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ALGEBRAIC THEORIES OVER NOMINAL SETS

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ABSTRACT. We investigate the foundations of a theory of algebraic data types with variable binding inside classical universal algebra. In the first part, a category-theoretic study of monads over the nominal sets of Gabbay and Pitts leads us to introduce new notions of finitary based monads and uniform monads. In a second part we spell out these notions in the language of universal algebra, show how to recover the logics of Gabbay-Mathijssen and Clouston-Pitts, and apply classical results from universal algebra.

1. INTRODUCTION

The nominal sets of Gabbay and Pitts [10] give an elegant and powerful treatment of variable binding which is, on the one hand, close to informal practice and, on the other hand, lends itself to rigorous formalisation in theorem provers or programming languages. Nominal sets have been extraordinarily successful as witnessed by a wide range of work.

Closely related, albeit less developed, are the models of variable binding based on presheaf categories $[\mathcal{I}, \mathsf{Set}]$. These are categories of functors $\mathcal{I} \to \mathsf{Set}$ where the indexing category $\mathcal{I}$ consists of contexts (=sets of free variables) and maps between them (such as weakenings and renamings). This started with [9, 13] and was axiomatised in [23] to treat different $\mathcal{I}$ in a uniform way. We focus on the indexing category $\mathbb{I}$ associated with nominal sets (more below) and leave the general theory for future work.

This paper presents the foundations of a theory of algebraic data types with variable binding. We do this inside standard many-sorted universal algebra. In particular, the logics arising are (fragments of) the standard ones based on equational logic. This enables us to leverage the existing theory of universal algebra and we illustrate this by transferring two classical theorems to nominal sets: Birkhoff's variety theorem (or HSP-theorem) characterising equationally definable classes of algebras; and the quasivariety theorem characterising implicationally definable classes (Section 5).

We proceed in the following way. Although the category Nom of nominal sets is not equationally definable itself, it embeds in a canonical way into a presheaf category $[\mathbb{I}, \mathsf{Set}]$, sorted over contexts. Like any presheaf category, $[\mathbb{I}, \mathsf{Set}]$ is a many-sorted variety, ie equationally definable. Thus, over $[\mathbb{I}, \mathsf{Set}]$, universal algebra can be done in the usual way, by adding operations and equations. Transferring this back to nominal sets, it turns out that the logic thus obtained is more general than what is usually intended when working with nominal sets. The reason is that over $[\mathbb{I}, \mathsf{Set}]$ we have access to individual contexts and can define theories which do not treat contexts in a uniform way. This is repaired by introducing uniform theories. We then show that the (quasi)variety theorems specialise to uniform theories (Section 4).

Three points are worth noting:

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Nominal sets and sets-in-context. There has been some debate on whether nominal sets or sets-in-context are preferable. We illustrate how both have their advantages. On the one hand, our concept of a uniform theory originates from Gabbay's discovery [11] that classes of algebras over nominal sets (in the sense of [11]) are closed under abstraction (Definition 3.16). On the other hand, the sets-in-context approach of $[\mathbb{I}, \mathsf{Set}]$ allows us to use universal algebra directly and we obtain Gabbay's HSP-theorem and novel variations as a corollary of the classical theorems.

Category theory (CT). Category theory appears in this work for several reasons. First, CT offers a widely accepted notion of algebraic theory over a category, namely that of a monad. Thus, an account of algebraic theories over nominal sets ignoring monads would be incomplete. Second, the relationship between nominal sets and sets-in-context is best formulated in CT, see for example the crucial 'transport theorems' of Section 3.3. Third, CT allows for proofs at the right level of abstraction, thus providing more general results and opening new directions, some of which we will discuss in the conclusions.

Fb-monads. The categorical analysis of monads on nominal sets leads us to add fb-monads to the powerful toolbox of CT in computer science. They arise because monads on nominal sets are too general to remain in the realm of equational logic and universal algebra. Whereas fb-monads are precisely those monads which can be presented in universal algebra. Moreover, they can be transported from nominal sets to $[\mathbb{I}, \mathsf{Set}]$ and back: Loosely speaking, universal algebra does not see the difference between the two categories.

The structure of the paper is as follows. Section 3 studies monads on nominal sets and $[\mathbb{I}, \mathsf{Set}]$ and introduces fb-monads and uniform monads. Section 4 develops universal algebra over $[\mathbb{I}, \mathsf{Set}]$ and gives a syntactic description of the notion of uniform theory. Section 5 applies these results to algebras over nominal sets and shows that the work of Gabbay and Mathijssen [12] and Clouston and Pitts [5] fit in our framework.

2. Preliminaries

Notations. If $\mathscr{A}$ is a small category and $\mathscr{K}$ an arbitrary category the functor category $[\mathscr{A}, \mathscr{K}]$ has as objects functors from $\mathscr{A}$ to $\mathscr{K}$ and as morphisms natural transformations between functors.

For an endofunctor L on a category $\mathscr{A}$, we consider the category of *L-algebras*, denoted by $\mathsf{Alg}(L)$, whose objects are defined as pairs (A, α) such that $\alpha : LA \to A$ is a morphism in $\mathscr{A}$. A morphism of L-algebras $f : (A, \alpha) \to (A', \alpha')$ is a morphism $f : A \to A'$ of $\mathscr{A}$ such that $f \circ \alpha = \alpha' \circ Lf$.

If $\mathscr{A}$ is a category and $\mathbb{M} = (M, \mu, \eta)$ is a monad on $\mathscr{A}$ then $\mathscr{A}^{\mathbb{M}}$ denotes the category of Eilenberg-Moore algebras for the monad $\mathbb{M}$. These are algebras for M that behave well with respect to the multiplication and unit of the monad, see [19] for a precise definition.

If L is either a functor or a monad we use the ad-hoc notation $L\text{-}Alg$ for algebras for L.

If S is a set and A an object in a cocomplete category $\mathscr{K}$, $S \bullet A$ denotes the copower, that is, the coproduct of S-copies of A.

Universal algebra (UA) and UA-presentations. A signature (Srt, Op) in the sense of UA, or a *UA-signature*, is given by a set Srt (of sorts) and a set Op of operation symbols $op : w \to s$ where w is a finite word over Srt and $s \in Srt$. A *UA-theory* $\langle Srt, Op, E\rangle$ is given by a UA-signature and a set E of equations and $\mathsf{Alg}(Srt, Op, E)$ is the class of its models. If a category $\mathscr{A}$ is isomorphic to $\mathsf{Alg}(Srt, Op, E)$ we say that $\langle Srt, Op, E\rangle$ is a

UA-presentation of $\mathscr{A}$ and call $\mathscr{A}$ a *variety*. A variety $\mathscr{A}$ comes with a forgetful functor $U_{\mathscr{A}} : \mathscr{A} \to \mathsf{Set}^{Srt}$, which has a left-adjoint $F_{\mathscr{A}}$.

Monads. Any adjunction $F \dashv U : \mathscr{K} \to \mathscr{X}$ gives rise to a monad $\mathbb{T} = UF$, which in turn determines the category $\mathscr{X}^{\mathbb{T}}$ of algebras for the monad. If $\mathbb{T}$ is finitary (=preserves filtered colimits [2]) and $\mathscr{X} = \mathsf{Set}^{Srt}$, then $\mathscr{X}^{\mathbb{T}}$ is a variety. Conversely, any variety $\mathscr{A} \cong \mathsf{Alg}(Srt, Op, E)$ is isomorphic to $(\mathsf{Set}^{Srt})^{\mathbb{T}}$ where $\mathbb{T} = U_{\mathscr{A}} F_{\mathscr{A}}$ is a finitary monad. We say that $\langle Srt, Op, E\rangle$ is a UA-presentation of the monad $\mathbb{T}$.

Nominal Sets. We consider a countable set $\mathcal{N}$ of names and the group $\mathfrak{S}(\mathcal{N})$ of finitely supported permutations on $\mathcal{N}$ (that is permutations that fix all but a finite set of names). Let $\cdot : \mathfrak{S}(\mathcal{N}) \times X \to X$ be a left action of the group $\mathfrak{S}(\mathcal{N})$ on a set X. We say that a finite subset $S \subset \mathcal{N}$ supports an element x of X, if for any permutation $\pi \in \mathfrak{S}(\mathcal{N})$ that fixes the elements of S we have $\pi \cdot x = x$. A *nominal set* is a left action $(X, \cdot)$ such that any element of X is supported by a finite set.

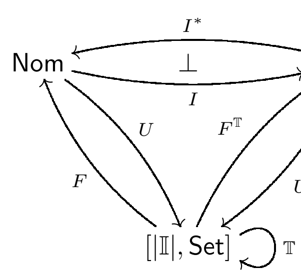
For each element x of a nominal set there exists a smallest set, in the sense of inclusion, which supports x. This set, denoted by $\mathsf{supp}(x)$, is called the *support* of x. We say that $a \in \mathcal{N}$ is *fresh* for x if $a \notin \mathsf{supp}(x)$.

A morphism of nominal sets $f : (X, \cdot) \to (Y, \circ)$ is an *equivariant* map between the carrier sets: $f(\pi \cdot x) = \pi \circ f(x)$ for all $x \in X$. Let Nom be the category of nominal sets and equivariant maps.

Nominal sets and the functor category $[\mathbb{I}, \mathsf{Set}]$**.** The notion of support equips Nom with a forgetful functor U, which in turn generates the variety $[\mathbb{I}, \mathsf{Set}]$ and the embedding $\mathsf{Nom} \to [\mathbb{I}, \mathsf{Set}]$. Here, $\mathbb{I}$ is the category whose objects are finite subsets of $\mathcal{N}$ and morphisms are injective maps. The underlying discrete subcategory is denoted by $|\mathbb{I}|$.

To define $U : \mathsf{Nom} \to [|\mathbb{I}|, \mathsf{Set}]$, we let, for a nominal set X, $UX(S)$ be the set of elements of X supported by S. U has a left adjoint $F : [|\mathbb{I}|, \mathsf{Set}] \to \mathsf{Nom}$.[1] Let $\mathbb{T}$ denote the monad on $[|\mathbb{I}|, \mathsf{Set}]$ generated by $F \dashv U$. The category of Eilenberg-Moore algebras for the monad $\mathbb{T}$ is equivalent to $[\mathbb{I}, \mathsf{Set}]$. The adjunction $F \dashv U$ is not monadic, but rather of *descent type*: this means that the comparison functor $I : \mathsf{Nom} \to [\mathbb{I}, \mathsf{Set}]$ is full and faithful.

(1)



Nom is equivalent to the full reflective subcategory of $[\mathbb{I}, \mathsf{Set}]$ consisting of pullback preserving functors, and this category is actually a Grothendieck topos. The comparison functor $I : \mathsf{Nom} \to [\mathbb{I}, \mathsf{Set}]$ has a left adjoint I^*. We know that I preserves filtered colimits and all limits, while I^* preserves finite limits and all colimits.

Abstraction. Let $(X, \cdot)$ be a nominal set. We consider the set $[\mathcal{N}]X$ consisting of equivalence classes of pairs $(a, x) \in \mathcal{N} \times X$ for the equivalence relation $\sim$ given by $(a, x) \sim (b, y)$ if and only if there exists $c \in \mathcal{N} \setminus \{a, b\}$, such that c is fresh for x and for y and

[1]The nominal sets FY are the strong nominal sets of [24].

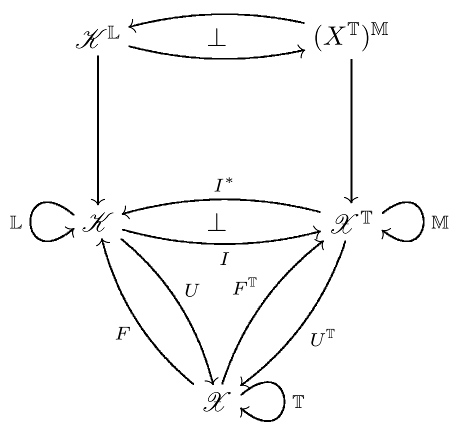
$(a\ c) \cdot x = (b\ c) \cdot y$. Let $[a]x$ denote the equivalence class of (a, x). There is a left action of $\mathfrak{S}(\mathcal{N})$ on $[\mathcal{N}]X$ given by $\pi \circ [a]x = [\pi(a)]\pi \cdot x$, so the set $[\mathcal{N}]X$ can be endowed with a nominal set structure. In fact, the above construction extends to a functor $[\mathcal{N}] : \mathsf{Nom} \to \mathsf{Nom}$, called *abstraction* or $\mathcal{N}$-abstraction in [10].

We have a similar notion of abstraction on $[\mathbb{I}, \mathsf{Set}]$, given by a functor $\delta : [\mathbb{I}, \mathsf{Set}] \to [\mathbb{I}, \mathsf{Set}]$ defined in Figure 3. As one might expect, $[\mathcal{N}]$ and δ are related to each other via the adjunction $I^* \dashv I$, see Section 3.3.

3. Finitary based and uniform monads

The aim of this section is two-fold: First, to study monads on Nom. Second, to show how to transport monads from Nom to $[\mathbb{I}, \mathsf{Set}]$. The category theoretic analysis is simplified by abstracting from (1) and studying instead

(2)



where $\mathscr{K}$ and $\mathscr{X}^{\mathbb{T}}$ replace Nom and $[\mathbb{I}, \mathsf{Set}]$. $\mathbb{L}$ and $\mathbb{M}$ are monads, $\mathscr{K}^{\mathbb{L}}$ and $(X^{\mathbb{T}})^{\mathbb{M}}$ are the associated categories of algebras.

Our assumptions are the following. $\mathscr{X}$ and $\mathscr{K}$ are locally finitely presentable (l.f.p.) categories [2] and $F \dashv U : \mathscr{K} \to \mathscr{X}$ is a finitary adjunction of descent type. This means that the comparison functor $I : \mathscr{K} \to \mathscr{X}^{\mathbb{T}}$ is full and faithful, where $\mathbb{T}$ is the monad generated by the adjunction. Equivalently, $F \dashv U$ is of descent type if every commutative diagram

(3)
$$FUFUA \underset{FU\varepsilon A}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow{\varepsilon A} A$$

is a coequalizer, where ε denotes the counit of $F \dashv U$.

The main contribution of this section is a notion of functors/monads that can comfortably be transported back and forth from $\mathscr{K}$ to $\mathscr{X}^{\mathbb{T}}$ using the above adjunction $I^* \dashv I$. These are exactly those functors/monads that are determined by their behaviour on finitely generated free objects. They can be presented by finitary signatures of a special kind: the only admissible arities are objects, free on finitely presentable (f.p.) objects of $\mathscr{X}$. As we will see in the next section, this means that they can be presented by operations and equations in the sense of universal algebra.

We recall first what is meant by signatures and equational presentations in category theory.

arities in $[|\mathbb{I}|, \mathsf{Set}]_{fp}$:
$N_{S,a} = |\mathbb{I}|(S \cup \{a\}, -)$ for $S \subseteq_f \mathcal{N}$ and $a \notin S$.

fb-signature:
$\Sigma_\delta : [|\mathbb{I}|, \mathsf{Set}]_{fp} \to [\mathbb{I}, \mathsf{Set}]$
$\Sigma_\delta(N_{S,a}) = \mathbb{I}(S, -)$, empty otherwise.

polynomial functor induced by the signature:
$H_{\Sigma_\delta} : [\mathbb{I}, \mathsf{Set}] \to [\mathbb{I}, \mathsf{Set}]$ given as $H_{\Sigma_\delta} = \mathrm{Lan}_F \Sigma_\delta$
$H_{\Sigma_\delta}(X) = \coprod_{N_{S,a}} X(S \cup \{a\}) \bullet \mathbb{I}(S, -)$.

equations omitted (but see Figure 3)

FIGURE 1. Kelly-Power (KP) presentation of δ

3.1. **Finitary (based) signatures.** In [15], Kelly and Power proved that finitary monads on a general l.f.p. category $\mathscr{K}$ indeed capture the idea of equational presentations of algebras on $\mathscr{K}$. Moreover, the monadic approach coincides with the UA-approach described in Section 2 in case when $\mathscr{K} = \mathsf{Set}^{Srt}$ where Srt is a set (of sorts). That is, the presentation (in the sense of Kelly and Power) of any finitary monad $\mathbb{T}$ on Set^{Srt} is a UA-presentation, i.e., $(\mathsf{Set}^{Srt})^{\mathbb{T}}$ is equivalent to a many-sorted variety in the sense of universal algebra. Figure 2 shows such a presentation where $\mathbb{T}$ is as in (1).

The concept of an equational presentation in a general l.f.p. category generalizes the triad

finitary signatures, terms of depth ≤ 1, equational theories

of universal algebra on (many-sorted) sets to the triad

finitary signatures, finitary endofunctors, finitary monads

of category theory.

The important ingredient of the presentation result of Kelly and Power [15] is the recognition of properties of the adjunction between the elements of the above triad: for every finitary monad $\mathbb{T}$ on $\mathscr{K}$, there exist two finitary signatures Γ and Σ and a coequalizer diagram

$$\mathbb{F}_\Gamma \rightrightarrows \mathbb{F}_\Sigma \longrightarrow \mathbb{T}$$

in the category of finitary monads on $\mathscr{K}$, where $\mathbb{F}_\Gamma$ and $\mathbb{F}_\Sigma$ are free (finitary) monads on Γ and Σ, respectively. A *finitary signature* Σ on $\mathscr{K}$ is a family Σn of objects of $\mathscr{K}$ indexed by f.p. objects n in $\mathscr{K}$. Similarly for Γ.

In fact, the above coequalizer expresses exactly the fact that $\mathbb{T}$-algebras are precisely those Σ-algebras satisfying equations specified by the parallel pair. We refer the reader to [15] for more details.

In what follows, a *special kind* of finitary signature on $\mathscr{K}$ will prove to be useful:

Definition 3.1. *Given an adjunction $F \dashv U : \mathscr{K} \to \mathscr{X}$ of descent type, an* fb-signature *on $\mathscr{K}$ is a family Σn of objects of $\mathscr{K}$, indexed by f.p. objects n in $\mathscr{X}$.*

Notice that every object of the form Fn is f.p. in $\mathscr{K}$. Hence fb-signatures are exactly those finitary signatures on $\mathscr{K}$ that have "nonempty" objects of operations only for arities of the form Fn, n f.p. in $\mathscr{X}$. That is, as opposed to finitary signatures, fb-signatures take arities in $\mathscr{X}$ instead of $\mathscr{K}$.

3.2. **Finitary and based functors/monads.** The functorial counterpart of fb-signatures is the following notion:

Definition 3.2. *A functor $L : \mathscr{K} \to \mathscr{K}$ is called* based *if L preserves all coequalizers of type (3). A monad $\mathbb{M} = (M, \mu, \eta)$ on $\mathscr{K}$ is called based if M is a based functor. A finitary and based functor/monad is called an* fb-functor/monad.

Remark 3.3. *It can be proved that fb-endofunctors of $\mathscr{K}$ are exactly those that are determined by their values on objects of the form Fn, where n is f.p. in $\mathscr{X}$.*

Let $\mathsf{End}_{fb}(\mathscr{K})$ denote the full subcategory of $[\mathscr{K}, \mathscr{K}]$ consisting of fb-functors, and let $\mathsf{Mnd}_{fb}(\mathscr{K})$ denote the category of fb-monads on $\mathscr{K}$. Any fb-monad on $\mathscr{K}$ can be presented by operations taking arities from finitely presentable objects of $\mathscr{X}$. To make this precise:

Theorem 3.4. *An fb-functor/monad on $\mathscr{K}$ can be presented by operations taking arities from f.p. objects of $\mathscr{X}$. Conversely, if a monad has such a presentation then it is finitary based.*

Remark 3.5. *Since any fb-functor/monad is* a fortiori *finitary, it can be equationally presented in the sense of Kelly and Power* [15] *using arities from $\mathscr{K}_{fp}$. The import of the above result is that arities are "finitely generated" free objects Fn. Therefore, one can work with arities n which are f.p. in $\mathscr{X}$.*

We can apply all the above results to endofunctors/monads on $\mathscr{X}^{\mathbb{T}}$. Fb-endofunctors on $\mathscr{X}^{\mathbb{T}}$ are exactly those that are *determined by values on finitely generated free algebras*, since based now means relative to the monadic adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \to \mathscr{X}$.

Example 3.6. *The presentation of the abstraction functor from Section 2 is given in Figure 1 and, using the notation from universal algebra, in Figure 3.*

The following two results will be used in the Section 4 to show that fb-monads have presentations in the sense of universal algebra. In a slogan, these results show that fb-monads are 'universal algebraic'.

Proposition 3.7. *Suppose $U : \mathscr{X}^{\mathbb{T}} \to \mathsf{Set}^{Srt}$ is a many-sorted variety. An endofunctor/monad on $\mathscr{X}^{\mathbb{T}}$ is finitary based iff it preserves sifted colimits*[2]*.*

Theorem 3.8 (monadic composition theorem)**.** *Suppose that $\mathbb{T}$ is a finitary monad on an l.f.p. category $\mathscr{X}$ and $\mathbb{M}$ an fb-monad on $\mathscr{X}^{\mathbb{T}}$. Then the composite*

$$(\mathscr{X}^{\mathbb{T}})^{\mathbb{M}} \longrightarrow \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$$

of the forgetful functors is monadic.

3.3. **Transporting monads and algebras.** Since fb-functors are exactly those determined by values on "finitely generated" free objects, they have nice properties w.r.t. transport back and forth along the adjunction $I^* \dashv I$. The reason for their nice behaviour is, essentially, that I is a comparison functor and such functors interact nicely with free objects.

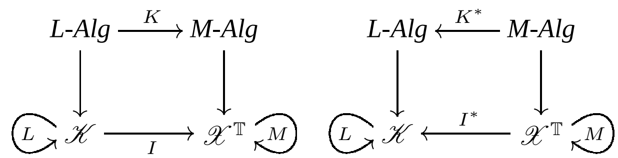
Theorem 3.9. *The assignment $L \mapsto ILI^*$ constitutes a functor $\Phi : \mathsf{End}_{fb}(\mathscr{K}) \to \mathsf{End}_{fb}(\mathscr{X}^{\mathbb{T}})$ that lifts to a functor $\widehat{\Phi} : \mathsf{Mnd}_{fb}(\mathscr{K}) \to \mathsf{Mnd}_{fb}(\mathscr{X}^{\mathbb{T}})$. Both Φ and $\widehat{\Phi}$ are full, faithful and have left adjoints. The left adjoint of Φ is given by $W \mapsto I^*WI$.*

Example 3.10. *δ and $[\mathcal{N}]$, as well as polynomial functors are transported to each other.*

[2]For an introduction to sifted colimits see [3].

Next, we consider the effect of transport on algebras. It turns our that the adjunction $I^* \dashv I$ lifts to an adjunction between the categories of algebras.

Theorem 3.11. *Consider a fb-functor/monad L on $\mathscr{K}$ and let $M = ILI^*$ be its "transport along I". Then there are diagrams*



commuting up to isomorphism, the left-hand one being a pseudopullback. Moreover, $K^* \dashv K$ holds.

Pseudopullbacks are a "bicategorical" notion of pullbacks. The pseudopullback condition means that every M-algebra with carrier from $\mathscr{K}$ is an L-algebra. This will be used in Section 5.

3.4. **Uniform monads.** An important feature of nominal sets, but also other categories for variable binding [23] is the presence of an abstraction functor, say D. It is therefore of interest to study functors (monads) H which have the property that D lifts to H-algebras, that is, there is a 'distributive law' $HD \to DH$: Given $HA \to A$ we obtain an H-algebra $HDA \to DHA \to DA$ over DA.

From now on, we instantiate $\mathscr{K}$ in (2) with Nom, hence D is either $[\mathcal{N}]$ or δ as in Section 2. We leave a more general development for future work.

Definition 3.12. *An endofunctor H on Nom (or $[\mathbb{I}, \mathsf{Set}]$) is called* uniform *if there exists a natural transformation $H[\mathcal{N}] \to [\mathcal{N}]H$ (or $H\delta \to \delta H$).*

Example 3.13. *Polynomial functors and δ are uniform. Figure 5 shows an fb-functor that is not uniform.*

In the case of monads, the natural transformation needs to satisfy an additional property and is then called a distributive law [14].

Definition 3.14. *A monad on Nom, respectively on $[\mathbb{I}, \mathsf{Set}]$, is called uniform if it has a distributive law over $[\mathcal{N}]$, respectively over δ.*

Example 3.15. *δ is uniform. In Figure 5 we describe a fb-functor that is not uniform.*

This allows us to define abstraction of algebras. We spell it out for δ and uniform functors, the remaining cases are analogous.

Definition 3.16. *Suppose H is a uniform functor by means of a distributive law $\tau : H\delta \to \delta H$. Then the abstraction of an H-algebra (A, a) is an H-algebra $(\delta A, Ha \circ \tau_A)$.*

Proposition 3.17. *If an fb-functor/monad L on Nom distributes over $[\mathcal{N}]$, then the transport M along I distributes over δ. Conversely, if M distributes over δ then $[\mathcal{N}]$ distributes over L.*

operation symbols $Op_{[\mathbb{I},\mathsf{Set}]}$:

$$\begin{array}{ll} (b/a)_S : S \cup \{a\} \to S \cup \{b\} & a \neq b,\ a \notin S,\ b \notin S \\ w_{S,a} : S \to S \cup \{a\} & a \notin S \end{array}$$

equations $E_{[\mathbb{I},\mathsf{Set}]}$:

$$\begin{array}{l} (a/b)_S(b/a)_S(x) = x \\ (b/a)_{S\cup\{d\}}(d/c)_{S\cup\{a\}}(x) = (d/c)_{S\cup\{b\}}(b/a)_{S\cup\{c\}}(x) \\ (c/b)_S(b/a)_S(x) = (c/a)_S \\ (b/a)_{S\cup\{c\}} w_{S\cup\{a\},c}(x) = w_{S\cup\{b\},c}(b/a)_S \\ (b/a)_S w_{S,a}(x) = w_{S,b}(x) \\ w_{S\cup\{b\},a} w_{S,b}(x) = w_{S\cup\{a\},b} w_{S,a}(x) \end{array}$$

FIGURE 2. UA-theory of $[\mathbb{I}, \mathsf{Set}]$

4. UNIVERSAL ALGEBRA OVER $[\mathbb{I}, \mathsf{Set}]$

In this section we see that fb-monads on $[\mathbb{I}, \mathsf{Set}]$ are given by universal algebra (UA) theories on $[\mathbb{I}, \mathsf{Set}]$. Corresponding to the concept of uniform monad we introduce the notions of uniform signature, uniform equations and uniform UA-theories. Similar to Birkhoff's variety theorem, we can characterise classes of algebras definable by uniform equations as those that are closed under images, subalgebras, products and abstraction. We also prove the uniform analogue of the quasivariety theorem.

4.1. **Equational theories.** As explained in Section 2, our notions of many-sorted signature (Srt, Op), equational theory $\langle Srt, Op, E\rangle$, algebras $\mathsf{Alg}(Srt, Op, E)$ are those of Universal Algebra. We are interested in $Srt = |\mathbb{I}|$. Referring to Figure 2, we call

$$\langle |\mathbb{I}|\ ,\ Op_{[\mathbb{I},\mathsf{Set}]} \uplus Op\ ,\ E_{[\mathbb{I},\mathsf{Set}]} \uplus E\rangle \tag{4}$$

a *theory over* $\mathbb{I}$. If equations in E do not contain nested occurrences of operation in Op we say that the theory is of *rank 1*, see Figure 1 for an example.

Proposition 4.1 ([17, 16]). *A theory $\langle Srt, Op, E\rangle$ over $\mathbb{I}$ of rank 1 determines a functor $M : [\mathbb{I}, \mathsf{Set}] \to [\mathbb{I}, \mathsf{Set}]$. Moreover, $\mathsf{Alg}(M) \cong \mathsf{Alg}(Srt, Op, E)$.*

In one-sorted universal algebra such a functor is typically a polynomial functor $X \mapsto LX = \coprod_{n\in\mathbb{N}} \mathsf{Set}(n, X) \bullet \Sigma n$, where $\mathsf{Set}(n, X) \bullet \Sigma n$ denotes the coproduct of $\mathsf{Set}(n, X)$-many copies of Σn. Hence Σn is the set of n-ary operations. Here, apart from polynomial functors, we are also interested in functors specifying operations involving binders, the most basic one being the δ of Figure 3.

Specifying additional operations by a functor has the advantage that the initial algebra of terms comes equipped with an inductive principle. For an example see how λ-terms form the initial algebra for a functor in [9, 10, 13].

4.2. **Relating KP- and UA-presentations.** We argue that the fb-monads from Section 3 are precisely those monads that have a UA-presentation.

Example 4.2. *Consider a UA-signature as in (4) with Op containing one operation $\mathsf{app} : \emptyset, \emptyset \to \emptyset$ and $E = \emptyset$. Consider $N : |\mathbb{I}| \to \mathsf{Set}$ defined as $N(\emptyset) = 2$ and empty otherwise. The corresponding fb-signature $\Sigma : |[|\mathbb{I}|, \mathsf{Set}]_{fp}| \to [\mathbb{I}, \mathsf{Set}]$ maps all f.p. objects in $[|\mathbb{I}|, \mathsf{Set}]$ to 0 with the exception of N which is mapped to $\mathbb{I}(\emptyset, -)$. The endofunctor presented by Σ*

operation symbols Op_δ:

$[a]_S : S \cup \{a\} \to S$
for all finite sets S and $a \notin S$

equations E_δ:

$$\begin{array}{ll} (c/b)_S[a]_{S\cup\{b\}}t = [a]_{S\cup\{c\}}(c/b)_{S\cup\{a\}}t & t : S \cup \{a,b\} \\ [a]_S t = [b]_S (b/a)_S t & t : S \cup \{a\} \\ w_{S,b}[a]_S t = [a]_{S\cup\{b\}} w_{S\cup\{a\},b} t & t : S \cup \{a\} \end{array}$$

FIGURE 3. UA-presentation of δ

then is $H_\Sigma(X) = (X(\emptyset) \times X(\emptyset)) \bullet \mathbb{I}(\emptyset, -)$. Going back from H_Σ to a UA-presentation gives us the theory of Figure 5. This theory is different from the one we started with, but the two theories are equivalent: they define isomorphic categories of algebras.

This example can be generalised and similar to Proposition 4.1 we have

Proposition 4.3. *Every UA-theory over $\mathbb{I}$ gives rise to an fb-monad on $[\mathbb{I}, \mathsf{Set}]$.*

Conversely, fb-functors/monads have UA-presentations.

Theorem 4.4. *Every fb-functor on $[\mathbb{I}, \mathsf{Set}]$ has a presentation as a UA-theory over $\mathbb{I}$ of rank 1.*

This is a consequence of Proposition 3.7 and [17, 16].

Theorem 4.5. *Every fb-monad on $[\mathbb{I}, \mathsf{Set}]$ has a presentation as a UA-theory over $\mathbb{I}$.*

This is a consequence of Theorem 3.8.

4.3. **Uniform UA-theories.** Let us give an intuitive motivation for the notions introduced in this section. Assume we want to investigate algebraic theories over nominal sets by studying their transport to $[\mathbb{I}, \mathsf{Set}]$. Suppose we have some notion of signature and equations over nominal sets, such as the nominal logics of [12, 5]. A nominal set X satisfies an equation, if for any instantiation of the variables, possibly respecting some freshness constraints, we get equality in X. Notice that the support of the elements of X used to instantiate the variables can be arbitrarily large. Let us think what this means in terms of the corresponding presheaf IX. For a finite set of names S, $IX(S)$ is the set of elements of X supported by S. So IX should satisfy not one, but a set of 'uniform' equations, (for an example, see Figure 6). This means that we should be able to extend in a 'uniform' way the operation symbols together with their arities, the sort of the equations and the sort of the variables. We formalize this below, following the same lines as in [18]. Moreover, we prove that this concrete syntax implements the notions introduced in Section 3.4.

Definition 4.6. *A UA-signature over $\mathbb{I}$ of the form $\langle |\mathbb{I}|\ ,\ Op_{[\mathbb{I},\mathsf{Set}]} \uplus Op\ \rangle$ is called uniform if the set Op of operation symbols can be organized as a presheaf, abusively also denoted by $Op \in [\mathbb{I}, \mathsf{Set}]$, such that any operation symbol $f \in Op(S)$ has arity of the form*

$$f : S_1, \ldots, S_n \to S_0$$

with $\cup S_i = S$. Additionally, we require that for any injective map $u : S \to T$ the operation symbol $Op(u)(f)$ has arity

$$Op(u)(f) : T \setminus u[S \setminus S_1], \ldots, T \setminus u[S \setminus S_n] \to T \setminus u[S \setminus S_0]$$

where $u[S \setminus S_i]$ denotes the direct image of $S \setminus S_i$ under u. For simplicity let $u \cdot f$ denote $Op(u)(f)$.

The intention here is that $S \setminus S_i$ is the set of names bound by f at the corresponding position. For example, the operations in Figure 3 form a uniform signature. They can be structured as a presheaf as follows:

$$\begin{array}{l} [a]_S \in Op(S \cup \{a\}) \\ w_b \cdot [a]_S = [a]_{S\cup\{b\}} \\ (b/a)_S \cdot [a]_S = [b]_S \end{array} \tag{5}$$

For such a signature we define the notions of uniform term and uniform equation. The intuition here is that a uniform equation generates a set of equations in the sense of universal algebra.

A *uniform term* $t : T$ for a uniform signature is a term t of type T formed according to the rules in Figure 4. Each rule can be instantiated in an infinite number of ways: T ranges over finite sets of names and a, b over names. The notation $T \uplus \{a\}$ indicates that an instantiation of the schema is only allowed for those sets T and those atoms a where $a \notin T$. A *uniform equation* is a pair of uniform terms of the same sort $u = v : T$, such that any variable X appears with the same type T_X in both u and v. A *uniform theory* consists of a set of uniform equations.

A uniform equation $u = v : T$ is not an equation in the sense of universal algebra, but it generates a set of equations indexed over all finite sets of names S that are disjoint from T. We will call these equations the translations of $u = v : T$ by S, and they are defined below. These translations should involve enlarging the sort of the variables. However this is not always possible, for example if we have a subterm $w_a X$ of an equation, then the sort of X cannot contain the name a.

Definition 4.7. *The freshness set of a variable X appearing with sort T_X in an equation E of the form $u = v : T$ is the set*

$$\mathsf{Fr}_E(X) = \bigcup_{t:T} T \setminus T_X$$

where the union is taken over all sub-terms t of either u or v that contain the variable X.

Example 4.8. *As an example, let us consider a set of operation symbols*

$$\begin{array}{l} a_S \ : \ S \cup \{a\} \\ \mathsf{app}_S \ : \ S, S \to S \\ [a]_S \ : \ S \cup \{a\} \to S \end{array}$$

In fact these operations subject to some equations give a presentation for the functor $LX = \mathcal{N} + \delta X + X \times X$, whose initial algebra is the presheaf of α-equivalence classes of λ-terms, see [18, Section 4] *for details on this.*

For the uniform equation $[a]_\emptyset \mathsf{app}_{\{a\}}(w_a X, a_\emptyset) = X$ the freshness set of X is $\mathsf{Fr}(X) = \{a\}$. In Figure 6 we see that this equation corresponds to equations in other nominal logics having as side condition that a is fresh for X.

Definition 4.9. *The translation of an equation E of the form $u = v : T_E$ by a name $a \notin T_E$ is an equation $tr_a(u) = tr_a(v)$ of sort $T \cup \{a\}$, where the translation $tr_a(t : T)$ (with $a \notin T$) of a sub-term t of either u or v is defined recursively by*

(6)
$$
\begin{aligned}
&tr_a\,(f(t_1,\ldots,t_n):T_0) \stackrel{(a\not\in T)}{=} (w_a\cdot f)(tr_a\,(t_1),\ldots,tr_a\,(t_n))\\
&tr_a\,(f(t_1,\ldots,t_n):T_0) \stackrel{(a\in T)}{=} w_a(f(t_1,\ldots,t_n))\\[1em]
&tr_a\,(w_b t : T\uplus\{b\}) = w_{S\cup\{a\},b}\, tr_a\,(t:T)\\[1em]
&tr_a\,((b/c)t : T\uplus\{b\}) \stackrel{(a\neq c)}{=} (b/c)_{T\cup\{a\}}\, tr_a\,(t:T\uplus\{c\})\\
&tr_a\,((b/a)t : T\uplus\{b\}) \;=\; w_a(b/a)_T\, t\\[1em]
&tr_a\,(X:T_X) = w_a X_{T_X} \quad \text{if } a\in \mathsf{Fr}_E(X)\\
&tr_a\,(X:T_X) = X'_{T_X\cup\{a\}} \quad \text{if } a\notin \mathsf{Fr}_E(X)
\end{aligned}
$$

where in the first two conditions $f : T_1,\ldots,T_n \to T_0$ is an operation symbol in Op. In the last condition $X'_{T_X\cup\{a\}}$ is a variable of sort $T_X\cup\{a\}$.

The translation of an equation E of the form $u = v : T_E$ by a set $S = \{a_1,\ldots,a_k\}$ disjoint from T_E is a defined as $tr_{a_1}\,(\ldots tr_{a_k}\,(u = v : T)\ldots) : T\cup S$. (The chosen order of the elements of S is irrelevant).

We will say that a set of (standard universal algebra) equations is *uniformly generated* by a uniform theory $\mathcal{U}$ if it consists of all possible translations of the uniform equations in $\mathcal{U}$.

Example 4.10. *The UA-theory expressing the eta-equivalence of the λ-calculus is uniformly generated by the uniform equation of the last line of Figure 6.*

Definition 4.11. *A uniform UA-theory over $\mathbb{I}$ is a theory $\langle|\mathbb{I}|\ ,\ Op_{[\mathbb{I},\mathsf{Set}]}\uplus Op\ ,\ E_{[\mathbb{I},\mathsf{Set}]}\uplus E\uplus E_{Op}\rangle$ such that the set of equations E is uniformly generated by a uniform theory and E_{Op} is the set of equations of the form:*

$$
\begin{aligned}
&(w_a\cdot f)(w_a x_1,\ldots,w_a x_n) = w_a f(x_1,\ldots,x_n)\\
&((a/b)_{S\setminus\{b\}}\cdot f)(\langle a/b\rangle_{S_1\setminus\{b\}}x_1,\ldots,\langle a/b\rangle_{S_n\setminus\{b\}}x_n) =\\
&\langle a/b\rangle_{S_0\setminus\{b\}} f(x_1,\ldots,x_n)
\end{aligned}
$$

for $f\in Op(S)$ having arity $S_1,\ldots,S_n\to S_0$, $a\notin S$ and $b\in S$, with the additional convention that $\langle a/b\rangle_{S_i\setminus\{b\}}$ denotes the identity on S_i if $b\notin S_i$ and $(a/b)_{S\setminus\{b\}}$ if $b\in S_i$.

Next, we will see that there is a strong connection between uniform UA-theories and the concept of abstraction. The reason for this is the existence of an isomorphism for every finite set S and $a\notin S$

$$A(S\cup\{a\})\cong \delta A(S)$$

that maps $x\in A(S\cup\{a\})$ to $[a]_S x$.

Consider a uniform signature as in Definition 4.6 and let A be an algebra for the uniform theory $\langle|\mathbb{I}|\ ,\ Op_{[\mathbb{I},\mathsf{Set}]}\uplus Op\ ,\ E_{[\mathbb{I},\mathsf{Set}]}\uplus E_{Op}\rangle$. We can define the *abstraction* of A to be an algebra with carrier δA and the interpretation of an operation symbol in $Op(S)$ of the form $f : S_1,\ldots,S_n\to S_0$ given by:

$$f^{\delta A}([a]_{S_1}x_1,\ldots,[a]_{S_n}x_n) = [a]_{S_0}(w_a\cdot f^A)(x_1,\ldots,x_n)$$

for some $a\notin S$.

The next proposition is based on the observation that an algebra δA satisfies an equation E if and only if the algebra A satisfies the translation $tr_a\, E$ of an equation by a *new* name a.

$$\frac{t_1 : T_1,\ \ldots,\ t_n : T_n}{f(t_1, \ldots t_n) : T_0}\ (f : T_1, \ldots, T_n \to T_0 \in Op(T))$$

$$\frac{t : T}{w_a t : T \uplus \{a\}} \qquad \frac{t : T \uplus \{a\}}{(b/a)t : T \uplus \{b\}} \qquad \frac{}{X : T_X}$$

FIGURE 4. Uniform terms

operations:

$\mathsf{app}_S : \emptyset, \emptyset \to S$ for all $S \subseteq_f \mathcal{N}$

equations:

$w_a \mathsf{app}_S(x, y) = \mathsf{app}_{S\cup\{a\}}(x, y)$
$(b/a)_S \mathsf{app}_{S\cup\{a\}}(x, y) = \mathsf{app}_{S\cup\{b\}}(x, y)$

FIGURE 5. UA-Presentation of a non-uniform functor $L_0(X) = (X(\emptyset) \times X(\emptyset)) \bullet \mathbb{I}(\emptyset, -)$

Proposition 4.12. *A class of algebras for a uniform UA-theory* $\langle |\mathbb{I}|\ ,\ Op_{[\mathbb{I},\mathsf{Set}]} \uplus Op\ ,\ E_{[\mathbb{I},\mathsf{Set}]} \uplus E_{Op}\rangle$ *defined by uniform equations* E *is closed under abstraction.*

From this it follows that a class of algebras for a uniform UA-theory defined by additional uniform equations is closed under abstraction. This means that the abstraction functor δ lifts to a functor $\tilde{\delta}$ on the categories of algebras for a uniform UA-theory. Therefore, similar to Proposition 4.1 we have:

Proposition 4.13. *The functor on* $[\mathbb{I}, \mathsf{Set}]$ *determined by a uniform UA-theory of rank 1 is uniform in the sense of Definition 3.12.*

Example 4.14. *The functor* δ *has a uniform presentation given in Figure 3. As a counterexample, consider the functor* L_0 *presented in Figure 5. Although the operations can be structured as a presheaf, the presentation is not uniform.*

Similar to Proposition 4.3 we obtain

Proposition 4.15. *Every uniform UA-theory over* $\mathbb{I}$ *gives rise to a uniform fb-monad on* $[\mathbb{I}, \mathsf{Set}]$*, see Definition 3.14.*

4.4. **Results from universal algebra.** In one-sorted universal algebra, Birkhoff's variety theorem characterizes equationally definable classes of algebras as those closed under HSP, that is, homomorphic images, subalgebras and products. The theorem is not true in general for many-sorted algebras, see [1]: An equationally definable class of many-sorted algebras is closed under homomorphic images, subalgebras, products and *directed colimits*. However, because of the special structure of the category $\mathbb{I}$, as pointed out in [18], we have:

Theorem 4.16. *Consider a UA-theory over* $\mathbb{I}$ *and let* $\mathcal{A}$ *denote its algebras. Then a class* $\mathcal{C} \subseteq \mathcal{A}$ *is equationally definable if and only if it is closed under HSP.*

There exists a similar characterization of finitary quasivarieties for many-sorted algebras. These are classes of algebras definable by implications, where by implication we mean here a formula

$$(u_1 = v_1) \wedge \cdots \wedge (u_n = v_n) \Rightarrow (u_0 = v_0)$$

where $u_i = v_i$ are equations. The next theorem is an instance of the well known quasivariety theorem.

Theorem 4.17. *Let $\mathcal{A}$ be the category of algebras for a UA-theory over $\mathbb{I}$. Then a class $\mathcal{C} \subseteq \mathcal{A}$ is implicationally definable if and only it is closed under subalgebras, products and filtered colimits.*

For the uniform UA-theories we can provide similar characterizations. The next theorem generalises [18, Theorem 5.23]. On a category of algebras $\mathcal{A}$ given by a uniform UA-theory we have an abstraction operator given by Proposition 4.12. We have

Theorem 4.18. *Consider a uniform UA-theory over $\mathbb{I}$ and let $\mathcal{A}$ denote its algebras. Then a class $\mathcal{C} \subseteq \mathcal{A}$ is equationally definable by additional uniform equations if and only if it is closed under HSPA, that is, homomorphic images, subalgebras, products and abstraction.*

Definition 4.19. *A uniform implication of type T is a formula*

$$(u_1 = v_1) \wedge \cdots \wedge (u_n = v_n) \Rightarrow (u_0 = v_0) : T$$

where $u_i = v_i : T_i$ are uniform equations for $i = 0, \ldots, n$ and $T = T_0 \cup \cdots \cup T_n$.

Each uniform implication of type T generates a set of standard universal algebra implications, indexed by finite sets S with $S \cap T = \emptyset$. We do this by translating each uniform equation $(u_i = v_i) : T_i$ as in (6), with the only difference being that in the last two relations, we use $\mathsf{Fr}_{u_0=v_0}(X) \cup \cdots \cup \mathsf{Fr}_{u_n=v_n}(X)$ instead of $\mathsf{Fr}_{u_i=v_i}(X)$.

Consider the category of algebras $\mathcal{A}$ for a uniform UA-theory. We say that $\mathcal{C} \subseteq \mathcal{A}$ is implicationally definable by uniform implications if there exits a set of uniform implications $\mathcal{I}$ such that $\mathcal{C}$ is definable by the set of UA-implications generated by all the elements of $\mathcal{I}$. Then we can prove:

Theorem 4.20. *Consider a UA-theory over $\mathbb{I}$ and let $\mathcal{A}$ denote its algebras. Then a class $\mathcal{C} \subseteq \mathcal{A}$ is implicationally definable by uniform implications if and only if it is closed under subalgebras, products, filtered colimits and abstraction.*

5. UNIVERSAL ALGEBRA OVER NOMINAL SETS

Building on the general theory developed in Section 3, we can now transfer properties and results obtained in universal algebra on $[\mathbb{I}, \mathsf{Set}]$ to nominal sets. To achieve this we use the next theorem, which can be derived from Theorem 3.11.

Theorem 5.1. *Any fb-monad/functor L on Nom induces a UA-theory Φ on $[\mathbb{I}, \mathsf{Set}]$, so that the category of L-algebras is the category of Φ-algebras 'restricted along I'.*

There are several approaches in the literature to develop algebraic theories over nominal sets: nominal (universal) algebra of [12] and NEL of [5]. These approaches fit in the general framework developed here, and more importantly, we can prove new results for them using our technique.

For example, the signatures defined in [12] are given by functors of the form $\mathcal{N} + [\mathcal{N}] + \Sigma$, where $\mathcal{N}$ is the constant functor, $[\mathcal{N}]$ is the abstraction functor and Σ is a polynomial functor. These functors are uniform and finitary based. In fact, in [18, Section 6] we have given syntactical translations of theories of nominal algebra and NEL into uniform theories, for

nominal algebra ([12]):
$a\#X \vdash [a]\mathsf{app}(X, a) = X$

NEL ([5, Fig. 4]):
$a \not\# x \vdash L_a(A\, x\, V_a) \approx x$

UA-theory:
$[a]_S\mathsf{app}_{S\cup\{a\}}(w_{S,a}X_S, a_S) = X_S$
for all finite S, $a \notin S$ and X_S variable of sort S

uniform UA-theory:
$[a]\mathsf{app}_{\{a\}}(w_aX, a) = X$

FIGURE 6. η-rule for untyped λ-calculus

an example see Figure 6. As anticipated in Example 4.8 we translate a freshness condition $a\#X$ by adding operations symbols of the form w_a in front of the variable X.

In our general setting, we can characterise the equationally definable subcategories of algebras on nominal sets. First, let us see what we mean by this.

Definition 5.2. *Let L be a functor on* Nom. *A full subcategory $\mathcal{C}$ of L-algebras is equationally definable by a UA-theory Φ on $\mathbb{I}$ if $\mathcal{C}$ consists of L-algebras (A, a) with $K(A, a) \models \Phi$, where K : L-Alg $\to$ Φ-Alg is the lifting of I as in Theorem 3.11.*

The next theorem follows from Theorem 4.16 and the observation that a Φ-algebra which lies in the closure under HSP of $I\mathcal{C}$ and has as carrier a nominal set is in fact an object of $I\mathcal{C}$. Here, $I\mathcal{C}$ is the subcategory of Φ-*Alg*, obtained as the image of $\mathcal{C}$ under I.

Theorem 5.3. *Let L be a fb-functor/monad on* Nom. *A class of L-algebras is equationally definable if and only if it is closed under homomorphic images of support-preserving maps, under subalgebras and under products.*

Remark 5.4. *We obtain closure under homomorphic images of support-preserving maps rather than all homomorphic images because I only preserves the former.*

But we can do better than that for algebras for a functor L, whose transport on $[\mathbb{I}, \mathsf{Set}]$ is given by a uniform UA-theory of rank 1. In the remainder of this section by algebras over nominal sets we understand algebras for such functors. From Theorem 4.18 we derive

Theorem 5.5. *A class of algebras over nominal sets is definable by uniform equations if and only if it is closed under homomorphic images, subalgebras, products, and abstraction.*

Similarly, using Theorem 4.20 we can prove a quasivariety theorem for algebras over nominal sets:

Theorem 5.6. *A class of algebras over nominal sets is definable by uniform implications if and only if it is closed under subalgebras, products, filtered colimits and abstraction.*

These theorems can be transferred to nominal algebras [12] and NEL [5], using the translations given in [18], for example we recover Gabbay's HSPA-theorem of [11]. Additionally we obtain new results such as:

Theorem 5.7. *Categories of nominal algebras in the sense of* [12] *are given by uniform monads on* Nom.

This is obtained using the fact that the translation of nominal algebra into uniform theories are semantically invariant ([18, Theorem 6.8]), and that uniform theories are given by uniform fb-monads.

6. Conclusions

We have shown how algebra with variable binding can be done inside standard many-sorted universal algebra. Our framework comprises nominal sets as well as the associated presheaf model of variable binding. Of particular importance here are the results of Section 3.3 which show that universal algebra can not detect the difference between the two. It also sheds new light on the different proposals of equational logic for nominal sets [5, 12], as they can be compared now as describing slightly different fragments of the uniform theories described in Section 4.3.

Future work:

- To extend by 'uniform implications' the logics of [12] and [5].
- To transfer more results of universal algebra and to develop applications to the theory of algebraic data types.
- To 'nominalise' other areas of theoretical computer science based on universal algebra.
- In particular, there is ongoing work on nominal regular languages and their automata. Appropriate notions of finite algebras are obtained via the named sets of [6].
- Applications to process algebras with name binders. For example, the logic developed in [4] falls into our framework, as do Stark's algebraic models of the π-calculus [22].
- Our general aims are related to those of Fiore and Hur [7], but instead of developing an abstract framework we focus on particular models and stay inside classical universal algebra. A precise relationship needs to be worked out.
- To extend our framework to other presheaf models of variable binding according to the general theory developed in [23].
- To deal with recursion, presheaf models over cpos have been studied in [8, 21]. Let us note that Section 3 as well as [20] work in the enriched setting, suggesting to replace Set by cpos. This raises the interesting question of what 'enriched equational logic' is.

References

[1] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic Theories: a Categorical Introduction to General Algebra*.

[2] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. CUP, 1994.

[3] Jiří Adámek and Jiří Rosický. On sifted colimits and generalized varieties. *Th. Appl. Categ.*, 8, 2001.

[4] M.M. Bonsangue and A. Kurz. Pi-calculus in logical form. In *LICS'07*.

[5] R. Clouston and A. Pitts. Nominal equational logic. In *Computation, Meaning and Logic, Articles dedicated to Gordon Plotkin*, volume 172 of *Electronic Notes in Theoretical Computer Science*. 2007.

[6] Gianluigi Ferrari, Ugo Montanari, and Marco Pistore. Minimizing transition systems for name passing calculi: A co-algebraic formulation. In *FoSSaCS'02*.

[7] M. Fiore and C.-K. Hur. Equational systems and free constructions. In *ICALP'07*.

[8] M. Fiore, E. Moggi, and D. Sangiorgi. A fully-abstract model for the π-calculus. In *LICS'96*.

[9] M. Fiore, G. Plotkin, and D. Turi. Abstract syntax and variable binding. In *LICS'99*.

[10] M. Gabbay and A. Pitts. A new approach to abstract syntax involving binders. In *LICS'99*.

[11] M.J. Gabbay. Nominal algebra and the HSP theorem. *J. Logic Computation*, 2008. doi:10.1093/logcom/exn055.

[12] Murdoch J. Gabbay and Aad Mathijssen. Nominal (universal) algebra: equational logic with names and binding. *J. Logic Computation*, 2009. In press.

[13] M. Hofmann. Semantical analysis of higher-order abstract syntax. In *LICS'99*.

[14] P. T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bull. London Math. Soc.*, 7, 1975.

[15] G. Kelly and J. Power. Adjunctions whose counits are coequalizers and presentations of enriched monads. *J.Pure Appl. Algebra*, 89, 1993.
[16] A. Kurz and D. Petrişan. Functorial coalgebraic logic: The case of many-sorted varieties. *Information and Computation*. Accepted.
[17] A. Kurz and J. Rosický. Strongly complete logics for coalgebras. July 2006.
[18] Alexander Kurz and Daniela Petrişan. On universal algebra over nominal sets. *Math. Struct. Comp. Sci.* Accepted.
[19] Saunders Mac Lane. *Category Theory for the Working Mathematician*. Springer, 1971.
[20] John Power and Miki Tanaka. Category theoretic semantics for typed binding signatures with recursion. *Fundam. Inform.*, 84, 2008.
[21] I. Stark. A fully-abstract domain model for the π-calculus. In *LICS'96*.
[22] Ian Stark. Free-algebra models for the pi -calculus. *Theor. Comput. Sci.*, 390, 2008.
[23] Miki Tanaka and John Power. Pseudo-distributive laws and axiomatics for variable binding. *Higher-Order Symb. Computat.*, 19, 2006.
[24] Nikos Tzevelekos. Full abstraction for nominal general references. In *LICS'07*.

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