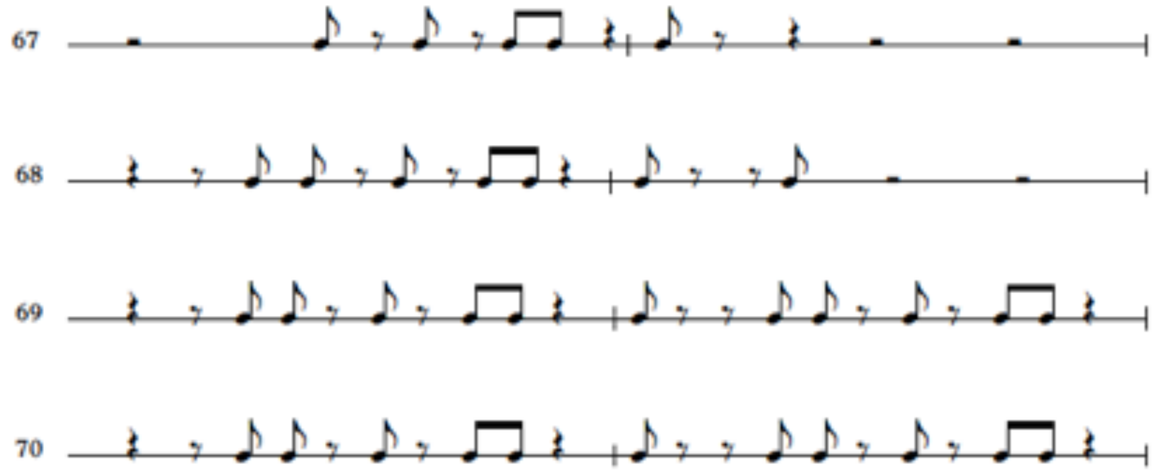


Fig. 3 – 12 Process of adding onsets to complete B. Cl. 10 rhythm at rehearsal 70

their process at R67, it is ambiguous as to which B. Cl. has the primary rhythm and which follows. Retrospectively speaking, when looking through the two completed rhythms and their interactions throughout the piece, it seems that it would make a great deal of sense to label B. Cl. 10 as the primary rhythm, because the onsets stay the same through the meter changes, which will be discussed.

At this point, because the rhythmic patterns are cyclic at the measure as opposed to the two-measure level, it is beneficial to look at both interval histograms for this rhythm: the 24-beat universe histogram so that we can compare it to earlier analyses, in addition to the simpler 12-beat universe histogram, as seen in Fig. 3 – 13. The rhythmic pattern used for the B. Cl. parts has 6 onsets that are spaced somewhat unevenly. At this point in the analysis, it is clear that the B. Cl. parts do not function in relation to a ME set; instead, it is more pertinent to compare the qualities of the rhythm that these histograms represent in relation those of the other clarinets' parts. The basic rhythm used as the canon and later the subsets of the resultant rhythms for the live clarinet had an interval ranking of 5/7/12 → 2/10 → 9 → 3/4 → 8/11 → 1/6, whereas the B. Cl. rhythm has an interval ranking of 12 → 3/6/9 →

4/8 → and then everything else. With these interval rankings in mind, we can clearly see that there is a connection to the ME 14 → 24 set in the basic rhythm but not in the bass clarinet rhythm.

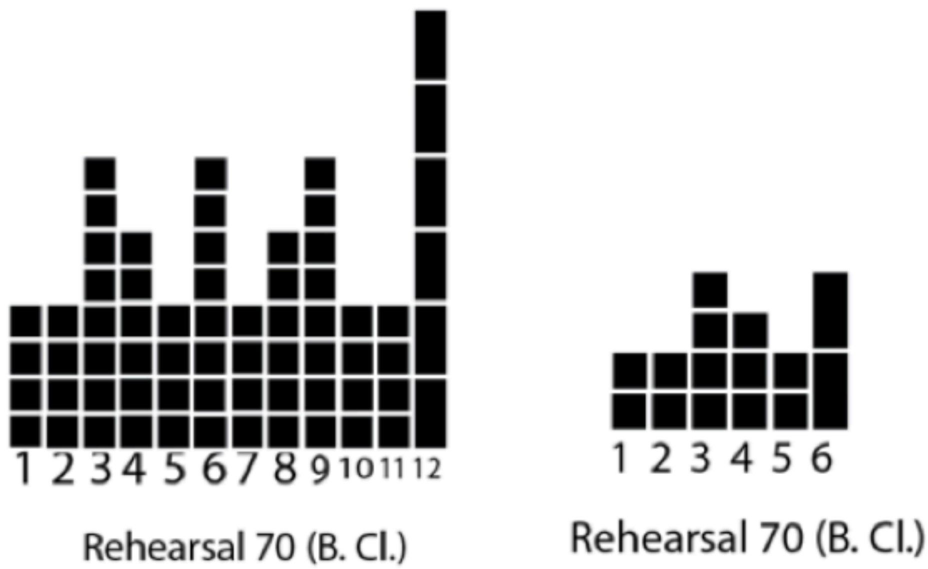


Fig. 3 – 13 Interval histograms of complete B. Cl. Rhythm in 24 and 12-beat universes

Reich speaks to how the meter of this movement is ambiguous, and the interval graphs from both the basic rhythm and the B. Cl. rhythm are a verification of this fact. If we are to compare the interval occurrences of the interval histogram to rhythmic values, then we can conclude, at the very least, what values each rhythm is highlighting. The basic rhythm has the values of 5, 7, and 12 at its highest ranking, which translates to the rhythmic values of 𝅗𝅥‿♪, 𝅗𝅥.‿♪ , and 𝅝‿𝅗𝅥, respectively, with the second highest rankings being 2 and 10, which translate to the rhythmic values of ♩ and 𝅝‿♩. With the exception of the measure-long interval, this rhythm then seems to be emphasizing intervals that do not fit the $\frac{6}{4} = \frac{3}{2}$ or $\frac{12}{8}$ meter, and instead highlights values that are a little under and over a quarter of the universe's length and are not multiples of 3 (with the exception of 12). On the other hand, the B. Cl. rhythm's interval histogram emphasizes the rhythmic values of 𝅝‿𝅗𝅥 , ♩. , 𝅗𝅥. , and 𝅝‿♪. These rhythmic values relate to a triple meter but, as discussed previously, do not necessarily imply a sense of meter. In the B. Cl. rhythm, intervals of 3 and 4 happen to line up with the metric grid, meaning that the ambiguity of duple verses triple is grounded in the bass voice. We can use the histograms as a guide to the overall relation to duple or triple meter, but not necessarily relate them to meter specifically because the intervals accounted for might not fall exactly on the metric grid. This means that this piece has a mixture of a relation to triple meter in the B. Cl. parts, while the other clarinet parts relate to odd-length rhythmic values that do not fit a specific meter.

In further investigation of the B. Cls.' parts, the pattern switches at the same time that the meter changes and the other Cl. parts switch, which occurs at R73. In the first version of the rhythm as seen at R70, the lower notes of each bass clarinet part occurred every 4 ♪, while the higher notes occurred every 3 ♪. R73 complicates this ambiguity further by switching the registral accents so that the lower notes now occur every 3 ♪ and the higher notes occur every 4 ♪. In terms of rhythm, only the B. Cl. 9 part changes the canonic interval to ♩. later than the B. Cl. 10 part at R73. The B. Cl. functions to highlight the ambiguity that this meter and rhythmic patterns Reich present within this movement. While still using the same rhythmic pattern between R70 and R73, changes in canonic delay and contour of notes affects the listener to perceive 3 groups of 4 ♪ or 4 groups of 3 ♪. This change in metrical accent achieves what Reich describes as a perception of change in something that is not changing. The B. Cl. parts are the most unambiguous part of the piece in terms of their metrical implications because the rhythm itself fits the meter. Reich plays with the idea that $\frac{6}{4} = \frac{3}{2}$ or $\frac{12}{8}$ highlight both duple and triple meters through the canon and contour relations of the B. Cl. parts. It is in these parts that the Reich achieves what the epigraph at the beginning of this section stated – to vary perception.

Reich's *New York Counterpoint* Mvt. 3 has a tight-knit construction. By consisting of only three types of rhythm: the basic rhythm, subset of resultant, and B. Cl. rhythm, it is simple and yet complex. The complexity comes from the construction of the canons through both the Cl. and B. Cl parts, in which Reich is concerned with metrical ambiguity between duple and triple meter. Relating his rhythmic choices to ME patterns, the analyst reveals, at a deeper level, the onset intervals that also highlight the triple and duple meter, along with the symmetrical rhythmic qualities they are related to. Reich uses the meters $\frac{6}{4} = \frac{3}{2}$ or $\frac{12}{8}$ to aid in his pursuit to change the perception of the audience while the music itself remains unchanged. This perception change is especially highlighted by the B. Cl. parts, while the other Cl. parts explore the implications of different canonic delays in their basic rhythm. The piece progresses from a simple basic rhythm to canonic and metrical complexity in all 11 clarinet parts and finishes with a simple canonical relationship. The effect of metrical accent and canon leaves Reich's audience to wonder what of their own perceptions are correct.

4. Conclusion

Music theorists are always searching for the answer to so many musical puzzles, and the way in which rhythm behaves is only one case. Although there have already been many discoveries in the rhythmic domain, this paper has aimed to contribute to the deeper understanding of how it can be constructed and function. By applying pitch theories to rhythm, as Babbitt first conceived of in 1962, a new world of possibilities opens up. This thesis sought to provide a different perspective on how to appreciate and comprehend rhythms. In order to do this, the concept of ME and rhythmic voice leading provided a basis to understand how a rhythmic pattern can change and transform throughout a piece while still relating to structure in the background through the progression of the music. It is through this lens that rhythms were then explored geometrically in order to conceptually ground the idea that rhythm exists in voice leading space.

The analysis of "Fém" helped to illustrate the concept of ME and rhythmic voice leading through a long ranging polyrhythm between the two hands of a pianist. It also showed that in interpreting rhythms in such a way, there is a clear conception of form and intimate relationships in the rhythmic choices that Ligeti has made. In these findings of the behavior of the rhythm, the piece can be explained as having variety through consistency, where the beat cycles remain but the relationship changes between them. Although surface level onsets change, the background structure can relate to ME and connections to voice leading space. By using Reich's *New York Counterpoint* Mvt. 3 as a piece of interest, the analysis sought to understand the rhythmic choices that were made, and to pinpoint the metrical ambiguity that Reich alludes to in his program notes. In relating the rhythms to the ME set, it is enlightening to think about the qualities that his rhythmic choices present. It is through this analysis that the ambiguity can be identified but nonetheless, will not affect the listener's change in perception throughout the piece.

I hope that in these particular attempts, my readers may now think about rhythm in a geometric way – in which the connections to pitch organization are much deeper than first thought. In the future, I wish to further investigate rhythm and its interaction with voice leading space, and, in doing so, I hope to expand on the conception of rhythm that I have developed for this project.

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