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Three-Dimensional Arm Movements at Constant Equi-Affine Speed

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Three-Dimensional Arm Movements at Constant Equi-Affine Speed

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Abstract

It has long been acknowledged that planar hand drawing movements conform to a relationship between movement speed and shape, such that movement speed is inversely proportional to the curvature to the power of one-third. Previous literature has detailed potential explanations for the power-law's existence as well as systematic deviations from it. However, the case of speed-shape relations for three-dimensional (3D) drawing movements has remained largely unstudied. In this paper we first derive a generalization of the planar power law to 3D movements, which is based on the principle that this power law implies motion at constant equi-affine speed. This generalization results in a 3D power law where speed is inversely related to the one-third power of the curvature multiplied by the one-sixth power of the torsion. Next, we present data from human 3D scribbling movements, and compare the obtained speed-shape relation to that predicted by the 3D power law. Our results indicate that the introduction of the torsion term into the 3D power law accounts for significantly more of the variance in speed-shape relations of the movement data and that the obtained exponents are very close to the predicted values.

Keywords: Arm movement, Laws of motion, Motor Control

Three-Dimensional Arm Movements at Constant Equi-Affine Speed

Introduction

The characterization of spatial and temporal properties of human movement has long been one avenue of investigation into human movement production. This approach is important because characterization of movements not only allows one to form qualitative models of the mechanisms behind movement production, but also permits close examination of the predictions of quantitative models of motion production. In this paper we investigate a characteristic of human movement – the speed of drawing movement is related to its geometry. This empirical relationship has been described in general terms as an inverse relationship between movement speed and curvature (Binet & Courtier, 1893; Jack, 1895; Abend, Bizzi, & Morasso, 1982). More specifically, for planar drawing movements, it was termed the *two-thirds power law* (denoted *2/3-PL* below), relating speed to curvature (Lacquaniti, Terzuolo, & Viviani, 1983; Viviani & McCollum, 1983; Viviani & Cenzato, 1985), or more formally:

$$v = \alpha\kappa^{-1/3} \tag{1}$$

Here v is the movement speed, κ is the curvature of the path, and α is some constant, termed the *velocity gain factor* (or *speed gain factor*). The law acquired its name from its expression in terms of angular speed, A, i.e. $A = \alpha\kappa^{2/3}$.

As detailed below, various types of drawing movements have been demonstrated to obey the power law (e.g. Viviani & McCollum, 1983; Viviani & Cenzato, 1985). This power-law has also been suggested to have a role in locomotion (Vieilledent, Kerlirzin, Dalbera, & Berthoz, 2001; Hicheur, Vieilledent, Richardson, Flash, & Berthoz, 2005) and motion perception (Viviani & Stucchi, 1989, 1992; Levit-Binnun, Schechtman, & Flash, 2006). Moreover, possibly more importantly, it has been related to neural coding of motor commands during various types of drawing movements according to the population vector model (Georgopoulos, Kalaska, Caminiti, & Massey, 1982; Georgopoulos, Schwartz, & Kettner, 1986; Georgopoulos, Kettner, & Schwartz, 1988; Schwartz, 1992, 1993, 1994; Moran & Schwartz, 1999; Schwartz & Moran, 1999, 2000) as well as to proprioceptive feedback (Viviani, Baud Bovoy, & Redolfi, 1997; Albert, Ribot-Ciscar, Fiocchi, Bergenheim, & Roll, 2005).

One incomplete aspect of these relations between kinematics and geometry is the question of whether speed is related to geometrical features of three-dimensional (3D) movements such as curvature and torsion. Previous research has not revealed an obvious relationship between speed and torsion (Morasso, 1983), and it was actually argued that 3D hand movements tend to be piecewise planar (Morasso, 1983). Moreover, there is evidence that the *2/3-PL* describing planar movements does not satisfactorily describe 3D movements (Pollick & Ishimura, 1996; Schaal & Sternad, 2001). In the following, we propose a mathematically derived novel relationship between speed and 3D geometry for drawing movements, evaluate it empirically on spatial scribbling hand movements, and compare it to the *2/3-PL*. By scribbling movements we mean spontaneous unconstrained movements of the hand, where the subject is guided by no template or instructions but those of her or his own wish (see Figure 2).

The starting point for this research is a recent theoretical description of the *2/3-PL*, which showed that the covariation of speed and curvature that this power-law describes in planar drawing motion is consistent with the mathematical interpretation of motion at constant equi-affine speed[1] (Flash & Handzel, 1996; Pollick & Sapiro, 1997; Handzel & Flash, 1999; Flash & Handzel, 2007).

This interpretation has implications for the representation of figural forms (see next section). But, more importantly for us here, the conditions for motion at constant equi-affine speed in two dimensions can be generalized to 3D movements, where they entail a new relationship between speed and geometry (Pollick, Flash, Giblin, & Sapiro, 1997). The details of this derivation are presented below (in the Appendix) and provide a predicted relationship: that the speed (v) of a 3D movement is proportional to the inverse of the product of the curvature (κ) raised to the 1/3 power and the absolute value of the torsion (τ) raised to the 1/6 power[2]. More formally:

$$v = \alpha\kappa^{-1/3}|\tau|^{-1/6} = \alpha\left(\kappa^2|\tau|\right)^{-1/6}, \tag{2}$$

where α is once again a constant, termed the velocity gain factor (or speed gain factor). We name this relationship between the speed, curvature and torsion the *one-sixth power law* (usually designated *1/6-PL* below).

The empirical thrust of this paper is to examine unconstrained self-paced spatial scribbling movements, to see whether they conform to this new 3D power law. We will also compare between the adherence of our data to both of the above power-laws.

Planar movement

It has been found that for production of continuous planar hand movements such as drawing movements, the relationship between speed and curvature is rather well described by a power law where speed is proportional to the curvature to the power of $-1/3$, or equivalently to the 1/3 power of the radius of curvature (Lacquaniti et al., 1983). This power law has been found in a variety of drawing tasks (Viviani & McCollum, 1983; Viviani & Cenzato, 1985; Wann, Nimmo-Smith, & Wing, 1988; Massey, Lurito, Pellizzer, & Georgopoulos, 1992; Viviani & Flash, 1995), and has been shown to evolve with the development of drawing skills (Viviani & Schneider, 1991). In addition to describing the production of planar drawing movements, the visual perception of planar form (Viviani &

Stucchi, 1989) and movement uniformity (Viviani & Stucchi, 1992; Levit-Binnun et al., 2006) were both found to be influenced by deviations from the two-thirds power law. Why one would expect the two-thirds power law relation for perception, let alone production of human movement, is an open question that has received considerable attention, and is discussed directly below.

The two-thirds power law as an emergent property.

The tendency of human drawing movements to obey the power-law may reflect the internal neural representation of movement by the central nervous system (CNS) (Schwartz, 1992, 1994; Schwartz & Moran, 1999; Moran & Schwartz, 1999; Schwartz & Moran, 2000). However, the fact that planar drawing movements obey the two-thirds power law does not necessarily imply that this particular speed-curvature relation is made explicit in movement planning by the CNS. The possibility does exist that it arises as a byproduct of more basic motor-planning principles or from some peripheral factors. Several researchers have pursued this approach in examining the origins of the two-thirds power law. Accounts of the two-thirds power law were provided by suggesting that the CNS may in fact wish to maximize movement smoothness (i.e. minimizing jerk or some other higher time derivatives of position; Flash & Hogan, 1985; Viviani & Flash, 1995; Richardson & Flash, 2002), smoothness along a predetermined path (Todorov & Jordan, 1998), or possibly minimizing endpoint variability under signal-dependent noise control (Harris & Wolpert, 1998).

Another suggestion was that given an input signal to draw an ellipse at constant Euclidean speed, mechanical properties of muscles may give rise to the observed behavior, namely that the movement output will obey the two-thirds power law (Viviani & Schneider, 1991). Such suggestions have been based on an implementation of equilibrium point control (Gribble & Ostry, 1996) and purely sinusoidal control (Sternad & Schaal, 1999). It has also been recently suggested that the power-law might be due, at least in part, to correlated noise in the human motor system (Maoz, Portugaly, Flash, & Weiss, 2006). In summary, the two-thirds power law of movement production has been demonstrated to possibly arise from a variety of principles.

The two-thirds power law and equi-affine representation of planar shape.

It has been established that the two-thirds power law of drawing planar shapes is equivalent to tracing out a curve at constant equi-affine speed (Flash & Handzel, 1996; Pollick & Sapiro, 1997; Flash & Handzel, 2007). The essence of this statement is that a particular representation of planar shape is suggested – one that represents length by equi-affine arc-length rather than by Euclidean arc-length. This proposition of the use of equi-affine arc-length to parameterize a curve is of practical interest since results from the visual representation of shape show that equi-affine properties of plane curves can be exploited to some degree to obtain recognition of a planar shape approximately independently of viewing direction (Munich & Perona, 1999; Olver, Sapiro, & Tannenbaum, 1999; Sato & Cipolla, 1997). When drawing a plane curve in compliance with the two-thirds power law, it is possible that this visual invariance could be exploited.

Given the role of the two-thirds power law in motion perception (Viviani & Stucchi, 1989, 1992; Levit-Binnun et al., 2006), it should also be evident that what the equi-affine interpretation of the two-thirds power law provides is a basis for the common coding of perception and action (Levit-Binnun et al., 2006). This suggestion can be contrasted with other approaches which suggested that the visual system learned the two-thirds power law from the motor system (Viviani & Stucchi, 1992). Generally speaking, the notion of common coding implies that at some level of processing the motoric and visual representations share the same principles of coding (Prinz, 1997). For the case of tracing planar figures, the parameterization of shape by equi-affine arc-length grants plausibility to the idea that the common coding originates in a common visuomotor equi-affine representation of the shape to be drawn (Flash & Handzel, 2007).

Properties of three-dimensional movement

The two-thirds power law relates movement kinematics, or speed, to the geometrical properties, or the shape, of the curve being drawn. Examining the geometry of curves in two and three dimensions reveals that, whereas the shape of a curve in the plane is uniquely specified by its curvature, for a curve in three-dimensional space an additional parameter, torsion, is required to completely specify shape (O'Neill, 1997; Oprea, 1997). Thus, we see that the power law given in Equation (1) is, in the sense of relating movement speed to geometry, complete for planar movements, but mathematically incomplete for the drawing of 3D space curves. However, speed-shape relations in 3D drawing movements have hardly been examined, and therefore the implications of this theoretical limitation is unknown.

One study that examined potential relationships between the curvature, torsion and speed of 3D drawing movements was performed by Morasso (1983). He found that while a clear inverse relationship between curvature and speed of movement exists for 3D scribbling movements, a similar result was not found for torsion. Instead, torsion appeared to indicate segmentation of movements into piecewise planar segments, and no obvious relation was found between torsion and either curvature or speed.

A further study into the segmentation of 3D drawing movements has suggested that these may be piecewise planar (Soechting & Terzuolo, 1987). Piecewise planarity would suggest that torsion is generally approximately zero and would tend to occur only in the transitions between different planar segments of a movement. However, deviations from piecewise-planar drawing in 3D have been observed (Gusis, 1995), and indeed this hypothesis does not seem to be supported by a more recent study involving 3D shape tracing hand movements (Todorov & Jordan, 1998). Moreover, it was shown that when asked to draw ellipses of different sizes, participants draw ellipses which become less planar as they increase in size; and violations of the two-thirds power law relation between speed and curvature increase with this decreasing planarity (Schaal & Sternad, 2001).

A clear picture of the relationship between torsion and drawing speed does not arise from these previous studies. If 3D drawing movements are piecewise planar then it might be expected that the observed torsion of 3D drawing movements would serve only to separate a movement into its segments. Thus, we might expect the torsion within the planar segments to be nearly zero and therefore to have no effect on the drawing speed within the piecewise planar segments of a space curve. However, it is clear that in cases such as tracing a line on a curved surface, the movement need not be piecewise planar and could have both torsion and curvature that continuously vary. Finally, the results by Schaal and Sternad (2001), which demonstrated that as the reproductions of a planar shape become less planar they also obtain a worse fit to the *2/3-PL*, are at least consistent with the possibility that torsion might influence the speed of drawing movements.

Constant equi-affine speed and power-laws of spatial hand motion

The possibility that torsion influences the speed of 3D drawing movements raises the question of what is the effect of torsion on the speed of such movements. Intuitively, we can guess that because both curvature and torsion indicate deviation from straightness, then torsion, like curvature, would be inversely related to drawing speed. Moreover, the minor inverse correspondence between the **absolute value of torsion** and speed in the data of Morasso (1983) (see the "Torsion and piecewise planarity" subsection of the Discussion below) suggests that torsion would have less of an influence on speed than curvature. So it would be reasonable to guess that speed is inversely related to torsion to the power of some magnitude, which is less than the 1/3 of curvature.

While such intuitions are useful, the approach we will take to model the role of torsion in modulating drawing speed is based on equi-affine differential geometry, which was used to explain the two-thirds power law of planar drawing movements. In the plane, it was shown that this law is equivalent to movement along a curve at constant equi-affine speed. Movement at constant equi-affine speed can be obtained by re-parameterizing a curve using equi-affine arc-length parameterization and then moving along the curve in equal units of this equi-affine arc-length per unit time. The extension from a 2D curve to a 3D curve is mathematically straightforward. The result is expressed in Equation (2) and derived in the Appendix below. This relation is consistent with the intuitions discussed above.

Methods

We examined self-paced, unconstrained spatial scribbling movements for the predicted relationship between Euclidean speed, v, curvature, κ, and torsion, τ as formulated in Equation (2).

Participants

Ten subjects volunteered to participate in the experiment (2 of them were females, 3 of the 10 were left-handed), which was approved by the ethical committee of the Weizmann Institute of Science. All gave their informed consent to participate, and did not report any previous arm injuries. They are designated below by their initials: AB, DF, DS, ED, ES, FP, LK, MD, RF and SL.

Apparatus

Participants were seated on a high-back chair and restrained with a shoulder harness to reduce effects of body sway. Their wrist was braced to minimize motion at the wrist joint. The Polhemus Liberty magnetic movement measurement system was used to measure the 3D position and orientation of three sensors placed on the forearm, upper-arm and chest of the subjects (see Figure 1). The system does not require line-of-sight between the magnetic source and the sensors, so it does not suffer from occlusions. The sampling rate was 240 Hz.

Experimental procedure

Subjects were instructed to move at a comfortable pace and to make free unconstrained scribbling or drawing movements in 3D space. They began their motion, and after a few seconds the data recording began. Movement was recorded for a period of 15 sec and a total of 6 trials were obtained for each subject. Subjects made a pause between movements, and were then allowed to rest if they were so inclined. Examples of the recorded paths from each of the subjects is given in Figure 2.

Data preprocessing

Rigorous testing that we performed proved that the Liberty system has a measurement error of roughly 1 mm for the distance from the magnetic source that we used and for the environment in which movement recording was performed. The motion data were approximated with splines using the Mathworks, Matlab implementation of the GCVSPL package for generalized, cross-validatory spline smoothing and differentiation (Woltring, 1986). The 1 mm measurement error was used to compute the appropriate tolerance for these splines. The quantities of speed, v, curvature, κ, and torsion, τ, could then be calculated by analytically differentiating these splines. Figure 3 displays the speed, curvature, torsion and absolute value of torsion versus the path arc-length for one experimental movement trial.

Extensive simulations were performed with this smoothing and differentiation method. We took trajectories very similar to those traced by our subjects, whose geometric properties were analytically computed. The characteristic noise of our measurement system was then added to these trajectories. We could then test how close our approximations of the geometric parameters of the noisy trajectories were to the original parameters. These demonstrated that for scribbling trajectories such as ours, torsion is calculated to within about 10% overall error for each sample, whereas speed and curvature are calculated to within 2-3% error. Figure 4 presents a typical example of the torsion and its approximation. It is apparent that the approximation is close to the true signal and well preserves its characteristic form.

Processing of the data involved examining whether the predicted relationship between the speed, curvature and absolute value of torsion in Equation (2) was obtained. There were two initial considerations when examining this relationship. The first was whether or not parts of the movement were planar. In regions where a movement was rather planar, the torsion was approximately zero and the relationship between speed, curvature and torsion, as presented in Equation (2), becomes ill-defined. A second consideration is that subjects tend to trace 3D paths whose signed torsion is composed of negative and positive regions (Figure 3 is typical in that respect). Taking the absolute value of this torsion thus gives rise to physically unrealizable cusps at locations that are transition regions between positive and negative values for signed-torsion (compare third from top and bottom panels in Figure 3). These non-realizable transition regions must be omitted before any correspondence between speed, curvature and torsion can be examined[3].

In order to deal with the planar regions and the torsion cusps we placed a threshold of 2 meters^{-1} on the absolute torsion for the power-laws fitting procedures described below. We thus effectively utilized only data that had absolute torsion values of at least 2 meters^{-1} for power-laws fitting. This threshold value was the lowest one that enabled us to considerably get rid of the planar regions of the movement as well as the cusps' transition regions; 66% of the data, which were above threshold, were maintained for the power-law fitting.

Fitting the power-laws to the data

Three types of power-law fitting methods were used on the data. In the first, we wanted to best-fit the exponents of the one-sixth power law to the experimental data. This was done in order to ascertain that the exponents are compatible with motion at constant equi-affine speed – i.e. approximately $-1/3$ for the curvature and $-1/6$ for the torsion, as described in Equation (2). We thus introduce a new equation:

$$v = \alpha \kappa^{\beta} \tau^{\gamma}, \tag{3}$$

termed the *unconstrained curvature-torsion power-law* and denoted *uc-$\kappa\tau$-PL*, for which we computed the best-fit α, β and γ. More formally, this method was based on simultaneously finding the α, β and γ values in Equation (3), which minimize the square error between the experimentally measured speed and the speed predicted according to the equation (using the curvature and torsion computed from the experimental data).

For the second type of fit, only the best-fit speed gain factor α (in the least-squares sense) of each of Equations (1) and (2) was found using linear regression. In the third power-law fitting, the power laws' exponents were not taken as fixed constants, but rather the least-squares-error speed gain factor and exponents were sought together. More formally, instead of the two-thirds power law we have:

$$v = \alpha \kappa^{\beta}. \tag{4}$$

We name this the *curvature power-law* and denote it by the acronym *κ-PL*. Similarly, instead of the *1/6-PL* we have:

$$v = \alpha \left(\kappa \cdot \sqrt{|\tau|} \right)^{\beta}. \tag{5}$$

This is termed the *constrained curvature-torsion power-law* and is denoted *c-$\kappa\tau$-PL* below, because here assume a constraint on the relationship between the curvature and torsion of the *uc-$\kappa\tau$-PL* of Equation (3). The five power-laws discussed above are summarized in Table 1.

The constraint between the curvature and torsion in the *c-$\kappa\tau$-PL* is the same as that which exists between the curvature and torsion exponents for constant equi-affine speed and hence the same as that which appears in Equation (2). It was introduced so that the *c-$\kappa\tau$-PL* would have the same number of free parameters as the *1/6-PL* (i.e. α and β in both Equation (4) and (5)). Had we tried to compare the explanatory power of the *uc-$\kappa\tau$-PL* of Equation (3) with that of the *κ-PL* of Equation (4) we would have encountered a problem because the *uc-$\kappa\tau$-PL* has an extra free parameter (γ) over those of the *κ-PL*.

For the first and third methods (Equations (3), (4), and (5)) non-linear regression had to be utilized to find the best-fit exponents. Taking the log of both sides of the two equations, we found the approximate least-squares regression values using linear regression in this log-space. We then took these values as starting points when utilizing the subspace trust region non-linear regression method (Coleman & Li, 1994, 1996). Last, the result was double-checked using an exhaustive regional search with iteratively increasing resolution. We always used the the R^2 goodness of fit statistic to compare between the power-laws (Rao, 1973).

Results

Visual inspection of Figure 3 suggests that the speed of the movement is inversely correlated to the curvature, as predicted by a two-thirds power law. It is also apparent that while an inverse relationship between torsion and speed is possibly present, it is not as strong as that of curvature and speed. In order to quantify these informal observations, we performed an analysis that was aimed at assessing the influence of including torsion in the relationship between the speed and the geometry, as far as this relationship accounts for the variance of the speed.

Testing for 3D constant equi-affine speed

If our subjects moved at constant equi-affine speed, we could expect that when fitting the *uc-$\kappa\tau$-PL* to the data we would obtain $\beta \approx -1/3$ and $\gamma \approx -1/6$. The results of this fitting are presented in Table 2. Pooling these results over all subjects, we cannot reject the hypothesis that the mean of the exponent distributions for the curvature and torsion exponents are -1/3 and -1/6, respectively (both tested with a one-sample t-test at the 0.05 significance level). Moreover, the standard deviations are rather small.

Comparing between 2/3-PL *and* 1/6-PL

After checking that our movement data is compatible with the one-sixth power-law, we wish to compare between the fit quality of this power-law and the two-thirds power-law. The only difference between the *2/3-PL* and the *1/6-PL* models (Equations (1) and (2)) is that the second has the right hand side of the first multiplied by $|\tau|^{-1/6}$. Similarly, the only difference between the *κ-PL* and the *c-$\kappa\tau$-PL* models (Equations (4) and (5)) is that the second has the right hand side of the first multiplied by $\sqrt{|\tau|}^{\beta}$ in the second case. To examine whether incorporating torsion resulted in a better power-law, we compared the explanatory power of the *2/3-PL* versus *1/6-PL* and the explanatory power of the *κ-PL* versus the *c-$\kappa\tau$-PL*.

We evaluated the distributions of the R^2 goodness-of-fit statistics for the fit of the speed predicted by each of the two power-laws, as they appear in Equations (1) and (2), to the experimentally measured speed. Here α was the only regression parameter in both cases. This was done for every subject separately as well as by pooling over all the subjects together. The averages and standard deviations of these R^2 distributions as well as the p-value of the test of the statistical significance of their difference (Kruskal-Wallis nonparametric one-way ANOVA) are shown in Table 3.

As is apparent, the *1/6-PL* explains the data significantly better than the *2/3-PL* for 8 out of 10 subjects (at the 0.05 significance level) and surely does so significantly better when all subjects are pooled together. On average, the *1/6-PL* explains 17±8% more of the data variance than the *2/3-PL*. Moreover, computations prove that the R^2 score of the *1/6-PL* is higher than that of the *2/3-PL* for all trials of all subjects. Last, no statistically significant differences among subjects were found for the R^2 score distributions of the *1/6-PL* for all subject-pairs except for AB & ES and AB & SL (i.e for 43 of the 45 subject-pairs, 96%; Kruskal-Wallis non-parametric one-way ANOVA and multiple comparison test of means using Tukey's honestly significant difference criterion at the 0.05 significance level).

Comparing between κ-PL *and* c-$\kappa\tau$-PL

We wanted to test the influence of including torsion on the power-law exponents as well as on the goodness of fit. We therefore compared the distributions of the β and R^2 values for the best least-squares fit of each of the two power-laws as they appear in Equations (4) and (5) (with α and β as the regression parameters in both cases). Figure 5 depicts some examples of the speed-profiles that these *κ-PL* and the *c-$\kappa\tau$-PL* models predict versus the actual measured speed.

The comparison between the power-laws' fitting was carried out for each subject separately as well as over all subjects together. The results are given in Table 4 and Table 5. Table 4 holds the results of the R^2 of the fit, similarly to Table 3. Table 5 holds the mean and standard-deviation of the power-laws' exponents (β of Equations 4 and 5) that were found in the fitting procedure.

As is apparent from Table 4, the *c-$\kappa\tau$-PL* explains the data significantly better than the *κ-PL* for 7 of the 10 subjects (at the 0.05 significance level), and certainly does better for the pool of all subjects together. On average the *c-$\kappa\tau$-PL* explains 14±7% more of the data variance than the *κ-PL*[4]. Moreover, for all trials of all subjects the computed R^2

score of the *c-$\kappa\tau$-PL* was higher than that of the *κ-PL*. Last, no statistically significant differences were found between subjects for the R^2 score distributions of the *1/6-PL* for all subject-pairs except AB & DF, AB & ES and AB & SL (i.e for 42 of the 45 subject-pairs, 93%; Kruskal-Wallis non-parametric one-way ANOVA and multiple comparison test of means using Tukey's honestly significant difference criterion at the 0.05 significance level).

The distributions of the exponent values (the β's of equations (4) and (5)) across subjects are also rarely significantly different in pairs. Statistically significant differences were found only between the distributions of subjects AB & DF, AB & ES, AB & LK and AB & SL (i.e only for 4 of the 45 subject-pairs, 9 %; Kruskal-Wallis non-parametric one-way ANOVA and multiple comparison test of means using Tukey's honestly significant difference criterion at the 0.05 significance level) [5]. When pooled over all subjects, the exponent values are significantly different between the two power-laws (Kruskal-Wallis nonparametric one-way ANOVA at the 0.002 confidence level). Moreover, whereas the mean of the distribution of the exponents of the *κ-PL* is significantly different than $-1/3$, we cannot reject the hypothesis that the mean of the exponent distribution of the *c-$\kappa\tau$-PL* is $-1/3$ (both tested with a one-sample t-test at the 0.05 significance level). The statistical analysis above thus seems to support rather robust and stable behavior across subjects in obeying the *c-$\kappa\tau$-PL* for 3D scribbling hand movements.

Torsion and piecewise planarity

Morasso (1983) and others (e.g. Soechting & Terzuolo, 1987) suggest that 3D hand movements are executed in a piecewise planar fashion. According to this hypothesis, torsion should be generally approximately zero, with high torsion regions appearing at the boundaries of planar sections – i.e. when shifting between planes. To test whether this is the case in our hand movements, we segmented our entire signed torsion data (without taking its absolute value and when not suppressing any data due to torsion-thresholding) for all subjects and trials into consecutive 0.2 sec bins in time (each segment thus contained 48 samples at our 240 Hz sampling rate). According to the piecewise planarity hypothesis, we would expect the great majority of the torsion bins to hold roughly zero values, with some exceptions for bins residing around inter-planar transition regions. However, when testing for each bin whether its median is significantly different from zero (sign-test at the 0.05 confidence level), we found that when averaged over all subjects $73 \pm 6\%$ of the bins have a median *significantly different* from zero. Therefore, for almost three-quarters of the time, the torsion cannot be said to be zero with random fluctuations.

Discussion

This paper has examined a proposal for extending the planar two-thirds power law for drawing movements to a relationship that can account for local speed-shape relations in 3D drawing movements, which we name the "one-sixth power-law". The derivation of this relationship was based on the equi-affine geometrical account of the planar two-thirds power law, which reveals that this power law implies motion at constant equi-affine speed. We demonstrated that constraining motion to constant equi-affine speed in three dimensions, leads to a straightforward relation that introduces a torsion component into the speed-shape relation, resulting in a new power-law of spatial hand motion (see the Appendix for the full mathematical derivation). Our results suggest that the one-sixth power-law explains spatial scribbling movements rather well, and certainly better than the two-thirds power-law.

The geometry of CNS movement planning

We behave and interact within an environment that is usually locally rather-well described by Euclidean geometry. This may have given rise to our intuition that the functional geometry of our sensory and motor systems is Euclidean. But there is persistent evidence that could be taken to suggest a different underlying geometric structure.

Some observed regularities of planar hand drawing movements have been demonstrated to be more naturally described using the non-Euclidean equi-affine geometry than Euclidean geometry. Firstly, the two-thirds power-law was revealed to be equivalent to motion at constant equi-affine speed (Flash & Handzel, 1996; Pollick & Sapiro, 1997; Handzel & Flash, 1999; Flash & Handzel, 2007). Secondly, there is the "local isochrony" principle (Viviani & Cenzato, 1985; Viviani, 1986; Viviani & Flash, 1995), which describes the modulation of speed within movement segments according to their extent in a manner that preservers the duration of each individual segment relatively insensitively to its length. This too was shown to be a possible consequence of moving at piecewise constant equi-affine speed (with the ratio between the speeds over the segments determined by the equi-affine arc-length of these segments; Flash & Handzel, 2007). A third result of motion at constant equi-affine speed is the time-scaling property of human movements (Hollerbach & Flash, 1982; Atkeson & Hollerbach, 1985) – the fact that the instantaneous hand speed is scaled by a multiplicative factor, whose value depends on the ratio of the overall movement duration to that of some reference movement (Flash & Handzel, 2007). The framework of equi-affine geometry was even used to identify possible elementary building blocks from which complex planar movements are composed (often termed "motor primitives") (Flash & Handzel, 2007). Last, it was shown that this framework can accommodate the internal neural coding of motor commands according to the population vector model (Georgopoulos et al., 1982, 1986, 1988; Schwartz, 1992, 1993, 1994; Moran & Schwartz, 1999; Schwartz & Moran, 1999, 2000).

These results, together with indications of the possible role of affine geometry in perception (e.g. Todd, Oomes, Koenderink, & Kappers, 2001), provides a possible mechanism of joint internal representation of perception and action (Viviani & Stucchi, 1992; Levit-Binnun et al., 2006), which may explain the role of the two-thirds power law in both (Polyakov et al., 2003; Polyakov, 2001, 2006; Flash & Handzel, 2007). What is more, the various studies that lend support to the two-thirds power-law go beyond hand movement and movement perception. It was found to apply in eye-motion (deSperati & Viviani, 1997) and movement prediction based on biological motion (Kandel, Orliaguet, & Viviani, 2000). Recent studies have also found it in locomotion (Vieilledent et al., 2001; Hicheur et al., 2005) and even for proprioceptive feedback in the leg (Albert et al., 2005) and hand (Viviani et al., 1997). The fact that this power-law is a direct consequence of motion at constant equi-affine speed, suggests that all these observed phenomena may be incorporated into the equi-affine framework.

The results of the present paper suggest that spatial scribbling is traced at constant equi-affine speed. In this they further contribute to the investigation of the role of non-Euclidean geometry, and more specifically that of equi-affine geometry, in neural representations subserving planning, controlling and perceiving motion. It was demonstrated that generalizing motion at constant equi-affine speed from two- to three-dimensions results in a viable invariant of spatial scribbling movement – the one-sixth power-law. This has, on the one hand, incorporated spatial scribbling into the growing collection of movement regularities whose derivation is straightforward under the framework of equi-affine geometry as a possible functional geometry of motion planning in the CNS. On the other hand, this derivation has demonstrated the power of the seemingly abstract equi-affine formulation in allowing to generate novel ideas (Pollick et al., 1997), and hence to experimentally test regularities of the sensorimotor system.

Torsion and piecewise planarity

As noted above, Morasso (1983) found that while a clear inverse relationship between curvature and speed of movement exists for 3D scribbling movements, a similar result was not found for torsion. In fact, he claimed that no obvious relation was found between torsion and either curvature or speed for his spatial movement data. It should be noted though, that Morasso examined the relationship between movement speed and signed torsion rather than the absolute value of torsion. When reexamining some of his figures, replacing the torsion values he obtained with their absolute value, a limited inverse correspondence between this value and speed is apparent (see Figure 3). Moreover, torsion measurement is not trivial with today's 3D measurement equipment, and thus considerable efforts had to be made to ensure the quality of our torsion approximation (see "Data Processing" subsection of the Methods above). So the noise level in Morasso's measurements (which were carried out in the early 80's, and he himself indirectly acknowledges as significant in one of his footnotes) could have influenced his results.

Morasso (1983) and Soechting and Terzuolo (1987) further suggested that 3D movements are piecewise planar. For this hypothesis to hold, torsion would have to be generally approximately zero with high torsion regions appearing at the boundaries of planar sections. Yet, as demonstrated in Figure 3, this type of behavior is not more typical for torsion than it is for curvature, at least in our data. Moreover, our more rigorous analysis suggested that for about three-quarters of the time torsion does not fluctuate around zero. Therefore the torsion signals of our data do not seem to conform to the piecewise planar hypothesis. This all leads us to suggest that until more is known about the torsion of 3D drawing movements, it would seem premature to make strong claims about a possible connection between torsion and segmentation of movements into planar subsections.

Points for further research

Our results suggest that about two-thirds of the variance in the speed profile of unconstrained spatial scribbling movements can be explained by their geometry according to the one-sixth power law. Nevertheless, we only used one value for the speed gain factor for the entire 15 sec of each movement trial. Although previous studies proved the equivalence of the speed gain factor of the two-thirds power law (α of Equation (1)) with the equi-affine speed along the planar path described by the curvature (κ), it was demonstrated that the speed gain factor of the two-thirds power-law is only *piecewise constant* for motion along complex planar paths, including scribbling (Viviani & Cenzato, 1985; Viviani, 1986; Viviani & Flash, 1995). This means that such paths are traced with piecewise constant equi-affine speed (Handzel & Flash, 1999; Flash & Handzel, 2007). It was further demonstrated that the shifts in the otherwise roughly constant speed gain factor correspond to natural shifts between consecutive segments of the drawing motion (Viviani & Cenzato, 1985; Viviani, 1986; Viviani & Flash, 1995). Nevertheless, additional claims that this segmentation may reflect segmented control by the CNS (Viviani & Cenzato, 1985; Viviani, 1986) were contrasted with results suggesting this segmentation to be epiphenomenal to continuous smooth minimum-jerk control at the hand level (Richardson & Flash, 2002) or to nonlinear transformations of the forward kinematics of human arms that perform smooth multi-joint rotations, which are governed by continuous control (Sternad & Schaal, 1999). It would thus be interesting to examine, which, if any, segmentation is implied by the one-sixth power-law, and how much more of the variance in the speed profile might be explained by the geometry in a segmented manner. Moreover, it would be important to test whether this new power-law and any segmentation scheme it may imply can be derived from some smoothness criteria in joint- or hand-space.

The Appendix lays out the proof that the speed gain factor of the one-sixth power-law, α of Equation (2), is the equi-affine speed along the path described by the curvature and torsion (κ and τ, respectively) there. When we constrain α to be constant throughout the entire scribbling trial as we did above (effectively assuming motion at globally constant equi-affine speed), our results suggest that about two-thirds of the variance of the Euclidean speed over the path can be explained by Equation (2). Given that complex planar paths including scribbling were traversed with piecewise constant equi-affine speed, a plausible point for further research would be to test wether 3D scribbling is carried out with piecewise constant equi-affine speed too. If this is the case, it may explain much of the remaining one-third of the speed variance left unexplained when assuming globally constant equi-affine speed.

Previous research demonstrated that there are systematic errors in the two-thirds power-law's account of planar movement in correlation with the overall shape of the path (Wann et al., 1988) and for spatial movement in correlation with the global non-planarity of the path (Schaal & Sternad, 2001). Incorporating torsion into the two-thirds power-law created the one-sixth power-law for spatial movement, which explained the data significantly better than the former law. The systematic errors found for the two-thirds power law for non-planar paths may thus be due to the assumption of motion at planar equi-affine speed for paths where spatial equi-affine speed is more appropriate. This still remains to be tested. Yet what about the systematic modification of the two-thirds power-law with the global planar shape? Does this occur for the one-sixth power law over spatial shapes? Further research is required in order to investigate what aspects of movement shape systematically modulate the predicted spatial speed-shape relation. If systematic corrections must be introduced into the two-thirds and one-sixth power-laws in correlation with planar and spatial shapes, respectively, it may be that the equi-affine framework is not broad enough to account for the intricacies of the functional geometry of the sensorimotor space. We must allow for the possibility that this framework is only a first-order approximation to some more extensive framework.

Conclusions

In this paper we have proposed and experimentally tested a model of speed-shape covariation, which conforms to motion at constant spatial equi-affine speed. Our experimental dataset was composed of free 3D scribbling movements. Our results show that these scribbling movements fit the model's predictions rather well, with unconstrained spatial drawing movements slowing down not only in regions of high curvature but also in regions of high torsion, though to a lesser extent, as expected. This new power-law describes the relationship between spatial speed and geometry significantly better than the two-thirds power law, resulting in a more accurate speed profile prediction for a given path. Moreover, a utility of the equi-affine geometric approach to describing human movement geometry was demonstrated by extending the previous treatment of planar movements to the three-dimensional case.

Appendix

This appendix presents the mathematical foundations of the one-sixth power-law. It expands upon the work of Pollick, Flash, Handzel and others (Flash & Handzel, 1996; Pollick & Sapiro, 1997; Pollick et al., 1997; Flash & Handzel, 2007), regarding the equivalence of the two-thirds power-law with motion at constant equi-affine speed and its generalization to three dimensions.

Euclidean arc-length, length and speed

A space curve, r, may be regarded as the trajectory of a point $q \in [0,1]$ in 3D space, $\mathbb{R}^3$. This means that $r : [0,1] \to \mathbb{R}^3$ (i.e. for each value of q it matches a point $r(q) = [x(q), y(q), z(q)] \in \mathbb{R}^3$ on the curve). The velocity of the trajectory would thus be given by $\frac{dr}{dq}$. Different parameterizations of this curve result in different velocity profiles over the curve, though they all define the same path (i.e. the same image of r in $\mathbb{R}^3$).

One particular and important parameterization is the *Euclidean arc-length parameterization*, p, in which the curve is traversed with unit speed, i.e. $\|\frac{dr}{dp}\| \equiv 1$ (where $\|\cdot\|$ is the Euclidean vector norm). In order to transform from an arbitrary parameterization q to p, the operation

$$p(q) = \int_0^q \left\| \frac{dr(w)}{dw} \right\| dw$$

is used. Utilizing this Euclidean parameterization, the Euclidean length of a curve between points p_1 and p_2 is

$$l_e(p_1, p_2) := \int_{p_1}^{p_2} dp.$$

The Euclidean arc-length parameterization is invariant under Euclidean transformations (translations and rotations). Formally this means the following: let us define $\tilde{r} = \mathbf{R}r + \mathbf{T}$, where $\mathbf{R} \in \mathbb{R}^{3\times 3}$ is a rotation matrix and $\mathbf{T} \in \mathbb{R}^3$ is a translation vector. Let us further designate by $\tilde{p}_1$ and $\tilde{p}_2$ the results of the Euclidean transformation of points p_1 and p_2. The Euclidean invariance of the arc-length p means that $dp = d\tilde{p}$, hence $l_e(\tilde{p}_1, \tilde{p}_2) = l_e(p_1, p_2)$.

Once we have a definition for Euclidean arc-length, we can continue and define Euclidean velocity by

$$v_e := \frac{dr}{dt} = \frac{dr}{dp}\frac{dp}{dt},$$

where t stands for time. The Euclidean speed is therefore:

$$\|v_e\| = \left\| \frac{dr}{dp}\frac{dp}{dt} \right\| = \left\| \frac{dp}{dt} \right\|.$$

Due to the fact that p is Euclidean invariant, so is $\|v_e\|$. This speed is the one measured in the power-laws above.

Equi-affine arc-length and speed

Yet what would happen if we wanted invariance not only to translation and rotation, but also to stretches with different values in the different axes so long as the "volume" of the curve remains the same (e.g. stretching the curve r by 5 in the x direction, 0.5 in the y direction and 0.4 in the z direction; or, more generally, stretching r

by a, b and α in the directions x, y and z, where $a \cdot b \cdot c = 1$)? These transformations are collectively named *equi-affine transformations*, so we would thus want invariance to equi-affine transformations. More formally, the 3D curves r and $\tilde{r}$ are invariant under equi-affine transformations if there exist some 3×3 matrix $\mathbf{A}$, whose determinant is equal to 1 and a 3×1 translation vector $\mathbf{T}$ for which $\tilde{r} = \mathbf{A}r + \mathbf{T}$.

The Euclidean arc-length is obviously not invariant under arbitrary equi-affine transformations. Yet we can define the notion of *equi-affine arc-length*, σ, based on the "volume" of the curve, which is equi-affine invariant.[6] This would enable us to further define the *equi-affine length*, l_a, which would be equi-affine invariant. We define the equi-affine arc-length of any regular parameter on the space curve r to be the simplest equi-affine invariant[7]. That is, the equi-affine arc-length is not affected by a change of parameter of the curve and is invariant under equi-affine transformations (Davis, 2006):

$$\frac{d\sigma}{dt} = \left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{\frac{1}{6}},$$

where "$|u, v, w|$" denotes the scalar triple product between vectors $u, v, w \in \mathbb{R}^3$; $|u, v, w| = u\dot{(}v \wedge w)$, where $x\dot{y}$ denotes the dot-product and $(x \wedge y)$ denotes the cross-product between vectors x and y.

This is in fact the signed volume of the parallelepiped created by the vectors $\frac{dr}{dt}$, $\frac{d^2r}{dt^2}$ and $\frac{d^3r}{dt^3}$ raised to the power of $\frac{1}{6}$. Therefore:

$$\sigma = \int_{t_0}^{t} \frac{d\sigma}{dt} dt = \int_{t_0}^{t} \left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{\frac{1}{6}} dt.$$

Hence the equi-affine length of a curve between σ_1 and σ_2, $l_a(t_1, t_2) := \left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{\frac{1}{6}} dt.$ As for the Euclidean case above, this would make the equi-affine speed:

$$v_a := \frac{d\sigma}{dt} = \left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{\frac{1}{6}},$$

where t denotes time.

Constant equi-affine speed

Given the above, constant equi-affine speed parameterization of a 3D curve would entail:

$$v_a = \left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{\frac{1}{6}} = const,$$

where t stands for time.

Now, from standard formulae of the differential geometry of space curves (Oprea, 1997) we know that for curvature:

$$\kappa = \frac{\left\| \frac{dr}{dt} \wedge \frac{d^2r}{dt^2} \right\|}{\left\| \frac{dr}{dt} \right\|^3} \tag{6}$$

and for torsion:

$$\tau = \frac{\left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|}{\left\| \frac{dr}{dt} \wedge \frac{d^2r}{dt^2} \right\|^2}, \tag{7}$$

where "u∧v" denotes the cross product between vectors $u, v \in \mathbb{R}^3$.

Combining Equations (6) and (7) with the knowledge that $v = \left\| \frac{dr}{dt} \right\|$, where v is the Euclidean speed[8], we get[9]:

$$v = \frac{\left| \frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3} \right|^{1/6}}{\kappa^{1/3} \cdot |\tau|^{1/6}} = v_a \kappa^{-1/3} |\tau|^{-1/6}. \tag{8}$$

Therefore motion at constant equi-affine speed in 3D space would directly entail Equation (2) above.

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Notes

[1] "Equi-affine transformations" are a special case of the more general "affine transformations", in which the determinant of the affine transformation matrix is strictly 1. See the Appendix for more.

[2] By definition, the curvature is the normal of the unit tangent vector to the curve (Oprea, 1997) and is hence non-negative. Intuitively it is the amount of local deviation from straightness of the curve. Torsion, on the other hand, is the negative of the dot product of the curve's principle normal vector and the derivative of the binormal vector, and can thus take positive or negative values (Oprea, 1997). Intuitively it is the amount of local deviation from planarity. In more technical terms, it is the signed rate of change of the osculating plane, where the sign is determined by the direction of movement. In the setting of our power-law we are interested in the rate of change of the osculating plane, rather than in its direction. We therefore use the absolute value, or magnitude, of torsion.

[3] Similar singularities and cusps in power law relationships have been noted for the planar two-thirds power law at inflection points (e.g. Viviani & Flash, 1995). There too these regions were discarded.

[4] If we remove outliers (here defined as data points that are two inter-quartile range estimations of the standard-deviation or more from the distribution's median) from the c-$\kappa\tau$-PL's R^2 distribution, its mean increases to 0.67, making it roughly equal to the median.

[5] Interestingly, *all* differences between subject pairs for the R^2 values in Table 3 and 4 as well as the exponents in Table 5 involve subject AB. This likely stems from that subject's power-law exponent being so low (its average value is closer to -1/2 than to -1/3 and is distanced more than 1.5 inter-quartile range approximations of the standard deviation from the median of the distribution of all subject mean exponent values). Nevertheless, even this possibly anomalous subject cannot be said to have significantly

different R^2 and exponent values when compared with most other subjects.

[6]Such a parameterization always exists for every curve, except for the infinitesimal neighborhood of a point of zero curvature (a straight line or an inflection point) on that curve. Because such points are of zero measure over the curves we are interested in, this is not problematic for us.

[7]Simplest here means lowest derivative.

[8]Denoting the Euclidean-*speed* by 'v' here when above 'v_e' and 'v_a' denoted Euclidean or equi-affine *velocity* is certainly confusing. However, in various formulations of the two-thirds power law and other such laws, v historically denotes the speed. We thus stick to this problematic method here.

[9]We replace τ by $|\tau|$ because if $\tau < 0$, $\tau^{1/6}$ is not a real number and neither is the speed it predicts, v.

Curvature Power-Laws				Curvature-Torsion Power-Laws			
Name	Acronym	Formula	Eq. No.	Name	Acronym	Formula	Eq. No.
Two-Thirds Power Law	*2/3-PL*	$v = \alpha\kappa^{-1/3}$	(1)	One-Sixth Power Law	*1/6-PL*	$v = \alpha\kappa^{-1/3}\tau^{-1/6}$	(2)
Curvature Power Law	κ-*PL*	$v = \alpha\kappa^{\beta}$	(4)	Constrained Curvature-Torsion Power-Law	*c*-$\kappa\tau$-*PL*	$v = \alpha\left(\kappa \cdot \sqrt{\lvert\tau\rvert}\right)^{\beta}$	(5)
				Unconstrained Curvature-Torsion Power-Law	*uc*-$\kappa\tau$-*PL*	$v = \alpha\kappa^{\beta}\tau^{\gamma}$	(3)

Table 1: A table summarizing the five power-laws that are discussed in the paper.

Subjects:	AB	DF	DS	ED	ES	FP	LK	MD	RF	SL	All
β	-0.42 ± 0.03	-0.31 ± 0.06	-0.34 ± 0.02	-0.36 ± 0.02	-0.29 ± 0.05	-0.35 ± 0.03	-0.31 ± 0.04	-0.37 ± 0.04	-0.38 ± 0.03	-0.30 ± 0.03	-0.34 ± 0.05
γ	-0.17 ± 0.03	-0.14 ± 0.06	-0.15 ± 0.02	-0.18 ± 0.01	-0.17 ± 0.03	-0.14 ± 0.03	-0.14 ± 0.02	-0.21 ± 0.05	-0.13 ± 0.02	-0.17 ± 0.04	-0.16 ± 0.04
R^2	-0.81 ± 0.05	-0.54 ± 0.22	-0.71 ± 0.04	-0.73 ± 0.04	-0.58 ± 0.06	-0.66 ± 0.11	-0.61 ± 0.11	-0.67 ± 0.11	-0.72 ± 0.07	-0.55 ± 0.07	-0.66 ± 0.13

Table 2: The table presents the exponent values of the *uc-$\kappa\tau$-PL* (β and γ of Equation (3)) and the R^2 goodness of fit statistic for each subject as well as pooled together over all subjects.

Subjects:	AB	DF	DS	ED	ES	FP	LK	MD	RF	SL	All
R^2 *2/3-PL*	0.69 ± 0.08	0.33 ± 0.29	0.46 ± 0.08	0.52 ± 0.09	0.29 ± 0.14	0.54 ± 0.17	0.39 ± 0.18	0.46 ± 0.13	0.64 ± 0.06	0.35 ± 0.16	0.47 ± 0.19
R^2 *1/6-PL*	0.78 ± 0.04	0.48 ± 0.30	0.71 ± 0.04	0.74 ± 0.04	0.52 ± 0.11	0.65 ± 0.12	0.60 ± 0.14	0.66 ± 0.12	0.70 ± 0.06	0.51 ± 0.10	0.63 ± 0.15
Signif. diff.	p=0.03	No	$p=4 \cdot 10^{-3}$	$p=4 \cdot 10^{-4}$	p=0.02	No	p=0.05	p=0.03	p=0.04	p=0.05	$p=8 \cdot 10^{-8}$

Table 3: The table presents an assessment of the explanatory powers of the *2/3-PL* compared with that of the *1/6-PL* across subjects using the R^2 statistic. The top and bottom parts of the table assign a column to each subject, which holds the average and standard deviation of R^2 values over that subject's six scribbling trials. The 'All' column on the bottom right gives the value when averaging over all the R^2 values of all the subjects pooled together. In the top and bottom parts of the table, the first row contains R^2 values pertaining to the *2/3-PL*, the second row values pertaining to the *1/6-PL*. The last row contains the p-values of the comparisons of the R^2 distributions between the power-laws, if these are statistically significantly different at the 0.05 significance level, (Kruskal-Wallis nonparametric one-way ANOVA) or the word "No" if they are not (i.e. if p-value>0.05).

Subjects:	AB	DF	DS	ED	ES	FP	LK	MD	RF	SL	All
R^2 κ-*PL*	0.69 ± 0.08	0.42 ± 0.19	0.57 ± 0.05	0.53 ± 0.07	0.38 ± 0.07	0.56 ± 0.13	0.44 ± 0.11	0.47 ± 0.13	0.63 ± 0.06	0.41 ± 0.11	0.51 ± 0.14
R^2 c-$\kappa\tau$-*PL*	0.79 ± 0.04	0.54 ± 0.21	0.70 ± 0.04	0.73 ± 0.04	0.57 ± 0.06	0.65 ± 0.10	0.61 ± 0.11	0.66 ± 0.12	0.70 ± 0.07	0.54 ± 0.08	0.65 ± 0.12
Signif. diff.	p=0.02	No	$p=4 \cdot 10^{-3}$	$p=4 \cdot 10^{-4}$	$p=7 \cdot 10^{-3}$	No	p=0.04	p=0.04	No	p=0.04	$p=1 \cdot 10^{-7}$

Table 4: The table presents the explanatory powers of the κ-*PL* compared with that of the c-$\kappa\tau$-*PL* (i.e. with the exponents as free parameters as well), in terms of the R^2 fit statistic of the speed profiles of the power-laws versus the actual measured speed profile.

Subjects:	AB	DF	DS	ED	ES	FP	LK	MD	RF	SL	All
β κ-*PL*	-0.35 ± 0.02	-0.28 ± 0.07	-0.27 ± 0.02	-0.32 ± 0.02	-0.29 ± 0.05	-0.33 ± 0.03	-0.29 ± 0.05	-0.38 ± 0.08	-0.35 ± 0.02	-0.30 ± 0.04	-0.315 ± 0.054
β c-$\kappa\tau$-*PL*	-0.42 ± 0.02	-0.30 ± 0.06	-0.35 ± 0.02	-0.36 ± 0.02	-0.30 ± 0.05	-0.35 ± 0.03	-0.31 ± 0.04	-0.38 ± 0.04	-0.37 ± 0.03	-0.31 ± 0.04	-0.344 ± 0.051

Table 5: The table presents the exponent values of the κ-*PL* (i.e. β of Equation 4) and the c-$\kappa\tau$-*PL* (i.e. β of Equation 5) for all subjects separately and pooled together.

Figure Captions

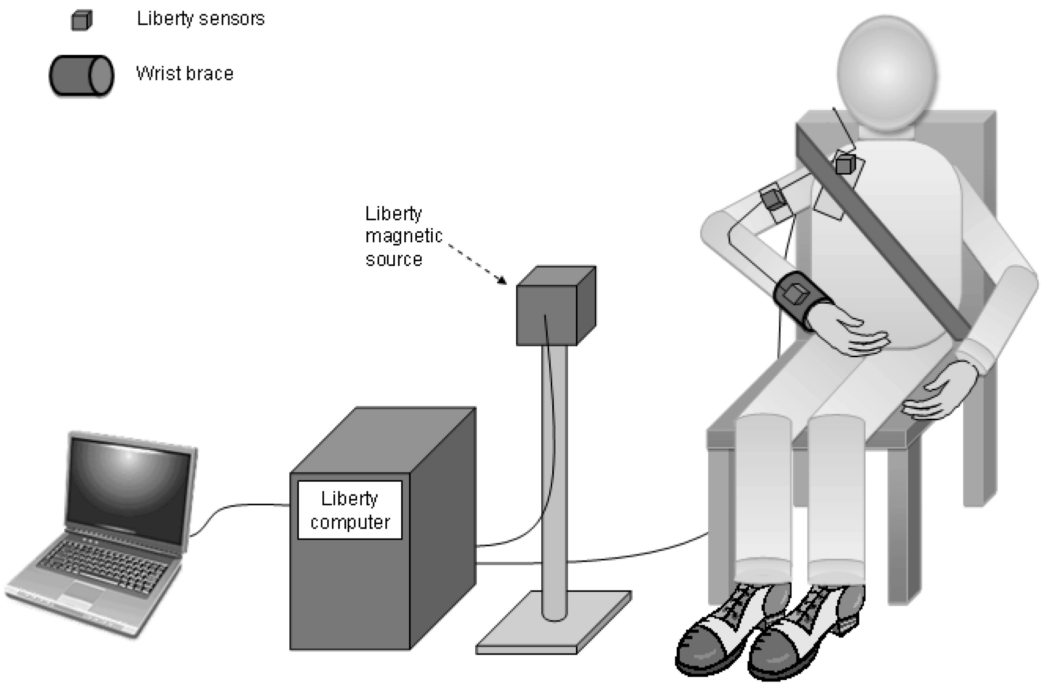
Figure 1. An illustration of the experimental apparatus depicting a subject and the Liberty magnetic motion measurement system.

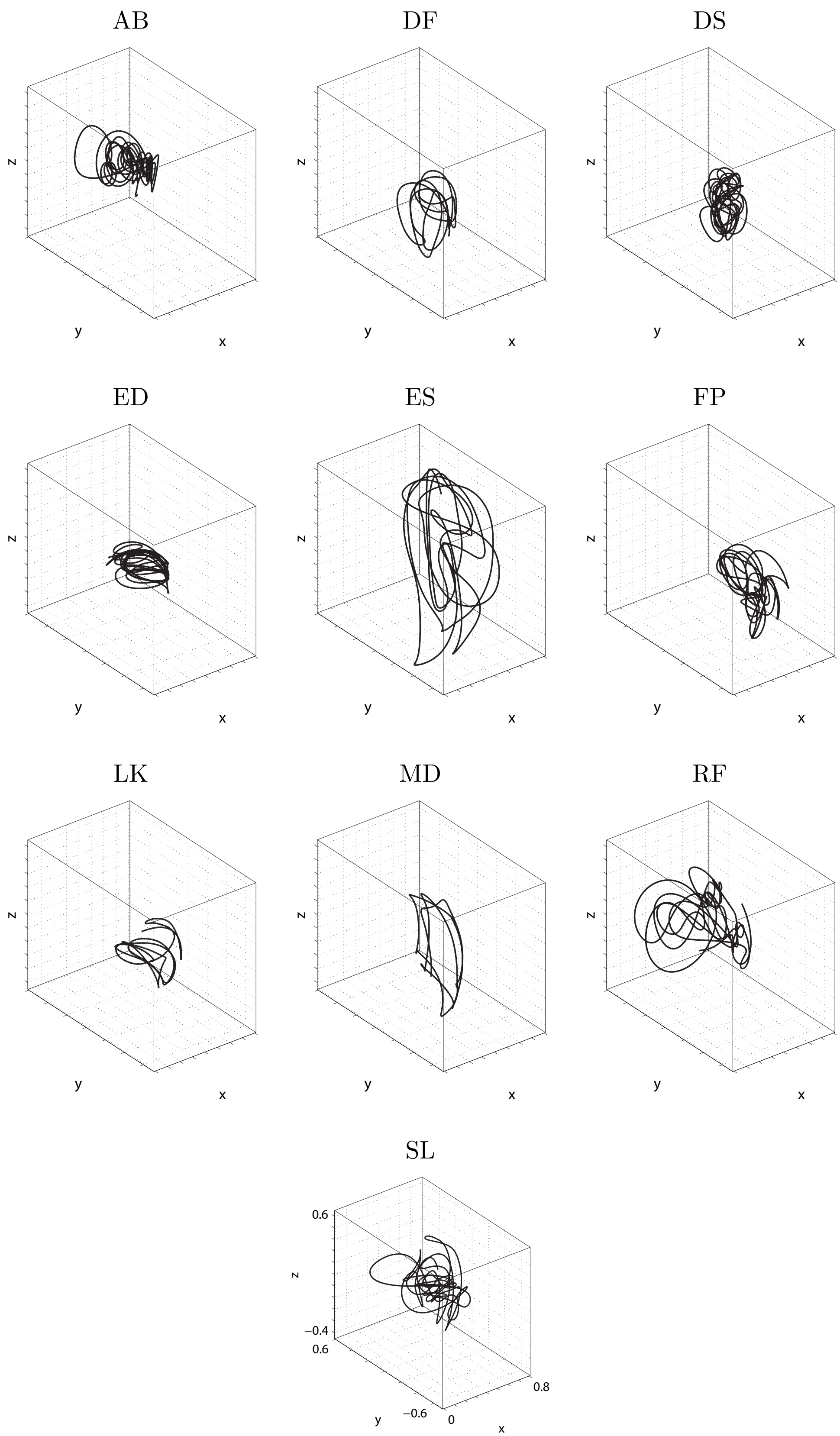
Figure 2. Ten examples of 15 sec scribbling paths, one from each subject. Subjects are designated by their initials. Axes limits are the same across all subjects and are presented for subject SL.

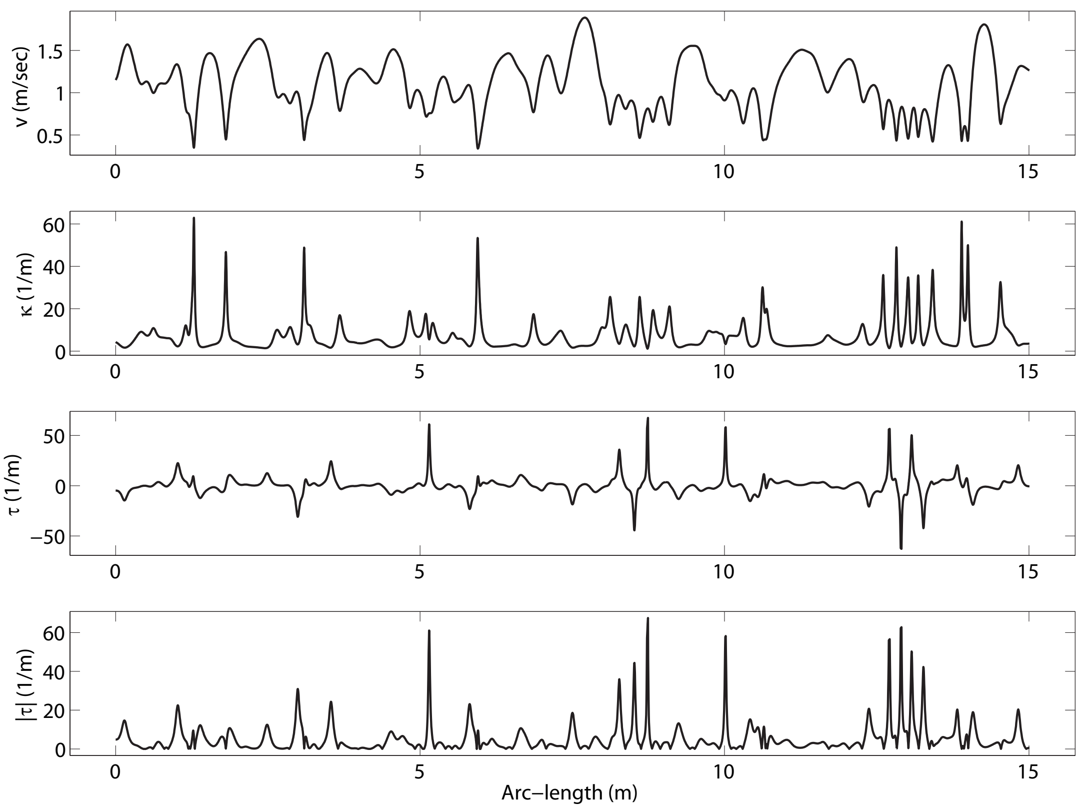
Figure 3. An example of speed (v), curvature (κ), torsion (τ) and absolute value of torsion ($|\tau|$) profiles over the path arc-length, from top to bottom respectively. The corresponding path is depicted in Figure 1 for subject RF.

Figure 4. An example of torsion approximation. The path of subject AB in Figure Figure 2 was strongly smoothed to remove noise (a smoothing spline with tolerance 0.01m was passed through it). This was designated the "true" path and its torsion was computed analytically. Characteristic noise of the Liberty device was then added to this true path, and our smoothing and differentiation routine was used to extract an approximation for the true torsion. The approximated and true torsion signals are displayed. The bottom panel is a zooms in on the dotted rectangular area of the top panel.

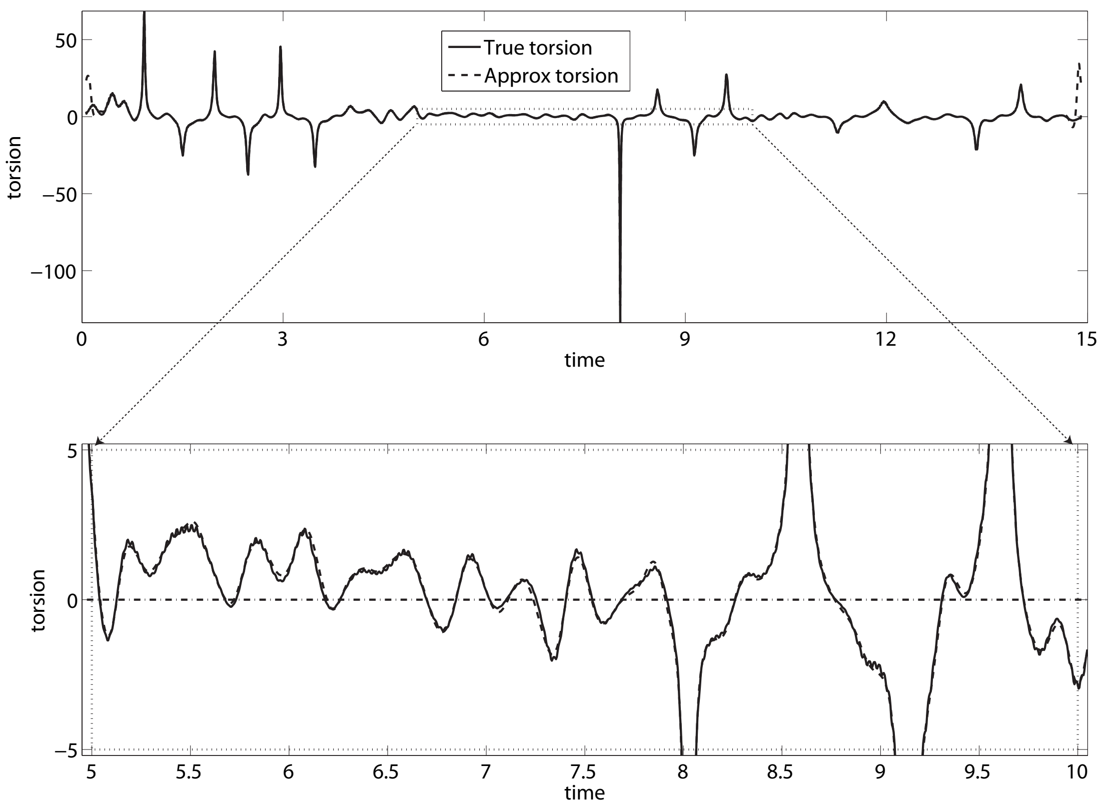
Figure 5. Four examples depicting the speed approximations of the κ-*PL* and the c-$\kappa\tau$-*PL* as well as the actual measured speed, all versus the arc-lengths of the paths. They are taken from subjects AB, DS, FP and RF. Solid lines appear in regions where the torsion is above its threshold ($2\ meter^{-1}$). The dashed lines are below threshold regions for the measured speed, and cubic interpolations for the power-law speed predictions.


Liberty sensors
Wrist brace
Liberty magnetic source
Liberty computer

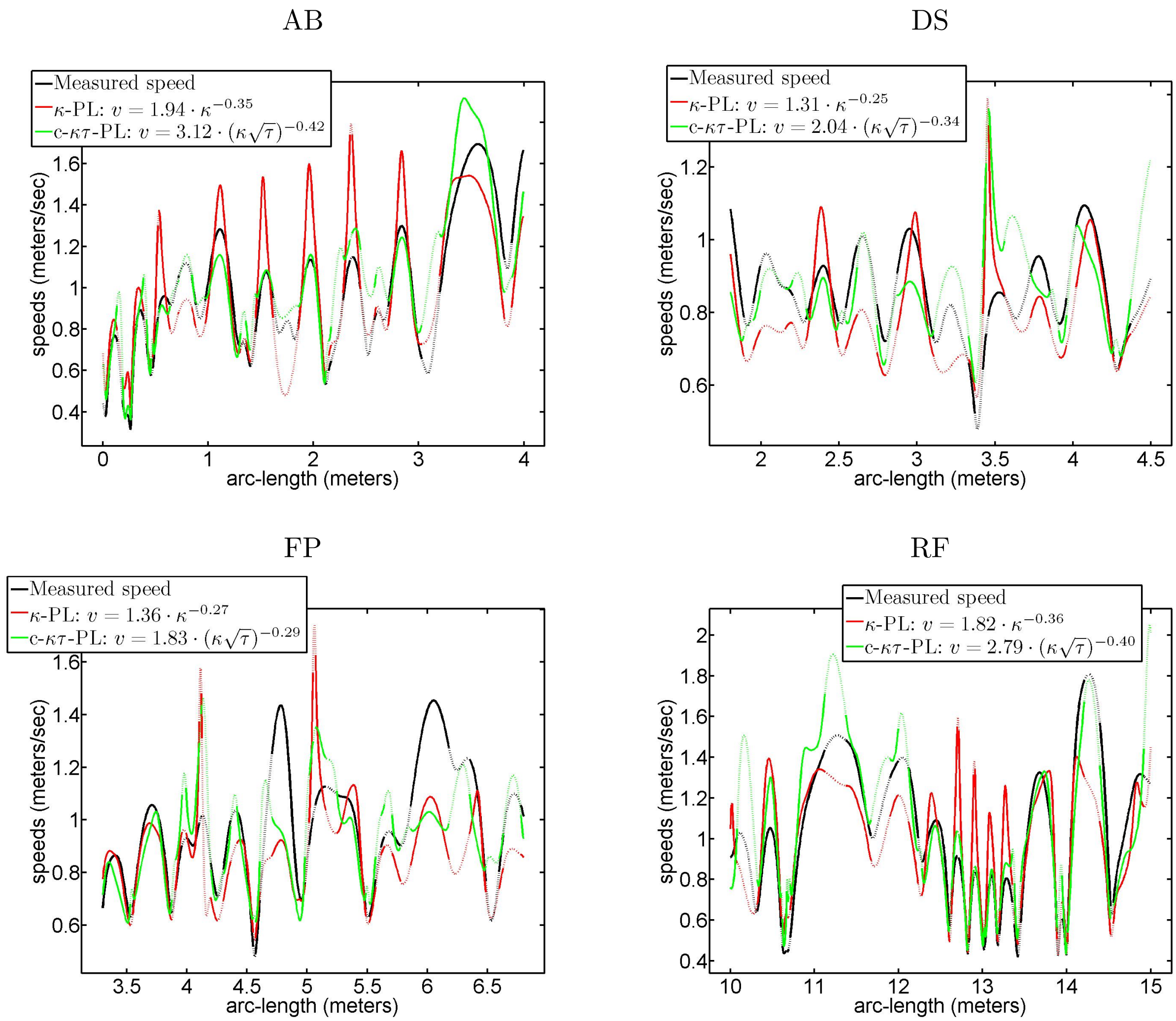

AB
DF
DS
ED
ES
FP
LK
MD
RF
SL
x
y
z
0.6
−0.4
0.6
−0.6
0
0.8



v (m/sec)
κ (1/m)
τ (1/m)
|τ| (1/m)
Arc−length (m)



True torsion
Approx torsion
50
0
−50
−100
torsion
0
3
6
9
12
15
time
5
0
−5
torsion
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
time


AB
Measured speed
κ-PL: $v = 1.94 \cdot \kappa^{-0.35}$
c-$\kappa\tau$-PL: $v = 3.12 \cdot (\kappa\sqrt{\tau})^{-0.42}$
speeds (meters/sec)
arc-length (meters)
DS
Measured speed
κ-PL: $v = 1.31 \cdot \kappa^{-0.25}$
c-$\kappa\tau$-PL: $v = 2.04 \cdot (\kappa\sqrt{\tau})^{-0.34}$
speeds (meters/sec)
arc-length (meters)
FP
Measured speed
κ-PL: $v = 1.36 \cdot \kappa^{-0.27}$
c-$\kappa\tau$-PL: $v = 1.83 \cdot (\kappa\sqrt{\tau})^{-0.29}$
speeds (meters/sec)
arc-length (meters)
RF
Measured speed
κ-PL: $v = 1.82 \cdot \kappa^{-0.36}$
c-$\kappa\tau$-PL: $v = 2.79 \cdot (\kappa\sqrt{\tau})^{-0.40}$
speeds (meters/sec)
arc-length (meters)