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Abstract

Discounted Expected Utility theory has been a workhorse in economic analysis for over half a century. However, it cannot explain empirical violations of 'dimensional independence' demonstrating that risk interacts with time preference and time interacts with risk preference, nor does it explain present bias or magnitude-dependence in risk and time preferences, or correlations between risk preference, time preference, and cognitive reflection. We demonstrate that these and other anomalies are explained by a dual system model of risk and time preferences that unifies models of a rational economic agent, models based on prospect theory, and dual process models of decision making.

Keywords: Risk; Time; Dimensional Independence

JEL Codes: D01, D03, D81, D90.

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1 Introduction

Many decisions in life involve uncertain outcomes that materialize at different points in time. For example, the struggle to kick an addiction involves a tradeoff between short term gratification and an increased risk to future health. Saving for retirement involves delaying immediate consumption to guard against uncertainty about future income. Whether to pursue a long-term project involves consideration of the time the project is expected to take and the likelihood of project success. The decision to purchase a warranty on an appliance involves a higher immediate cost, but reduced product breakdown risk. The decision to take a 'buy-it-now' option on eBay or wait until the auction ends for the chance of a better deal, the decision to purchase a laptop today or wait for a potential Black Friday sale, and the decision to take out a mortgage on a home or wait for a possibly lower interest rate each involve a tradeoff between a certain, immediate payoff and a risky, delayed payoff.

As these examples illustrate, decisions often involve both risk and delays. Yet the domains of risk and time have traditionally been studied separately. In cases where risk and time preferences are both considered, the discounted expected utility (DEU) model remains a major workhorse for analyzing individual behavior. There are, however, a variety of basic shortcomings of DEU. For instance, it implies that risk has no effect on time preference, that delays have no effect on risk preference, that risk and time preferences are generated by the same utility function, that people are risk-seeking toward lotteries over uncertain payment dates, that people discount the future exponentially, and that discounting does not depend on the magnitude of outcomes. All of these predictions have been contradicted by experimental evidence.

In this paper, we introduce a dual process model of choices under risk and over time that resolves each of these limitations of the DEU model. The model generalizes both rank-dependent utility theory (Quiggin, 1982), and Mukherjee's (2010) dual system model of choice under risk to develop a unified model that accounts for attitudes toward risk, attitudes toward time, and systematic interaction effects between risk and time preferences. The proposed model also provides a unification of three classes of decision models – rank-dependent utility, expected utility, and dual process (or dual selves) theories. We refer to the model developed here as Dual Process Utility (DPU) theory.

The DPU model introduces a new parameter into the analysis of economic decision models which represents the decision maker's 'cognitive type' or 'thinking style'. Essentially, an agent's 'cognitive type' identifies whether a person naturally engages in more intuitive and feeling-based processing or relies on more analytical and calculation-based processing. While cognitive types have been found to correlate with a wide variety of economic behaviors including risk and time preferences (e.g., Frederick, 2005; Burks et al., 2009), saving behavior (Ballinger et al., 2011), strategic sophistication (Carpenter et al., 2013), and efficiency in experimental asset markets (Corgnet et al., 2015), they appear nowhere in the conventional economic models of individual choice.

After providing some background in §2, the DPU model is introduced in §3. In §4 we demonstrate that DPU explains present bias and that DPU resolves a long-standing paradox in decision theory by simultaneously predicting both the magnitude effect for choice over time and peanuts effect for choice under risk. In §5, we show that DPU explains empirically observed interaction effects between risk and time preferences that violate the principle of dimensional independence which is implicitly assumed by the leading rational and behavioral models of decision making. In §6 we demonstrate that DPU predicts (i) a separation between risk and time preferences, (ii) a preference for diversifying payoffs across time, (iii) risk aversion toward timing risk, (iv) observed correlations between risk preference, time preference, and cognitive reflection, and (v) the observed non-monotonic relation between cognitive reflection and present bias. The behaviors implied by the model are summarized in §7. Related literature is discussed in §8. Concluding remarks are provided in §8. Proofs are provided in the appendix. To preview our results, we note that this paper makes the following contributions:

1. General formulation of System 1 and System 2 processes in decision making: We provide a formal link between the System 1 / System 2 paradigm often discussed qualitatively and predictions regarding economic behavior by postulating that System 1 has standard 'behavioral' preferences whereas System 2 has standard 'rational' preferences, and that these two systems interact when making decisions (modeled by a convex combination of rational and behavioral preferences).

2. Explaining empirical violations of dimensional independence: A byproduct of our formulation is that the DPU model provides a simple and general approach to predicting and explaining empirical violations of the dimensional independence axiom that cannot be explained by standard rational preferences or behavioral preferences. Surprisingly, under our approach the violations of dimensional independence are predicted in the direction observed in experiments by combining the standard rational and behavioral models. Moreover, each violation of dimensional independence explained by DPU, can only be explained due to the interaction between System 1 and System 2 (convex combination parameter strictly between 0 and 1), and thus cannot be explained by any multiplicatively separable model that multiplies a discount function by a probability weighting function by a utility or value function.

3. Explaining correlations between preferences and 'cognitive types': The convex combination parameter that places a weight on the System 2 value function can be viewed as indexing the decision maker's thinking style or 'cognitive type' – the extent to which the decision maker naturally relies on reflective System 2 processes or intuitive System 1 processes in decision making. The DPU model then implies that the agent's cognitive type is correlated with risk and time preferences. In particular, agents with higher 'cognitive types' are predicted to be more patient and to be closer to risk-neutrality, These predictions are supported by recent experimental evidence. In doing so, the DPU model incorporates a factor (thinking style or cognitive type) that appears nowhere in conventional economic models of decision making but which accounts for observed heterogeneity in economic behavior.

4. Resolving the peanuts effect/magnitude effect paradox: Since the work of Prelec and Loewenstein (1991), it has been generally assumed that no model of risk and time preferences can explain the peanuts effect in choices under risk (switching from risk seeking at low stakes to risk aversion at high stakes) and the magnitude effect for choice over time (increasing patience at larger stakes). In fact, the most commonly used specification of prospect theory with a power value function does not explain the peanuts effect for any probability weighting function and the model of quasi-hyperbolic discounting does not explain the magnitude effect. The challenge becomes even greater when developing a model of both risk and time preferences since the peanuts effect appears to reveal decreasing sensitivity to payoffs at larger stakes while the magnitude effect appears to reveal increasing sensitivity to payoffs at larger stakes. We show that this paradox is resolved under our approach and that the DPU model predicts both the peanuts effect and the magnitude effect.

5. Explaining general violations of discounted expected utility theory: We demonstrate that DPU also allows for a separation of risk preferences and inter-temporal substitution, and that it predicts present bias, aversion to timing risk, and a preference for diversifying risks across time.

2 Background

The study of risk preferences and time preferences, both analytically and empirically, has been the primary focus of research on individual choice for over half a century. However, although expected utility theory was axiomatized by von Neumann and Morgenstern in 1947, and discounted utility theory was axiomatized by Koopmans in 1960, it was not until 1991 when researchers identified remarkable parallels between anomalous behaviors across both domains – such as a common ratio effect in choice under risk and a common difference effect in choice over time (Prelec and Loewenstein, 1991). However, even in pointing out parallel behaviors between risk and time, Prelec and Loewenstein also presented a kind of impossibility result, indicating that no model could resolve both the peanuts effect in choice under risk (Markowitz, 1952) and the magnitude effect in choice over time (Prelec and Loewenstein, 1991). Prototypical examples of both of these effects are illustrated in Table I. For the peanuts effect, preferences switch from risk-seeking at small stakes (e.g., preferring a 1% chance of winning \$100 to \$1 for certain) to risk-averse at larger stakes (preferring \$100 for certain over a 1% chance of winning \$10,000). For the magnitude effect, behavior switches from impatient at small stakes (e.g., preferring \$7 now to \$10 in one year) to more patient at larger stakes (preferring \$1,000 in one year over \$700 now). Both effects involve scaling outcomes up by a common factor. The peanuts effect is not explained by the most widely used specification of cumulative prospect theory due to Tversky and Kahneman (1992) with a power value function, even when allowing for any probability weighting function. A more fundamental challenge when relating risk and time preferences is that the peanuts effect seems to reveal decreasing sensitivity to payoffs at larger stakes, while the magnitude effect seems to reveal increasing sensitivity to payoffs at large stakes. Thus, any conventional approach to explaining the peanuts effect should predict the opposite of the magnitude effect (and vice versa). Prelec and Loewenstein could not explain both effects, and this challenge has remained unresolved over the subsequent twenty-five years, posing an apparent impossibility result that no common approach to modeling risk and time preferences can capture both of these basic behaviors. Somewhat surprisingly, we will demonstrate that the model presented here simultaneously predicts both effects. Since the 'common approach' to risk and time preferences pioneered by Prelec and Loewenstein (1991), other models have emerged to explain behaviors across both domains. For instance, models of similarity judgments apply the same cognitive process to explain anomalies for risk and time (Rubinstein 1988; Leland 1994; Leland 2002; Rubinstein, 2003). However, this approach does not address another basic question of how risk and time preferences interact.

2.1 Violations of Dimensional Independence

It has been only fairly recently that attention has shifted to explaining interactions between risk and time preferences. This research direction was partially spurred by experimental studies from Keren and Roelofsma (1995) and Baucells and Heukamp (2010) who each observed different and systematic interactions between risk and time preferences. For instance, Keren and Roelofsma

(1995) observed that uncertainty induces more patient behavior. Baucells and Heukamp (2010) and Abdellaoui et al. (2011) both observed that time delays induce more risk-taking behavior. Andersen et al. (2011) and Miao and Zhong (2015) observed a preference for diversifying risks across time. Onay and Onculer (2007) and DeJarnette et al. (2015) observed risk aversion to lotteries over uncertain payment dates. These behaviors are illustrated in Table I.

A standard approach to modeling risk and time preferences is to multiply a time discount function by a probability weighting function by a utility or value function. However, this approach does not explain the finding that time delay reduces risk aversion (see Table I) since both alternatives used by Baucells and Heukamp (2010) are delayed by the same amount (three months) and so the discount weights cancel when comparing options A and B. This approach also does not explain the finding that uncertainty reduces impatience, since both payoffs used by Keren and Roelofsma (1995) have the same probability (50%), and so the probability weights cancel when comparing options A and B. In addition, this approach does not explain the finding of subendurance in the example by Baucells et al. (2009), since both options have the same payoffs (€100) and so the utilities cancel when comparing options A and B. It is then not obvious how to model such interaction effects between time delays, probabilities, and payoffs. It may be even less clear how to derive behaviors in the direction observed in experiments, or whether the same approach that explains interaction effects for time delays can also explain interaction effects for probabilities and payoffs. We will show that a unified approach to these interaction effects is possible and has a simple and intuitive interpretation.

The risk-time-money interaction effects in Table I each provide a test of the same general principle. This principle, called dimensional independence (Keeney and Raiffa, 1993; Bhatia, 2016) states that two attribute dimensions x and y are independent if for all x, y, x', y', an alternative (x, y) is chosen over $(x'$, y) if and only if (x, y') is chosen over (x', y'). This principle reflects the intuition that identical attribute values in the same dimension will cancel in the evaluation process and not affect decisions. This principle is a general feature of the leading normative and behavioral decision models.

Table I. Choices between Options involving Risk and Time

Observation	Option A vs.	Option B
Peanuts Effect**** (Markowitz, 1952)	**(100, with 1%, now)** (10,000, with 1%, now)	(1, for sure, now) **(100, for sure, now)**
Magnitude Effect**** (Prelec & Loewenstein, 1991)	**(7, for sure, now)** (700, for sure, now)	(10, for sure, 1 year) **(1,000, for sure, 1 year)**
Common Ratio Effect* (Baucells & Heukamp, 2010)	**(9, for sure, now)** (9, with 10%, now)	(12, with 80%, now) **(12, with 8%, now)**
Common Difference Effect** (Keren & Roelofsma, 1995)	**(100, for sure, now)** (100, for sure, 26 weeks)	(110, for sure, 4 weeks) **(110, for sure, 30 weeks)**
Delay reduces risk aversion* (Baucells and Heukamp, 2010)	**(9, for sure, now)** (9, for sure, 3 months)	(12, with 80%, now) **(12, with 80%, 3 months)**
Risk reduces impatience** (Keren and Roelofsma, 1995)	**(100, for sure, now)** (100, with 50%, now)	(110, for sure, 4 weeks) **(110, with 50%, 4 weeks)**
Subendurance* (Baucells et al., 2009)	**(100, for sure, 1 month)** (5, for sure, 1 month)	(100, with 90%, now) **(5, with 90%, now)**
Correlation Aversion**** (Miao and Zhong, 2015)	**If Heads: 100, now** **If Tails: 100, 1 week**	If Heads: 100 now, 100, 1 week If Tails: 0 now, 0, 1 week
Aversion to Timing Risk**** (Onay and Onculer, 2007)	If Heads: 100, 5 weeks If Tails: 100, 15 weeks	**If Heads: 100, 10 weeks** **If Tails: 100, 10 weeks**

Adapted from Baucells and Heukamp (2012). Complementary probabilities for all options correspond to payoffs of 0. Sources of experimental results are in parentheses. Majority responses are in bold font.
*Currency in Euros; ** Currency in Dutch Guilders; *** Prototypical examples.

2.2 Dual Processes in Decision Making

Recent literature in cognitive science argues that people do not have a single mental processing system, but rather have two families of cognitive processes. Stanovich and West (2000), and Kahneman and Frederick (2002) label these families neutrally as System 1 and System 2 processes where System 1 includes automatic, intuitive and affective processes and System 2 includes more deliberative, logical, and reflective processes. Kahneman (2011) simply distinguishes between processes that are 'fast' and 'slow.' Rubinstein (2007, 2013) distinguishes between "instinctive" and "cognitive" processes. Hsee and Rottenstreich (2004) posit two qualitatively different types of valuation processes – valuation by feeling and valuation by calculation. We adopt the standard System 1/System 2 distinction in our analysis. The relation between our approach and alternative dual system or dual selves models in economics is discussed in §8. Despite their recent rise to theoretical prominence, two-system (dual process) theories date back to the early days of scholarly thought. The conflict between reason and passion, for instance, features prominently in Plato's *Republic* and in Adam Smith's *Theory of Moral Sentiments*.

3 Dual Process Utility Theory

We let there be a finite set, T, of time periods, a set, $\mathscr{M}$, of outcomes with $\mathscr{M} \subseteq \mathbb{R}$, and a finite set X, of consumption sequences. We index consumption sequences by $j \in \{1, 2, ..., n\}$ and we index time periods by $t \in \{0, 1, ..., m\}$. A consumption sequence $x_j := [x_{j0}, ..., x_{jm}]$ is a sequence of dated outcomes. A stochastic consumption plan is a probability distribution over consumption sequences. We denote a stochastic consumption plan by a function $f : \mathrm{X} \to [0, 1]$, with $f(x_j)$ denoting the probability assigned to consumption sequence x_j, where $\sum_{x \in \mathrm{X}} f(x) = 1$. Denote the set of stochastic consumption plans by $\mathscr{F}$.

Our main assumption is that System 1 has behavioral preferences (which we formalize as discounted rank-dependent utility preferences), while System 2 has standard rational preferences (which we formalize as discounted expected utility preferences). We will demonstrate that a by-product of our approach is that it presents a simple way to predict and explain the empirical violations of the dimensional independence axiom in Table I, while resolving other systematic limitations of the DEU model.

The assumption of discounted expected utility preferences seems particularly appropriate for System 2 which may be intuitively thought to resemble the rational economic agent. However, in addition to differences in the content of risk and time preferences between systems (i.e., that the systems differ in their degrees of risk aversion and impatience), one might further propose that the two systems differ in the structure of their risk and time preferences, with System 2 having normative DEU preferences, and with System 1 having behavioral preferences based on prospect theory (PT) or rank dependent utility (RDU) theory. Supporting this, Rottenstreich and Hsee (2001) find that inverse S-shaped probability weighting (as assumed in RDU theory (Quiggin 1982) and PT (Tversky and Kahneman 1992)) is more pronounced for affect-rich outcomes. Support for assuming System 1 has PT preferences also comes from Barberis et al. (2013) who use PT to model "System 1 thinking" for initial reactions to changes in stock prices. Reflecting on PT three decades later, Kahneman (2011, pp. 281-282) remarks, "It's clear now that there are three cognitive features at the heart of prospect theory... They would be seen as operating characteristics of System 1."

To formalize the risk and time preferences of System 1, let $\pi : [0,1] \rightarrow [0,1]$, be a standard rank-dependent probability weighting function with $\pi(0) = 0$ and $\pi(1) = 1$, that is defined as:

$$\pi(f(x_{jt})) = w(f(x_{jt}) + \cdots + f(x_{1t})) - w(f(x_{j-1,t}) + \cdots + f(x_{1t})),$$

for $j \in \{1, 2, ..., n\}$, where consumption sequences are ranked according to the discounted utility for System 1 for each sequence such that $\sum_t \delta_1^t u_1(x_{nt}) \leq$ ...$\leq \sum_t \delta_1^t u_1(x_{1t})$. We assume System 1 has discounted RDU preferences. That is, System 1 evaluates stochastic consumption plans according to (1):

$$\mathrm{V}_1(f) = \sum_{\mathrm{t}} \sum_{\mathrm{j}} \delta_1^t \pi(f(x_{jt})) u_1(x_{jt}) \tag{1}$$

We assume System 2 has discounted expected utility preferences in (2):

$$\mathrm{V}_2(f) = \sum_{\mathrm{t}} \sum_{\mathrm{j}} \delta_2^t f(x_{jt}) u_2(x_{jt}) \tag{2}$$

Definition 1 (Dual Process Utility Theory): In Dual Process Utility (DPU) theory, there exists utility functions u_1, u_2, discount factors, $\delta_1 < \delta_2$, probability weighting function, π, and a unique parameter, $\theta \in [0, 1]$, such that for any $f, g \in \mathscr{F}$, $f \succsim g$ if and only if $\mathrm{V}(f) \geq \mathrm{V}(g)$, where $\mathrm{V}(f)$ is given by (3), V_1 is given by (1), and V_2 is given by (2):

$$\mathrm{V}(f) = (1 - \theta)\mathrm{V}_1(f) + \theta \mathrm{V}_2(f) \tag{3}$$

In (3), the parameter θ may be interpreted as the degree to which an agent is hard-wired to rely on System 2. We will refer to θ as the decision maker's *cognitive type*, with one's cognitive type becoming less based on feeling and intuition and more reliant on logic and calculation as θ increases. From a neuro-economic perspective, there are tight neural connections between the prefrontal cortex, a brain region implicated in planning, analytical thinking, and executive function and the limbic system, an evolutionarily older brain region involved in the generation of emotions and the experience of pleasure. One might view θ as indexing the strength of neural connections in the prefrontal areas relative to the strength of neural connections in the limbic areas.

Note that DPU adopts the view of cognitive types implicit in the interpretation of the cognitive reflection test (Frederick, 2005) and models of level-k thinking (Camerer et al., 2004), namely that there are reflective thinkers or those with high levels of reliance on System 2, and there are intuitive thinkers or those with high levels of reliance on System 1. It is in this sense in which we view θ as reflecting a decision maker's 'cognitive type' which allows for heterogeneity across agents. Within agents, DPU reflects a compromise between the System 1 preference for immediate gratification and the more patient preferences of System 2. This 'compromise' is consistent with the findings of Andersen et al. (2008) who "...observe what appears to be the outcome of a decision process where temptation and long-run considerations are simultaneously involved."

To apply the model, we make simplifying assumptions regarding the preferences of System 2. Mukherjee (2010) and Loewenstein et al. (2015) both argue that risk-neutrality is a plausible, and even natural, assumption for System 2. To the extent that System 2 characterizes an idealized rational agent, it appears at least plausible that it does not have a pure rate of time preference which some authors have argued to be irrational (e.g., Harrod, 1948; Traeger, 2013), and that it maximizes expected value. In Sections 4, 5, and 6, we illustrate DPU under the risk-neutrality and delay-neutrality assumptions for System 2. Formally, this means that we set $u_2(x_{jt}) = x_{jt}$, and $\delta_2 = 1$. This simplification also serves to restrict the number of free parameters. Let $\mathbb{E}[f]$ denote the undiscounted expected value of a stochastic consumption plan f. To simplify notation, when applying DPU in Sections 4, 5, and 6, we also now drop the subscripts on the System 1 utility function and discount factor. The DPU functional form in (3) now simplifies to (4):

$$\mathrm{V}(f) = \theta\,\mathbb{E}[f] + (1-\theta)\sum_{\mathrm{t}}\sum_{\mathrm{j}}\delta^{\mathrm{t}}\pi(f(x_{jt}))u(x_{jt}) \tag{4}$$

The DPU model in (4) is represented by a discount factor, probability weighting function, and utility function for System 1, as well as the decision maker's cognitive type, θ. Model (4) is thus a two-parameter generalization of the discounted expected utility model.

3.1 A Simple Dual Process Utility Theorem

Recent impossibility results (Mongin and Pivato, 2015; Zuber, 2016) have demonstrated difficulties in using axiomatic methods to aggregate non-expected utility preferences. For instance, Zuber (2016) considers a general class of non-expected utility preferences and concludes, "non-expected utility preferences cannot be aggregated consistently." As such, it is not clear whether the general form of the DPU model in (3) can be characterized from more primitive assumptions. As we show in this section, however, we can obtain a simple and natural characterization of the special case of (3) in which both systems have consistent DEU preferences and differ only in their degrees of risk aversion and impatience. This special case of the model delivers all of our results except for those that emerge due to non-linear probability weighting. Under this approach, the assumption of discounted expected utility preferences for each system can be viewed as postulating that each system has consistent risk and time preferences, and that inconsistencies that we observe are emergent phenomena that arise through the interaction between systems.

Formally, we take a semi-axiomatic approach. Let $\succsim_s$ and $\succ_s$ denote weak and strict preference, respectively, between pairs of stochastic consumption plans for system $s, s \in \{1,2\}$ that satisfy the non-triviality conditions $f \succ_1 g$ and $f' \succ_2 g'$ for some $f, g, f', g' \in \mathscr{F}$.

Assumption 1 (Preferences of Systems 1 and 2): Each system $s, s \in \{1,2\}$has consistent risk and time preferences represented by discounted expected utility theory. That is, there exists utility functions$u_s, s, s \in \{1,2\}$ and discount factors $\delta_s, s, s \in \{1,2\}$ such that for each $s, s \in \{1,2\}$ and for all $f, g \in \mathscr{F}$, $f \succsim_{\mathrm{s}} g \Longleftrightarrow \mathrm{V_s}(f) \geq \mathrm{V_s}(g)$, where:

$$\mathrm{V_s}(f) = \sum_{\mathrm{t}} \sum_{\mathrm{j}} \delta_s^t f(x_{jt}) u_{\mathrm{s}}(x_{jt}) \tag{5}$$

Let $\succsim$ and $\succ$ represent, respectively, weak and strict preferences of the decision maker over stochastic consumption plans. We minimally constrain the agent's time preferences, allowing for a very general time preference functional. In particular, we do not impose stationarity, nor do we impose that time preferences are multiplicatively separable into a discount function and a utility function. Formally, our Assumptions 2 and 3 can be viewed as special cases of Assumptions 2 and 3 in Harsanyi's (1955) theorem, as presented in Keeney and Nau (2011) in the domain of choice under uncertainty.

Assumption 2 (Preferences of the Decision Maker): There exists utility function u_t, such that for all $f, g \in \mathscr{F}$, $f \succsim g \Longleftrightarrow \mathrm{V}(f) \geq \mathrm{V}(g)$, where:

$$\mathrm{V}(f) = \sum_{\mathrm{t}} \sum_{\mathrm{j}} f(x_{jt}) u_{\mathrm{t}}(x_{jt}) \tag{6}$$

Our final assumption is a consistency condition which relates the preferences of each system to the preferences of the decision maker:

Assumption 3 (Pareto Efficiency): For all $f, g \in \mathscr{F}$, if both systems weakly prefer f to g, then $f \succsim g$, and if, in addition, one system strictly prefers f to g then $f \succ g$.

Proposition 1 (Dual Process Utility Theorem): *Given Assumptions 1, 2, and 3, there exists a unique constant*[1] $\theta \in (0,1)$, *utility functions* u_1, u_2, *and discount factors,* δ_1, δ_2, *such that for all* $f, g \in \mathscr{F}$, $f \succsim g$ *if and only if* $\mathrm{V}(f) \geq \mathrm{V}(g)$, *where* $\mathrm{V}(f)$ *is given by (7) and* $\mathrm{V_s}(f)$, $s \in \{1, 2\}$ *is given by* (5):

$$\mathrm{V}(f) = (1-\theta)\mathrm{V}_1(f) + \theta \mathrm{V}_2(f) \tag{7}$$

Proposition 1 derives the existence and uniqueness of the θ parameter and provides a formal justification for the convex combination approach. Proposition 1 provides a more general preference aggregation result than Harsanyi (1955), in that Proposition 1 simultaneously aggregates risk and time preferences. Although including both risk and time, and applying Harsanyi's theorem from social choice theory to model individual choice behavior are new, the proof for Proposition 1 follows straightforwardly from the proof of Theorem 1 in Keeney and Nau (2011). A related finding for decisions involving time but not risk was obtained in the context of group decision making by Jackson and Yariv (2015).

3.2 Basic Properties of DPU

Consider two stochastic consumption plans f and g, where $f(x_j)$ and $g(x_j)$ are the probabilities which f and g assign to consumption sequence x_j, respectively. Since a decision maker either receives one consumption sequence or another and so cannot interchange components of any arbitrary sequences, we first seek a means of objectively ranking different consumption sequences, analogous to how one would rank individual outcomes. We can then extend the standard definition of stochastic dominance from lotteries over outcomes to

[1] In Harsanyi's theorem, the weights on individual utilities are positive and unique up to a common scale factor. Without loss of generality, the weights can be scaled to sum to 1 in which case $\theta \in (0,1)$ is uniquely determined.

lotteries over consumption sequences. In particular, we say sequence x_j dominates sequence x_k if $x_{jt} \geq x_{kt}$ for all $t \in \{0,1,\ldots,T\}$, with a strict inequality for at least one t. We say that consumption sequences $x_1, ..., x_n$ are monotonically ordered if x_j dominates x_{j+1} for all $j \in \{1,\ldots,n-1\}$. For any monotonically ordered consumption sequences $x_1, ..., x_n$, we say f (first-order) stochastically dominates g if $\mathrm{F}(x_j) \leq \mathrm{G}(x_j)$ for all $j \in \{1,\ldots,n\}$, where F and G are the cumulative distribution functions for f and g, respectively. Note that this reduces to the standard definition of stochastic dominance in an atemporal setting.

Proposition 2: *Let $\succsim$ have a DPU representation as in* (3). *Then for any fixed $\theta \in [0,1]$, $\succsim$ is transitive and complete and satisfies first order stochastic dominance.*

The proof of transitivity and completeness in Proposition 3 are standard so we prove only stochastic dominance. Recall that sequenes are ranked such that $\sum_t \delta_1^t u_1(x_{nt}) \leq ... \leq \sum_t \delta_1^t u_1(x_{1t})$ prior to applying the $\pi(.)$ transformation. Note that if consumption sequences $x_1, ..., x_n$ are monotonically ordered, then $u_1(x_{\mathrm{1t}}) \geq ... \geq u_1(x_{\mathrm{nt}})$ for all $t \in \{0,1,\ldots,T\}$, and for any increasing function u_1. Thus, $\pi(.)$ preserves the monotonic ordering of the sequences. If f stochastically dominates g, then $\delta_1^t \sum_j \pi(f(x_j))u_1(x_{jt}) > \delta_1^{\mathrm{t}} \sum_{\mathrm{j}} \pi(g(x_j))u_1(x_{jt})$, for each period $t \in \{0,1,\ldots,T\}$, which implies that $\mathrm{V}_1(f) > \mathrm{V}_1(g)$. Since $\mathrm{V}_2(f) > \mathrm{V}_2(g)$, the convex combination of V_1 and V_2 ranks f higher than g for all $\theta \in [0,1]$.

4 Expected and Discounted Utility Violations

In this and the following sections, all propositions assume the decision maker has dual process utility preferences (given by (4)). Proofs of Propositions in §4 and §5 are given in the appendix. Each of these results (Propositions 3, 4, 5, 6, 7, and 8) do not hold when $\theta = 0$ or $\theta = 1$, indicating the need for a dual process paradigm in our setup. First, we show that DPU resolves two empirical violations of discounted utility theory - present bias and the magnitude effect.

4.1 Present Bias

Systematic empirical violations of the stationarity axiom of discounted utility theory (Koopmans, 1960), such as present bias, have been well-documented in experiments (Frederick et al., 2002), and are thought to reveal time-inconsistent preferences (Laibson 1997; O'Donoghue and Rabin 1999). Formal accounts of present bias and hyperbolic discounting have often directly assumed such behavior in the functional form of the agent's preferences (e.g., Laibson 1997). Present bias emerges as a general property of DPU without any explicit assumptions regarding hyperbolic discounting or diminishing sensitivity to delays. In fact, present bias is predicted by DPU even though each system has time consistent preferences. A similar result in the context of group decision making was shown by Jackson and Yariv (2015).

Let (c, p, t) denote a stochastic consumption plan which has one non-zero outcome, c, to be received with probability p at time t. We have the following definition:

Definition 2 (Present Bias): *Present bias* holds if for all $y \in (0, c)$, and $t, \Delta > 0$, $(y, p, 0) \sim (c, p, \Delta)$ implies $(y, p, t) \prec (c, p, t + \Delta)$.

Proposition 3: *Under DPU, present bias holds if and only if* $\theta \in (0, 1)$.

Proposition 3 implies that DPU explains the example of the common difference effect illustrated in Table I demonstrated by Keren and Roelofsma (1995). In particular, a decision maker indifferent between 100 Dutch guilders for sure now and 110 Dutch guilders for sure in 4 weeks will strictly prefer 110 Dutch guildrs for sure in 30 weeks over 100 Dutch guilders for sure in 26 weeks.

It is also clear from the proof of Proposition 3 that present bias does not hold if $\theta = 0$ or $\theta = 1$. Thus, under DPU, present bias arises due to the interaction between System 1 and System 2. This implication of DPU generates a strong empirical prediction: While time consistent discount rates will be positively correlated with cognitive types, present bias will not be correlated with cognitive types since it is observed for intermediate values of θ. Although this prediction may seem counter-intuitive, it has experimental support. Bradford et al. (2014) tested for a relationship between performance on the cognitive reflection test of Frederick (2005) and the time consistent and present bias parameters in Laibson's (1997) model of present bias. Bradford et al. report, "... we find that more patient individuals are more likely to answer each question correctly. However, the present bias discount factor β is uncorrelated with the CRT questions" (p. 19). Indeed, they find that the time consistent component parameter of the quasi-hyperbolic model, δ, is positively correlated with cognitive types at a p-value less than 0.01, although the present bias parameter β is not related to cognitive types. Both of these findings confirm the predictions of DPU.

4.2 The Magnitude Effect

The DPU model also offers an explanation of the magnitude effect in intertemporal choice. The magnitude effect is the robust observation that behavior is more patient for larger rewards than for smaller rewards (Prelec and Loewenstein, 1991). Formally:

Definition 3 (Magnitude Effect): The *magnitude effect* holds if for all $y \in (0, c)$, $s > t$, and $r > 1$, $(y, p, t) \sim (c, p, s)$ implies $(ry, p, t) \prec (rc, p, s)$.

Proposition 4: *For any concave power utility function u, the magnitude effect holds under DPU, if and only if $\theta \in (0, 1)$.*

4.3 The Peanuts Effect

While PT and RDU explain violations of expected utility theory (EU) such as the Allais paradoxes, standard specifications of PT or RDU do not explain the 'peanuts' effect. An example of this behavior is a willingness to pay \$1 for a one-in-ten million chance of \$1 million, but prefer a sure \$1000 over a one-in-ten million chance of \$1 billion. Under a power value function for PT, indifference in the former choice implies indifference in the latter for any probability weighting function and the peanuts effect does not hold. The problem is more challenging when incorporating both risk and time because, since Prelec and Loewenstein (1991), it has not been clear how the magnitude effect and the peanuts effect coexist. Yet DPU simultaneously predicts both effects. The peanuts effect holds since risk-seeking at small stakes is due to overweighting low probabilities (the domain where the peanuts effect is observed) while scaling payoffs up shifts more weight on the System 2 value function (if u_1 is more concave than u_2) which shifts preferences toward risk neutrality (if $u_2 = x$) or toward risk aversion (if u_2 is concave).

Definition 4 (Peanuts Effect): Let $\mathbb{E}[(y,p,t)] > \mathbb{E}[(c,q,t)]$. The *peanuts effect* holds if for all $y \in (0,c)$, $p > q$, and $r > 1$, $(c,q,t) \sim (y,p,t)$ implies $(c,q,t) \prec (ry,p,t)$.

Proposition 5: *For any concave power utility function* u, *the peanuts effect holds under DPU, if and only if* $\theta \in (0,1)$.

If System 2 is even slightly risk-averse, the peanuts effect also holds when $\mathbb{E}[(y,p,t)] = \mathbb{E}[(c,q,t)]$. In addition to resolving the peanuts and magnitude effects, DPU explains the finding in Fehr-Duda et al. (2010) that probability weighting is more pronounced at low stakes than at high stakes. This observation holds naturally under DPU given the assumption that the System 2 value function is closer to risk-neutrality than the System 1 value function. However, this stake-size effect violates prospect theory which assumes probability weights and outcomes are independent.

5 Risk and Time Preference Interactions

In this section, we apply DPU to explain the systematic interaction effects between risk and time preferences in Table I. We show that the DPU predictions are systematic: DPU predicts the interaction effects in the direction observed in experiments and rules out interaction effects in the opposite direction.

5.1 Delay Reduces Risk Aversion

As displayed in Table I, Baucells and Heukamp (2010) found most respondents preferred a guaranteed 9 Euros immediately over an 80% chance of 12 Euros immediately, but chose the chance of receiving 12 Euros immediately when the probabilities of winning were scaled down by a factor of 10. This behavior is an instance of the common ratio effect (Allais, 1953). Baucells and Heukamp further observed that when the receipt of payment is delayed 3 months, most respondents preferred an 80% chance of 12 Euros over a guaranteed 9 Euros. This finding that people are less risk-averse toward delayed lotteries was also observed by Abdellaoui et al. (2011). The common ratio effect example from Baucells and Heukamp (2010) holds under DPU if the probability weighting function is sub-proportional. Here we confirm that DPU explains the finding that delay reduces risk aversion which holds even if System 1's value function is linear in probabilities. Let $\mathbb{E}[f]$ denote the (undiscounted) expected value of stochastic consumption plan f. We consider the case where the riskier lottery has the higher expectation as in Baucells and Heukamp (2010).

Definition 5: Let $\mathbb{E}[(c, \alpha p, t)] > \mathbb{E}\left[(y, p, t)\right]$. *Delay reduces risk aversion* if for all $y \in (0, c)$, $\alpha \in (0, 1)$, and $s > t$, $(y, p, t) \sim (c, \alpha p, t)$ implies $(y, p, s) \prec (c, \alpha p, s)$.

Proposition 6: *Under DPU, delay reduces risk aversion if and only if* $\theta \in (0, 1)$.

5.2 Uncertainty Reduces Impatience

As displayed in Table I, Keren and Roelofsma (1995) found most respondents preferred a guaranteed 100 Dutch guilders immediately over a guaranteed 110 Dutch guilders in 4 weeks, but chose the guaranteed 110 when the receipt of both payments was delayed an additional 26 weeks. This behavior is an example of present bias. Keren and Roelofsma further observed that when the chance of receiving each payment was reduced, most respondents preferred a 50% chance of 110 Dutch guilders in 4 weeks over a 50% chance of 100 now. That is, making both options risky leads to more patient behavior, analogous to the effect of adding a constant delay to both options.

Definition 6: *Uncertainty reduces impatience* if for all $y \in (0, c)$, $t, \Delta > 0$, and $q < p$, $(y, p, t) \sim (c, p, t + \Delta)$ implies $(y, q, t) \prec (c, q, t + \Delta)$.

Proposition 7: *Under DPU, for any convex weighting function*, w, *uncertainty reduces impatience if and only if* $\theta \in (0, 1)$.

In the example by Keren and Roelofsma in Table I, convexity of the weighting function in Proposition 8 implies $w(0.5) < 0.5$, which implies $\pi(0.5) < 0.5$. This condition is a general feature of observed probability weighting functions (Starmer, 2000; Wakker, 2010) and represents a form of pessimism. This condition holds for all convex probability weighting functions as well as for the familiar inverse-S-shaped weighting functions (such as those parameterized by Tversky and Kahneman (1992), Wu and Gonzalez (1996), Prelec (1998), and Gonzalez and Wu (1999)), Abdellaoui (2000), and Bleichrodt and Pinto (2000)). This condition is also a general property resulting from Prelec's (1998) axiomatic characterization of his one-parameter probability weighting function. This condition ($\pi(0.5) < 0.5$) will reappear in our analysis and is the only substantive property of the weighting function that is necessary for DPU to explain the experimental observations studied here. The more general convexity condition is only necessary for the generalization of the behavior observed by Keren and Roelofsma to all $q < p$ as formalized in Definition 6.

5.3 Time Dominates at Low Stakes, Risk at High Stakes

Baucells et al. (2009) found that 81% of respondents preferred €100 for sure in one month to €100 with 90% probability immediately, but 57% preferred €5, with 90% probability immediately over €5 for sure, in one month. Baucells and Heukamp (2012) refer to this behavior as *subendurance* and they define it more generally as follows:

Definition 7: *Subendurance* holds if for all $y \in (0, c)$, $t, \Delta > 0$, and $\lambda \in (0, 1)$, $(c, p, t + \Delta) \sim (c, \lambda p, t)$ implies $(y, p, t + \Delta) \prec (y, \lambda p, t)$.

Proposition 8: *For any concave utility function u such that $u(0) = 0$, subendurance holds under DPU if and only if $\theta \in (0, 1)$.*

Subendurance reflects behavior in which time dominates at low stakes and risk dominates at high stakes. In particular, at low stakes, people are more sensitive to the time dimension and choose the sooner reward, whereas at larger stakes, with the same objective tradeoff between time and risk (e.g., receiving a sure payoff in one month or an immediate payoff with probability 0.90), people are more sensitive to the risk dimension and choose the higher probability reward. Subendurance thus reveals one way in which people directly trade off time versus risk.

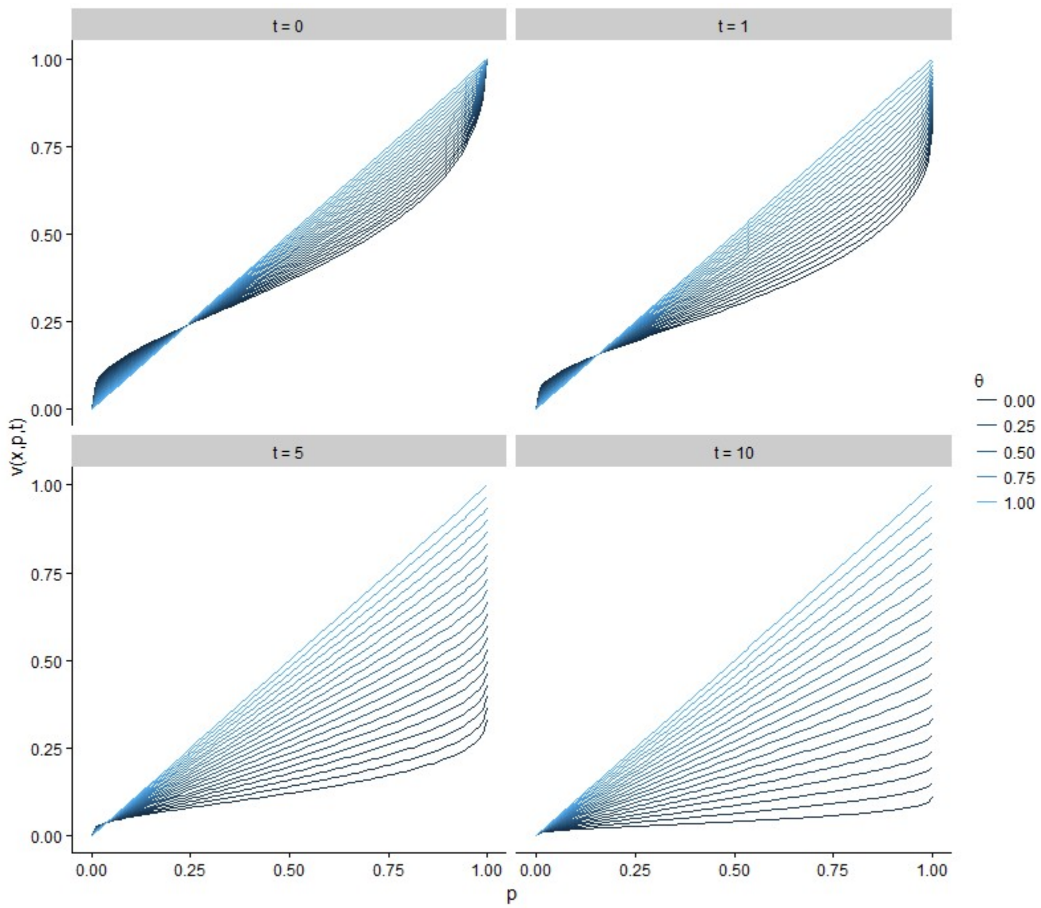
The interaction effects in this section challenge a larger class of time preferences than DEU. Indeed, they cannot be explained by any model of discounting in which evaluation of payoffs, probabilities, or delays is multiplicatively separable. As Baucells and Heukamp (2012) note, when evaluating a stochastic consumption plan (x, p, t), "One may be tempted to propose $\mathrm{V}(x, p, t) = \mathrm{w}(p)f(t)v(x)$. Unfortunately, this form is not appropriate because ... probability and time cannot be separated. One may then propose the more general form $\mathrm{V}(x, p, t) = g(p, t)v(x)$, but this fails to accommodate subendurance." Moreover, Ericson and Noor (2015) reject the assumption that discounting and utility functions are separable for nearly 70% of their participants. Given the necessity of a seemingly complex form for evaluating (x, p, t) to explain the observations in Table I, the DPU functional form in (4) is surprisingly simple.

5.4 Variations in Risk and Time

Figure I graphs (4) for different values of θ (within panels) and for different delays (across panels), as probabilities increase from 0 to 1. The figure employs Prelec's (1998) probability weighting function and evaluates a stochastic consumption plan paying $x > 0$ with probability p at time t and 0 otherwise, under the simplified case where $u_1(x) = u_2(x) := 1$ and $u_1(0) = u_2(0) := 0$. This specification may be viewed as a time-dependent probability weighting function that becomes flatter as the time horizon increases. In general, DPU does not have a separable probability weighting function that is independent of outcomes or time, but we can see how time affects the shape of the weighting function in the special case when $u_1(.) = u_2(.)$.

Figure I suggests people are less sensitive to variations in probability for longer time horizons. Also, relative to an event's probability p, the function over-weights low probability events with short horizons, such as weekly state lottery drawings (if $\delta^t w(p) > p$), but under-weights low probability events with long horizons, such as natural disasters and health risks from addictions (if $\delta^t w(p) < p$). Epper and Fehr-Duda (2016) also argue that accounting for time permits the coexistence of overweighting and underweighting tail events.

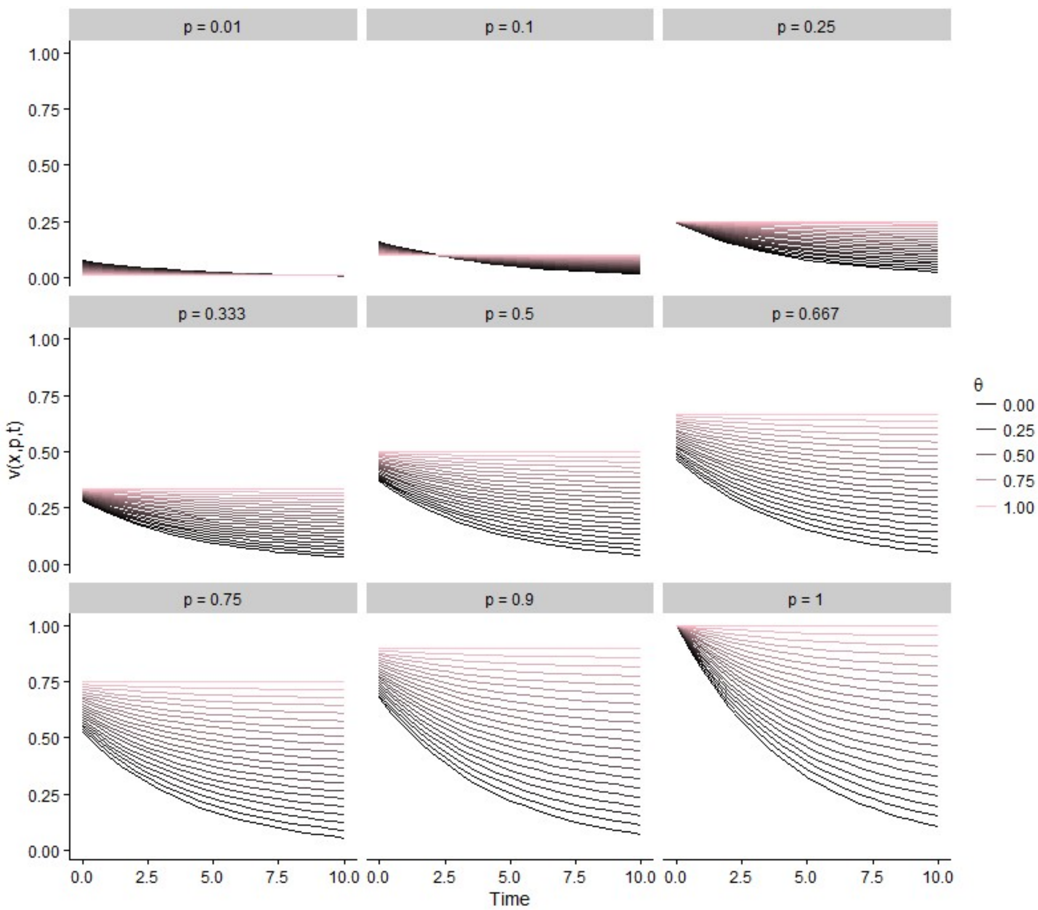
Figure I. Time Dependent Probability Weighting



For stochastic consumption plan (x, p, t) yielding $\$x$ with probability p at time t and 0 otherwise, Figure I plots (5) for different values of ϑ, different time delays (0 to 10 periods), and different probabilities. The parameters were set to $\delta = 0.8$, and $w(p) = \exp(-(-\ln(p))^{\alpha})$ (Prelec's one-parameter probability weighting function), with $\alpha = 0.5$. These parameter values are arbitrary and chosen for rough plausibility.

Figure II graphs the DPU function from (4) for different values of θ (within each panel) and for different probabilities (across panels), as the time horizon increases from 0 to 10 periods, using the same parametric specification as in Figure I. This specification may be viewed as a probability-dependent time discounting function for the special case where $u_1(.) = u_2(.)$. In Figure II, the function becomes steeper at higher probabilities, suggesting people are less patient as the outcome becomes more likely to be received, possibly reflecting anticipation prior to a reward.

Figure II. Probability Dependent Time Discounting



For stochastic consumption plan (x, p, t) yielding \$$x$ with probability p at time t and 0 otherwise, Figure II plots (5) for different values of ϑ, different time delays (0 to 10 periods), and different probabilities. The parameters were set to $\delta = 0.8$, and $w(p) = \exp(-(-\ln(p))^{\alpha})$ with $\alpha = 0.5$.

6 Other Relationships between Risk and Time

We next consider four other relationships between risk and time: risk preference and intertemporal substitution, a preference for diversifying risks across time, aversion to timing risk, and correlations with cognitive reflection.

6.1 Risk Preference and Intertemporal Substitution

The DEU model uses the same utility function for risk and time. However, risk preference and inter-temporal substitution are often observed to be distinct (e.g., Miao and Zhong, 2015). Consider the stochastic consumption plan, f, below, also considered by Miao and Zhong (2015), subject to $(1+r)c_1 + c_2 = 100$ and $(1+r)c_1^{'} + c_2^{'} = 100$, where $r \in (0,1)$ is an interest rate.

Figure III. A Simple Stochastic Consumption Plan

	t_1	t_2
p	c_1	c_2
$1-p$	c_1'	c_2'

The present equivalents $\mathrm{PE}(c_1, c_2)$ and $\mathrm{PE}^{'}(c_1^{'}, c_2^{'})$ of consumption $(\mathrm{c}_1, \mathrm{c}_2)$ and $(\mathrm{c}_1^{'}, \mathrm{c}_2^{'})$, respectively, are determined such that PE/PE' at t_1 is indifferent under V to receiving $(\mathrm{c}_1, \mathrm{c}_2)/(\mathrm{c}_1^{'}, \mathrm{c}_2^{'})$on the time horizon. They are defined as

$$\mathrm{PE}(c_1, c_2) := \mathrm{V}^{-1}((1-\theta)(u(c_1) + \delta u(c_2)) + \theta(c_1 + c_2)).$$

$$\mathrm{PE}'(c_1^{'}, c_2^{'}) := \mathrm{V}^{-1}((1-\theta)(u(c_1') + \delta u(c_2')) + \theta(c_1' + c_2')).$$

Employing rank-dependent probability weighting to aggregate the certainty equivalent as in the Chew-Epstein-Halevy approach (see Miao and Zhong, 2015), the certainty equivalent (CE) under DPU can be expressed as

$$\mathrm{CE}(f) = \mathrm{V}^{-1}(w(p)\mathrm{V}(\mathrm{PE}(c_1, c_2)) + w(1-p)\mathrm{V}(\mathrm{PE}(c_1^{'}, c_2^{'})))\ if\ \mathrm{PE} \geq \mathrm{PE}'.$$

$$\mathrm{CE}(f) = \mathrm{V}^{-1}(w(1-p)\mathrm{V}(\mathrm{PE}(c_1, c_2)) + w(p)\mathrm{V}(\mathrm{PE}(c_1^{'}, c_2^{'})))\ if\ \mathrm{PE} \leq \mathrm{PE}'.$$

This approach permits a separation between risk attitude (which is partially determined by w) and inter-temporal substitution (which does not depend on w).

6.2 Preference for Diversification across Time

Miao and Zhong (2015) provide a variant of the example shown below:

Figure IV. Preference for Diversification across Time (A $\succsim$ B)

Option A	$t = 0$	$t = 1$
$p = 0.5$	100	0
$1 - p = 0.5$	0	100

Option B	$t = 0$	$t = 1$
$p = 0.5$	100	100
$1 - p = 0.5$	0	0

We can think of the consumption sequences as being determined by the toss of a fair coin. Then Option A pays \$100 in period 0 if the coin lands heads, and it pays \$100 in period 1 if the coin lands tails. In contrast, Option B pays \$100 in both periods if the coin lands heads, and it pays \$0 in both periods if the coin lands tails. Miao and Zhong (2015) propose and find experimental support for the hypothesis that many people prefer Option A in which risks are diversified across time over Option B in which they are not. Such behavior has also been observed by Andersen et al. (2011) who refer to this preference pattern as 'correlation aversion' or 'intertemporal risk aversion.'

Correlation aversion is simply explained by DPU. Note that, for Option A, System 1 will rank consumption sequence $x := (100, t = 0; 0, t = 1)$ higher than the sequence $y := (0, t = 0; 100, t = 1)$ in order of preference for all $\delta_1 < 1$. Thus, DPU assigns weight $\pi(0.5)$ to x and $(1-\pi(0.5))$ to y, with weights assigned analogously for Option B. In most experimental studies of rank-dependent probability weighting functions (see references in §5.2), it has been found that $\pi(0.5)<0.5$. Under DPU, with $u(0) = 0$,

$$\text{V(A)} = (1-\theta)(\pi(0.5)u(100) + (1-\pi(0.5))\delta u(100)) + \theta(100).$$

$$\text{V(B)} = (1-\theta)(\pi(0.5)(u(100) + \delta u(100))) + \theta(100).$$

Hence, A is preferred to B if $\pi(0.5) < 0.5$.

6.3 Aversion to Timing Risk

Onay et al. (2007) and DeJarnette et al. (2015) experimentally investigate preferences over lotteries that pay a fixed prize at an uncertain date. For instance, in choices such as receiving \$100 in 10 weeks for sure (Option A), or receiving \$100 in either 5 or 15 weeks with equal probability (Option B), they find that people are generally risk-averse toward timing risk, preferring Option A. However, DEU and the standard models of hyperbolic and quasi-hyperbolic discounting imply people will be risk-seeking toward timing risk.

Consider a choice between receiving \$100 at time t (Option A), or \$100 at either time $t-r$ or time $t+r$ with equal probability (Option B). Under DPU, the values are:

$\mathrm{V(A)} = (1-\theta)\delta^{\mathrm{t}}u(100) + \theta(100).$

$\mathrm{V(B)} = (1-\theta)(\delta^{t-r}\pi(0.5)u(100) + \delta^{t+r}(1-\pi(0.5))u(100)) + \theta(100).$

For all $\theta \in [0,1)$, A is preferred to B if the following inequality holds:

$$1 > [\delta^{-\mathrm{r}}\pi(0.5) + \delta^{\mathrm{r}}(1-\pi(0.5))]$$

This inequality can hold if $\pi(0.5) < 0.5$, a robust finding, noted in §5.2.

6.4 Risk and Time Preferences and Cognitive Type

The DPU model also captures observed relationships between risk preference, time preference, and cognitive reflection. An agent's 'cognitive type', as parameterized by θ can be interpreted as a measure of reliance on System 2 processing and, since Frederick (2005), reliance on System 2 is thought to be correlated with cognitive reflection. DPU accommodates a continuum of types - any $\theta \in [0,1]$. Note that the DPU specification in (4) predicts the following:

Remark: *For a decision maker with preferences given by* (4):

(i) *The decision maker approaches risk-neutrality as θ increases.*

(ii) *The decision maker becomes more patient as θ increases.*

(iii) *Expected value maximization is negatively correlated with impatience.*

(iv) *Present bias is not monotonic in θ.*

Consistent with implications (i), (ii), and (iii), correlations between risk neutrality, patience, and cognitive types have been observed by Frederick (2005), Burks et al. (2009), Oechssler et al. (2009), Cokely and Kelley (2009), Dohmen et al. (2010), and Benjamin et al. (2013). Burks et al. (2009) report "those individuals making choices just shy of risk-neutrality have significantly higher CS [cognitive skills] than those making more either risk-averse or more risk-seeking choices" (p. 7747). However, Andersson et al. (2016) finds no correlation between risk preferences and cognitive skills. See Dohmen et al. (2018) for a recent review of this literature. Implication (iv) is supported by the study of Bradford et al. (2014) who observe a strong negative correlation between impatience and cognitive reflection but find no relation between present bias and cognitive reflection, as discussed in Section 4.1.

The notion that System 2 is closer to risk-neutrality and is more patient than System 1 is also supported by studies which employ other means of manipulating System 1 versus System 2 processing. Placing people under a high working memory load is one approach to inducing greater reliance on System 1. Studies have found that increased cognitive load (Deck and Jahedi, 2015; Holger et al., 2016) increases deviations from risk-neutrality such as increased small-stakes risk aversion and produces less patient and more impulsive behavior (Shiv and Fedorikhin, 1999). Leigh (1986), Anderhub et al. (2001), and Andersen et al. (2008) also find that risk aversion is positively correlated with impatience. In a large study of response times to the common ratio effect choices of Kahneman and Tversky (1979), Rubinstein (2013) observed slow responders to be significantly more likely to choose the expected value maximizing alternatives in both decisions than fast responders, and Kahneman (2011) has described the speed of cognition as a key difference between System 1 and System 2 processing in his book, "Thinking, fast and slow."

7 Summary of Results

The dual process utility model in (4) is sufficient to resolve each of the empirical violations of the discounted expected utility model in Table II. Moreover, the predictions of DPU are systematic: The predicted shifts in preference are in the direction observed in experiments. The reverse preference patterns are not predicted. In addition, given that no model in which preferences are represented by the product of a discount function, a probability weighting function, and a utility or value function can explain the risk-time-money interaction effects in Table II, the DPU model in (4) is surprisingly simple.

Table II. Risk and Time Anomalies Predicted by DPU Theory

Discounted Expected Utility Anomaly	Representative Reference
Allais paradox	Allais (1953)
Common ratio effect	Allais (1953)
Peanuts effect	Markowitz (1952)
Magnitude effect	Loewenstein and Prelec (1991)
Present bias	Laibson (1997)
Uncertainty reduces impatience	Keren and Roelofsma (1995)
Delay reduces risk aversion	Baucells and Heukamp (2010)
Time dominates at low stakes, risk dominates at high stakes	Baucells and Heukamp (2012)
Separation of risk and time preference	Epstein and Zin (1989)
Preference for diversifying risks across time	Miao and Zhong (2015)
Aversion to timing risk	DeJarnette et al. (2015)
Correlation between cognitive type and time preference	Frederick (2005)
Correlation between cognitive type and risk preference	Dohmen et al. (2018)

8 Related Literature

Many models for decisions under risk and for decisions over time have been developed in the past five decades and it is not feasible to review them all here. Since models developed for only decisions under risk or for only decisions over time cannot account for the majority of our results, we focus on models which consider both risk and time. Prelec and Loewenstein (1991) noted parallels between anomalies for decisions under risk and decisions over time. Rubinstein (1988, 2003) and Leland (1994, 2002) provided models of similarity judgments which explain key anomalies for decisions under risk and over time such as the Allais paradox and hyperbolic discounting as arising from the same cognitive process. However, these approaches treat risk and time independently, and thus cannot explain interaction effects between risk and time preferences.

Recent models by Halevy (2008), Walther (2010) and Epper and Fehr-Duda (2015) focus on implications of rank-dependent utility theory when extended to an intertemporal framework. Halevy (2008) and Walther (2010) focus primarily on relationships between hyperbolic discounting over time and non-linear probability weighting under risk. Halevy notes that his model is also consistent with the experimental evidence of Keren and Roelofsma (1995). The observations of Keren and Roelofsma and Baucells and Heukamp (2010) are both explained by the probability-time tradeoff model of Baucells and Heukamp (2012). However, this model applies only to a restrictive class of prospects offering a single non-zero outcome to be received with probability p at time t.

Aside from extensions of RDU to intertemporal choice, one other major literature stream is the class of dual-selves models motivated to explain temptation and self-control. A leading example in this is the model of Fudenberg and Levine (2006, 2011, 2012) and Fudenberg et al. (2014) which can explain the Allais paradox as well as the interactions between risk and time preferences identified by Keren and Roelofsma and Baucells and Heukamp. However, Fudenberg et al. (2014) comment "Unfortunately the model of Fudenberg and Levine (2011) is fairly complex, which may obscure some of the key insights and make it difficult for others to apply the model." (p. 56). In addition, a drawback of the model from both a normative and a descriptive viewpoint is that it violates transitivity (Fudenberg et al., 2014), even though transitivity is rarely violated in experiments (Baillon et al., 2014; Regenwetter et al., 2011).

Aside from the work of Fudenberg and Levine, most dual-selves models in economics are restricted to either risk or time. For decisions involving only risk, (5) reduces to a variant of the dual system model (DSM) of Mukherjee (2010).

The DPU model in (4) modifies the DSM by employing a rank-dependent probability weighting function for System 1, and extends the model to encompass both risk and time preferences. Rank-dependent weighting for System 1 eliminates the undesirable property that the DSM violates first order stochastic dominance. McClure et al. (2007) employ a two-system model of time preference with two discount factors but with a single utility function. Their approach can explain present bias, but not the magnitude effect or the interaction effects involving risk and time. Our results also relate to the finding in the social choice literature that group discount functions are present-biased (Jackson and Yariv, 2015). We show a similar phenomenon in a dual system model of individual choice. However, it should be clear that DPU does not capture all important behaviors for decisions over time. For instance, DPU is additively separable across time periods and so does not account for complementarities in consumption across time which is a hallmark of the classic model of habit formation (e.g., Constantinides, 1990).

9 Conclusion

The DPU model was developed to formalize behaviors based on System 1 and System 2 processes which are often discussed qualitatively. We have shown that one natural approach to constructing such a model (in which System 1 has behavioral preferences and System 2 has rational preferences) also predicts empirical violations of the dimensional independence axiom, as well as systematic interaction effects between risk and time preferences and observed correlations between risk preferences, time preferences, and cognitive types. Moreover, in Propositions 3, 4, 5, 6, 7, and 8 it is necessary to have the interaction between systems ($\theta \in (0,1)$) for the results to hold. Hence, these effects are not explained by standard rational or behavioral preferences alone. In addition to providing a unified approach to risk and time preferences, DPU provides a unification of models based on a rational agent, models based on prospect theory or rank-dependent utility and dual system or dual selves models of behavior.

Appendix: Proofs of Propositions

In the proofs of Propositions 3 – 8, the agent is assumed to have DPU preferences from equation (4).

Proposition 3: Present bias holds if and only if $\theta \in (0, 1)$.

Proof: (*Sufficiency*) We need to show that (8) implies (9):

(8) $\mathrm{V}(y, p, 0) = (1-\theta)w(p)u(y) + \theta py = \mathrm{V}(c, p, \Delta) = (1-\theta)\delta^{\Delta}w(p)u(c) + \theta pc.$

(9) $(1-\theta)\delta^{\mathrm{t}}w(p)u(y) + \theta py < (1-\theta)\delta^{(\mathrm{t}+\Delta)}w(p)u(c) + \theta pc.$

Note that since $c > y$, equation (9) implies that $w(p)u(y) > \delta^{\Delta}w(p)u(c)$. Also note that (8) can be rewritten as:

$(1-\theta)(w(p)u(y) - \delta^{\Delta}w(p)u(c)) = \theta p(c-y).$

In addition, (9) can be rewritten as:

$(1-\theta)\delta^{\mathrm{t}}(w(p)u(y) - \delta^{\Delta}w(p)u(c)) < \theta \mathrm{p}(\mathrm{c}-\mathrm{y}).$

Thus, $(1-\theta)\delta^{\mathrm{t}}(w(p)u(y) - \delta^{\Delta}w(p)u(c)) < (1-\theta)(w(p)u(y) - \delta^{\Delta}w(p)u(c)).$

The above inequality holds since $w(p)u(y) > \delta^{\Delta}w(p)u(c)$.

(*Necessity*) The agent has a constant discount factor if $\theta = 0$ or $\theta = 1$. ■

Proposition 4: For a concave power function u, the magnitude effect holds if and only if $\theta \in (0, 1)$.

Proof: (*Sufficiency*) We need to show that (10) implies (11):

(10) $\mathrm{V}(y, p, t) = (1-\theta)\delta^{\mathrm{t}}w(p)u(y) + \theta py = \mathrm{V}(c, p, s) = (1-\theta)\delta^{\mathrm{s}}w(p)u(c) + \theta pc$

(11) $(1-\theta)\delta^{\mathrm{t}}w(p)u(ry) + pry < (1-\theta)\delta^{\mathrm{s}}w(p)u(rc) + \theta prc$

Note that since $c > y$, equation (10) implies that$\delta^{\mathrm{t}}w(p)u(y) > \delta^{\mathrm{s}}w(p)u(c)$.

Also note that (10) can be rewritten as:

(12) $(1-\theta)w(p)(\delta^{\mathrm{t}}u(y) - \delta^{\mathrm{s}}u(c)) = \theta p(c-y)$

Inequality (10) can be written as: $(1-\theta)w(p)(\delta^{\mathrm{t}}u(ry) - \delta^{\mathrm{s}}u(rc)) < \theta pr(c-y)$

For concave power utility, (i.e., $u(z) = z^{\alpha}$, with $z > 0, \alpha < 1$), this becomes:

(13) $(1-\theta)r^{\alpha}(\delta^{\mathrm{t}}w(p)y^{\alpha} - \delta^{\mathrm{s}}w(p)c^{\alpha}) < \theta pr(c-y).$

Note that by (11), we have $(1-\theta)(\delta^{\mathrm{t}}w(p)y^{\alpha} - \delta^{\mathrm{s}}w(p)c^{\alpha})/\theta p(c-y) = 1.$

Thus, (13) reduces to $r > r^{\alpha}$, which is satisfied since $r > 1$ and $\alpha < 1$.

(*Necessity*) If $\theta = 0$ or $\theta = 1$, the scaling constant factors out. ■

Proposition 5: Let $\mathbb{E}[(\mathrm{y},\mathrm{p},\mathrm{t})] > \mathbb{E}[(\mathrm{c},\mathrm{q},\mathrm{t})]$. Then for any concave power function u, the peanuts effect holds under DPU if and only if $\theta \in (0,1)$.

Proof: (*Sufficiency*) We need to show that (14) implies (15):

(14) $\mathrm{V}(y,p,t) = (1-\theta)\delta^{\mathrm{t}} w(p)u(y) + \theta py = \mathrm{V}(c,q,t) = (1-\theta)\delta^{\mathrm{t}} w(q)u(c) + \theta qc$.

(15) $(1-\theta)\delta^{\mathrm{t}} w(p)u(ry) + \theta pry > (1-\theta)\delta^{\mathrm{t}} w(q)u(rc) + \theta qrc$.

For $\mathbb{E}[(y,p,t)] > \mathbb{E}[(c,q,t)]$, equation (14) implies that $\delta^{\mathrm{t}} w(p)u(c) > \delta^{\mathrm{t}} w(p)u(y)$.

Also note that (14) can be rewritten as:

(16) $(1-\theta)\delta^{\mathrm{t}}(w(q)u(c) - w(p)u(y)) = \theta(yp - cq)$.

In addition, the inequality in (15) can be rewritten as:

(17) $(1-\theta)\delta^{\mathrm{t}}(w(q)u(rc) - w(p)u(ry)) < \theta r(yp - cq)$

For a concave power utility function over gains, (i.e., $u(z) = z^{\alpha}$, with $z > 0, \alpha < 1$):

(18) $(1-\theta)\delta^{\mathrm{t}} r^{\alpha}(w(q)c^{\alpha} - w(p)y^{\alpha}) < \theta r(yp - cq)$.

Note that by (16), we have $(1-\theta)\delta^{\mathrm{t}}(w(q)u(c) - w(p)u(y))/\theta(yp - cq) = 1$.

Thus, (18) reduces to $r > r^{\alpha}$, which is satisfied since $r > 1$ and $\alpha < 1$.

(*Necessity*) If $\theta = 0$ or $\theta = 1$, the scaling constant factors out. ■

Proposition 6: Let $\mathbb{E}[(c,\alpha p,t)] > \mathbb{E}[(y,p,t)]$. Then delay reduces risk aversion if and only if $\theta \in (0,1)$.

Proof: (*Sufficiency*) We need to show that (19) implies (20):

(19) $(1-\theta)\delta^{\mathrm{t}} w(p)u(y) + \theta py = (1-\theta)\delta^{\mathrm{t}} w(\alpha p)u(c) + \theta\alpha pc$

(20) $(1-\theta)\delta^{\mathrm{s}} w(p)u(y) + \theta py < (1-\theta)\delta^{\mathrm{s}} w(\alpha p)u(c) + \theta\alpha pc$.

Since$\mathbb{E}[(c,\alpha p,t)] > \mathbb{E}[(y,p,t)]$, we have $\alpha cp > py$, in which case (19) implies that

$\delta^{\mathrm{t}} w(p)u(y) > \delta^{\mathrm{t}} w(\alpha p)u(c)$. Equation (19) can then be rewritten as:

$(1-\theta)\delta^{\mathrm{t}}(w(p)u(y) - w(\alpha p)u(c)) = \theta p(\alpha c - y)$.

In addition, note that the inequality in (20) can be rewritten as:

$(1-\theta)\delta^{\mathrm{s}}(w(p)u(y) - w(\alpha p)u(c)) < \theta p(\alpha c - y)$.

Thus, $(1-\theta)\delta^{\mathrm{s}}(w(p)u(y) - w(\alpha p)u(c)) < (1-\theta)\delta^{\mathrm{t}}(w(p)u(y) - w(\alpha p)u(c))$.

The above inequality holds since $w(p)u(y) > w(\alpha p)u(c)$.

(*Necessity*) If either $\theta = 0$ or $\theta = 1$, the discount factors in (19) and (20) cancel. ■

Proposition 7: For any convex weighting function w, risk reduces impatience if and only if $\theta \in (0, 1)$.

Proof: (*Sufficiency*) We need to show that (21) implies (22):

(21) $(1-\theta)\delta^{\mathrm{t}} w(p)u(y) + \theta py = (1-\theta)\delta^{(\mathrm{t}+\Delta)} w(p)u(c) + \theta pc$.

(22) $(1-\theta)\delta^{\mathrm{t}} w(q)u(y) + \theta qy < (1-\theta)\delta^{(\mathrm{t}+\Delta)} w(q)u(c) + \theta qc$.

Note that since $c > y$, equation (21) implies $\delta^{\mathrm{t}} w(p)u(y) > \delta^{(\mathrm{t}+\Delta)} w(p)u(c)$, and therefore $u(y) > \delta^{\Delta} u(c)$. Also note that (21) can be rewritten as:

$(1-\theta)\delta^{\mathrm{t}} w(p)(u(y) - \delta^{\Delta} u(c)) = \theta p(c-y)$.

Note that (22) can be rewritten as:

$(1-\theta)\delta^{\mathrm{t}} w(q)(u(y) - \delta^{\Delta} u(c)) < \theta q(c-y)$.

Then by (21), $((1-\theta)\delta^{\mathrm{t}}(u(y) - \delta^{\Delta} u(c)))/\theta(c-y) = p/w(p)$.

By (22), $((1-\theta)\delta^{\mathrm{t}}(u(y) - \delta^{\Delta} u(c)))/\theta(c-y) < q/w(q)$. Thus, if $w(q)/w(p) < q/p$ then (21) implies (22). Since $q \in (0, p)$, we can write $q = kp$, for $k \in (0, 1)$. For any convex w with $w(0) = 0$, we have $w(kp + (1-k)0) < kw(p) + (1-k)w(0)$, which implies $w(q)/w(p) < q/p$.

(*Necessity*) If $\theta = 0$ or $\theta = 1$, the probability weights in (21), (22) cancel. ■

Proposition 8: For any concave u, with $u(0) = 0$, subendurance holds if and only if $\theta \in (0, 1)$.

Proof: (*Sufficiency*) We need to show that (23) implies (24):

(23) $(1-\theta)\delta^{(\mathrm{t}+\Delta)} w(p)u(c) + \theta pc = (1-\theta)\delta^{\mathrm{t}} w(\lambda p)u(c) + \theta\lambda pc$

(24) $(1-\theta)\delta^{(\mathrm{t}+\Delta)} w(p)u(y) + \theta py < (1-\theta)\delta^{\mathrm{t}} w(\lambda p)u(y) + \theta\lambda py$

Since $pc > \lambda pc$, equation (23) implies $\delta^{\mathrm{t}} w(\lambda p)u(c) > \delta^{(\mathrm{t}+\Delta)} w(p)u(c)$. Also note that (23) can be rewritten as (25) and (24) can be rewritten as (26):

(25) $(1-\theta)(\delta^{\mathrm{t}} w(\lambda p)u(c) - \delta^{(\mathrm{t}+\Delta)} w(p)u(c)) = \theta pc(1-\lambda)$

(26) $\theta py(1-\lambda) < (1-\theta)(\delta^{\mathrm{t}} w(\lambda p)u(y) - \delta^{(\mathrm{t}+\Delta)} w(p)u(y))$.

From (25) and (26), for all $\theta \in (0, 1)$:

$(\delta^{\mathrm{t}} w(\lambda p)u(c) - \delta^{(\mathrm{t}+\Delta)} w(p)u(c))y < (\delta^{\mathrm{t}} w(\lambda p)u(y) - \delta^{(\mathrm{t}+\Delta)} w(p)u(y))c$.

For all $\theta \in (0, 1)$, the above inequality reduces to, $u(c)/c < u(y)/y$. Since $y \in (0, c)$, we can write $y = kc$, for $k \in (0, 1)$. For any concave u with $u(0) = 0$, we have $ku(c) + (1-k)u(0) < u(kc + (1-k)0)$ which implies $u(c)/c < u(y)/y$.

(*Necessity*) If $\theta = 0$ or $\theta = 1$, the utilities cancel in (23) and (24).■

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