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6-29-2017

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Recommended Citation

Greco G., Liang F., Moshier M.A., Palmigiano A. (2017) Multi-type Display Calculus for Semi De Morgan Logic. In: Kennedy J., de Queiroz R. (eds) *Logic, Language, Information, and Computation*. WoLLIC 2017. *Lecture Notes in Computer Science*, vol 10388. Springer, Berlin, Heidelberg. doi: 10.1007/978-3-662-55386-2_14

Multi-type display caculus for Semi-De Morgan Logic

Fei Liang

joint work with Giuseppe Greco, Andrew Moshier and Alessandra Palmigiano *

Preliminaries $\mathbb{A} = (A, \cap, \cup, ^*, 1, 0)$ is a *De Morgan algebra* (*DM-algebra*) if: (D1) $(L, \cap, \cup, 1, 0)$ is a bounded distributive lattice; (D2) $0^* = 1, 1^* = 0$; (D3) $(a \cup b)^* = a^* \cap b^*$ for all $a, b \in A$; (D4) $(a \cap b)^* = a^* \cup b^*$ for all $a, b \in A$; (D5) $a = a^{**}$ for every $a \in A$. It was originally introduced by Bialynicki-Birula and Rasiowa [1].

$\mathbb{A} = (A, \wedge, \vee, ', \top, \bot)$ is a *Semi-De Morgan algebra* (*SM-algebra*) if: (S1) $(A, \wedge, \vee, 1, 0)$ is a bounded distributive lattice; (S2) $\bot' = \top, \top' = \bot$; (S3) $(a \vee b)' = a' \wedge b'$ for all $a, b \in A$; (S4) $(a \wedge b)'' = a'' \wedge b''$ for all $a, b \in A$; (S5) $a' = a'''$ for every $a \in A$. It was first introduced by H.P. Sankappanavar [15] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices.

Problems D.Hobby [12] gave a topological duality based on Priestley spaces for semi-De Morgan algebras. Some subvarieties of semi-De Morgan algebras are also studied in the literature [14]. From a proof theoretical perspective, M. Ma and F. Liang [13] proposed a G3-style sequent calculus for semi-De Morgan algebras. In this talk, we will present a proper display calculi for semi-De Morgan logic. A formula is analytic iff it can be transformed into a structural rule which preserves cut-elimination in display calculi [3][9]. (S4) and (S5) are not analytic, that is to say, we cannot transform them into display structural rules which preserve cut-elimination.

Solutions We introduce a heterogeneous representation for semi-De Morgan algebra, in which all axioms are analytic. We proceed as follows:

For any SM-algebra $\mathbb{A} := (A, \wedge, \vee, ', \top, \bot)$, denoting by $\mathbb{L}$ the distributive lattice reduct of $\mathbb{A}$, there exists a De Morgan algebra $\mathbb{D} = (A'', \cap, \cup, ^*, 1, 0)$, where $A'' = \{a'' \mid a \in A\}$ such that there is a homomorphism h from $\mathbb{L}$ onto $\mathbb{D}$, and a one-to-one map f from $\mathbb{D}$ to $\mathbb{L}$ such that f is meet preserving and also preserves the bounded elements, moreover, $h \circ f = Id_D$.

An *heterogeneous SDM-algebra* [2] is a structure $(\mathbb{L}, \mathbb{D}, f, h)$ such that $\mathbb{L}$ is a distributive lattice, $\mathbb{D}$ is a De Morgan algebra, $f : \mathbb{D} \to \mathbb{L}$ and $h : \mathbb{L} \to \mathbb{D}$ such that ,

*This research is supported by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded to the second author in 2013.

h is a homomorphism from $\mathbb{L}$ onto $\mathbb{D}$, $h(f(\alpha)) = \alpha$ for every $\alpha \in \mathbb{D}$, and moreover, f is a one-to-one map, and it is meet preserving and also preserves the bounded elements. Then, we can define $\neg : \mathbb{L} \to \mathbb{L}$ by $\neg a := f((h(a))^*)$ for every $a \in \mathbb{L}$. Finally, we can show $(L, \neg)$ is a semi-De Morgan algebra. Then (S4) and (S5) will be:
(S4') $f((h(f((h(a \wedge b))^*)))^*) = f((h(f((h(a))^*)))^*) \wedge f((h(f((h(b))^*)))^*)$
(S5') $f((h(a))^*)) = f((h(f((h(f((h(a \wedge b))^*)))^*)))^*)$.
Now, both of them are analytic.

Since $h, f, ^*$ are normal operators, all of them can preserve canonicity see [7]. It suffices to show $(\mathbb{L}^\delta, \mathbb{A}^\delta, f^\pi, h^\pi)$ is a perfect heterogeneous SDM-algebra and $\neg^\delta := f^\pi \circ {^*}^\pi \circ h^\pi$ is the canonical extension of $\neg$. By general results of the theory of multi-type calculi [10, 11, 5, 8, 4, 6], these ingredients suffice to generate a conservative proper display calculus for semi-De Morgan algebras.

References

[1] A. Bialynicki-Birula and H. Rasiowa. On the representation of quasi-boolean algebras. *Journal of Symbolic Logic*, 22(4):370–370, 1957.

[2] Garrett Birkhoff and John D Lipson. Heterogeneous algebras. *Journal of Combinatorial Theory*, 8(1):115–133, 1970.

[3] Agata Ciabattoni and Revantha Ramanayake. Power and limits of structural display rules. *ACM Transactions on Computational Logic (TOCL)*, 17(3):17, 2016.

[4] Sabine Frittella, Giuseppe Greco, Alexander Kurz, and Alessandra Palmigiano. A multi-type display calculus for game logic. *In preparation*.

[5] Sabine Frittella, Giuseppe Greco, Alexander Kurz, and Alessandra Palmigiano. Multi-type display calculus for propositional dynamic logic. *Journal of Logic and Computation*, 2014. doi: 10.1093/logcom/exu064.

[6] Sabine Frittella, Giuseppe Greco, Alexander Kurz, Alessandra Palmigiano, and Vlasta Sikimić. A multi-type display calculus for dynamic epistemic logic. *Journal of Logic and Computation*, 2014. doi: 10.1093/logcom/exu068.

[7] Mai Gehrke and John Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.

[8] Giuseppe Greco, Alexander Kurz, and Alessandra Palmigiano. Dynamic epistemic logic displayed. In Huaxin Huang, Davide Grossi, and Olivier Roy, editors, *Proceedings of the 4th International Workshop on Logic, Rationality and Interaction (LORI-4)*, volume 8196 of *LNCS*, 2013.

[9] Giuseppe Greco, Minghui Ma, Alessandra Palmigiano, Apostolos Tzimoulis, and Zhiguang Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, page exw022, 2016.

[10] Giuseppe Greco and Alessandra Palmigiano. Lattice logic properly displayed. ArXiv: 1612.05930.

[11] Giuseppe Greco and Alessandra Palmigiano. Linear logic properly displayed. Submitted. ArXiv: 1611.04184.

[12] David Hobby. Semi-demorgan algebras. *Studia Logica*, 56(1-2):151–183, 1996.

[13] Minghui Ma and Fei Liang. Sequent calculi for semi-de morgan and de morgan algebras. *arXiv preprint arXiv:1611.05231*, 2016.

[14] C Palma and R Santos. On a subvariety of semi-de morgan algebras. *Acta Mathematica Hungarica*, 98(4):323–328, 2003.

[15] P Sankappanavar, Hanamantagouda. Semi-de morgan algebras. *The Journal of symbolic logic*, 52(03):712–724, 1987.