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ψ-Epistemic Models are Exponentially Bad at Explaining the Distinguishability of Quantum States

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The status of the quantum state is perhaps the most controversial issue in the foundations of quantum theory. Is it an epistemic state (state of knowledge) or an ontic state (state of reality)? In realist models of quantum theory, the epistemic view asserts that nonorthogonal quantum states correspond to overlapping probability measures over the true ontic states. This naturally accounts for a large number of otherwise puzzling quantum phenomena. For example, the indistinguishability of nonorthogonal states is explained by the fact that the ontic state sometimes lies in the overlap region, in which case there is nothing in reality that could distinguish the two states. For this to work, the amount of overlap of the probability measures should be comparable to the indistinguishability of the quantum states. In this Letter, I exhibit a family of states for which the ratio of these two quantities must be $\leq 2de^{-cd}$ in Hilbert spaces of dimension d that are divisible by 4. This implies that, for large Hilbert space dimension, the epistemic explanation of indistinguishability becomes implausible at an exponential rate as the Hilbert space dimension increases.

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The status of the quantum state is one of the most controversial issues in the foundations of quantum theory. Is it a state of knowledge (an *epistemic* state), or a state of physical reality (an *ontic state*)? Many realist interpretations of quantum theory employ ontic quantum states, but the rise of quantum information science has revived interest in the epistemic alternative because many of the puzzling phenomena employed in quantum information are explained quite naturally in terms of epistemic quantum states [1]. For example, consider the fact that two nonorthogonal quantum states cannot be perfectly distinguished. On the ontic view, the two states represent distinct arrangements of physical reality, so it is puzzling that this distinctness cannot be detected. However, on the epistemic view, a quantum state is represented by a probability measure over the physical properties of a system and nonorthogonal states correspond to overlapping probability measures. Indistinguishability is thus explained by the fact that preparations of the two quantum states sometimes result in the same physical properties of the system, in which case there is nothing existing in reality that could distinguish them.

Recently, several theorems have been proved aiming to show that the quantum state must be ontic [2–6]. These have been proved within the *ontological models* framework [7], which is a refinement of the hidden variable approach used to prove earlier no-go results, such as Bell's theorem [8] and the Kochen-Specker theorem [9]. Each of these theorems employs questionable auxiliary assumptions (see Ref. [10] for a review of these theorems and the criticisms of them). Without such assumptions, explicit counterexamples show that the epistemic view of quantum states can be maintained [11,12].

Within the ontological models framework, a model is ψ-ontic if the probability measures corresponding to every pair of pure quantum states have zero overlap, and it is ψ-epistemic otherwise. The recent no-go theorems aim to show that models must be ψ-ontic and the counterexamples show that, without auxiliary assumptions, ψ-epistemic models exist. However, being ψ-epistemic is an extremely permissive notion of what it means for the quantum state to be epistemic, since any nonzero amount of overlap between probability measures, however small, is enough to make a model ψ-epistemic. On the other hand, the ψ-epistemic explanation of indistinguishability requires that a significant part of the indistinguishability of $|\psi\rangle$ and $|\phi\rangle$ should be accounted for by the overlap of the corresponding probability measures, which means that the overlap should be comparable to a quantitative measure of the indistinguishability of $|\psi\rangle$ and $|\phi\rangle$. For this reason, it is interesting to bound the overlaps in ontological models, since this can be done without auxiliary assumptions, and it may still render ψ-epistemic explanations implausible.

Along these lines, Maroney showed that a measure of the overlap of probability measures must be smaller than $|\langle\phi|\psi\rangle|^2$ for some pairs of states in systems with Hilbert space of dimension $d \geq 3$ [13], which was later shown to follow from Kochen-Specker contextuality [14,15]. Following this, Barrett *et al.* showed that the ratio of an overlap measure derived from the variational distance to a comparable measure of the indistinguishability of quantum states must scale like $4/(d-1)$ in Hilbert spaces of dimension $d \geq 4$ [16]. In this Letter, I exhibit a family of states in Hilbert space dimensions d that are divisible by 4 for which the same ratio must be $\leq de^{-cd}$, where c is a positive constant. Hence, for large Hilbert space dimension, the ψ-epistemic explanation of indistinguishability becomes increasingly implausible at an exponential rate as the Hilbert space dimension increases.

We are interested in ontological models that reproduce the quantum predictions for prepare-and-measure experiments made on a system with Hilbert space $\mathbb{C}^d$, with no time evolution between preparation and measurement.

An ontological model for this set of experiments is an attempt to explain the quantum predictions in terms of some real physical properties—denoted λ and called ontic states—that exist independently of the experimenter. The set of ontic states is denoted Λ and we assume that it is a measurable space (Λ, Σ) with σ-algebra Σ. When the experimenter prepares a state $|\psi\rangle$, the preparation device might not fully control the ontic state, so $|\psi\rangle$ is associated with a probability measure $\mu_\psi\colon \Sigma \to [0,1]$. In general, the probability measure should be associated with the method of preparing $|\psi\rangle$ rather than with $|\psi\rangle$ itself because different preparation procedures for the same state may result in different probability measures. This phenomenon is known as *preparation contextuality* and it must occur for mixed states [17]. This complication does not affect the results presented here as the bounds derived apply equally well to any of the measures that can represent a preparation of $|\psi\rangle$.

Similarly, measurements might not reveal the value of λ exactly, so each element $|a\rangle$ of an orthonormal basis $M = \{|a\rangle, |b\rangle, \dots\}$ is associated with a positive measurable response function $\xi_M(a|\lambda)$, where $\xi_M(a|\lambda)$ is the probability of obtaining the outcome $|a\rangle$ when the measurement in the basis M is performed on the system and the ontic state is λ. Note that the response functions are allowed to depend on M to account for contextuality. Additionally, in order to form a well-defined probability distribution over the measurement outcomes, the response functions must satisfy

$$\sum_{|a\rangle \in M} \xi_M(a|\lambda) = 1. \tag{1}$$

Finally, the ontological model is required to reproduce the quantum predictions, which means that, for all pure states $|\psi\rangle$, all orthonormal bases M, and all $|a\rangle \in M$,

$$\int_\Lambda \xi_M(a|\lambda) d\mu_\psi = |\langle a|\psi\rangle|^2. \tag{2}$$

It will prove useful to define the sets of ontic states

$$\Gamma_M^a = \{\lambda | \xi_M(a|\lambda) = 1\}, \tag{3}$$

which always yield the outcome $|a\rangle$ with certainty when the measurement M is performed, and to note that, by Eq. (1), Γ_M^a and Γ_M^b are disjoint for $|a\rangle \neq |b\rangle$. Further, Eq. (2) implies

$$\int_\Lambda \xi_M(\psi|\lambda) d\mu_\psi = |\langle \psi|\psi\rangle|^2 = 1, \tag{4}$$

and in order to satisfy this $\xi_M(\psi|\lambda)$ must be equal to one on a set that is measure one according to μ_ψ. By definition, this must be a subset of Γ_M^ψ, so we also have $\mu_\psi(\Gamma_M^\psi) = 1$ for any $|\psi\rangle \in \mathbb{C}^d$ and any basis M that contains $|\psi\rangle$.

The goal of this work is to bound the overlap of probability measures in an ontological model, and this requires a quantitative measure. For this purpose, define the classical distance D_C between two quantum states in an ontological model to be the variational distance between the probability measures that represent them, i.e.,

$$D_C(\psi, \phi) = \sup_{\Gamma \in \Sigma} [\mu_\psi(\Gamma) - \mu_\phi(\Gamma)]. \tag{5}$$

This measure has the following operational interpretation. Suppose a system is prepared either in the state $|\psi\rangle$ or the state $|\phi\rangle$ with equal *a priori* probability. If you are told the exact value of λ then your optimal success probability of guessing which state was prepared is $\frac{1}{2}[1 + D_C(\psi, \phi)]$. When $D_C(\psi, \phi) = 1$, the ontic state λ effectively determines which quantum state was prepared uniquely, so the states have no overlap at the ontic level. Smaller values of D_C indicate a larger amount of overlap of the probability measures. For this reason, it is more convenient to work with the quantity

$$L_C(\psi, \phi) = 1 - D_C(\psi, \phi) = \inf_{\Gamma \in \Sigma} [\mu_\psi(\Gamma) + \mu_\phi(\Lambda \backslash \Gamma)], \tag{6}$$

which is called the classical overlap.

It is important to compare this quantity with a measure of the indistinguishability of quantum states that has an analogous interpretation, so that we are comparing like with like. Therefore, consider the trace distance D_Q between quantum states, which, for pure states, is given by

$$D_Q(\psi, \phi) = \sqrt{1 - |\langle \phi|\psi\rangle|^2}. \tag{7}$$

The operational interpretation of this quantity is the same as that of D_C except that now, instead of being told the ontic state λ, you must base your guess as to which state was prepared on the outcome of a quantum measurement. In this case, $\frac{1}{2}[1 + D_Q(\psi, \phi)]$ is your optimal success probability for guessing which of $|\psi\rangle$ or $|\phi\rangle$ was prepared. It is again more convenient to work with the quantity

$$L_Q(\psi, \phi) = 1 - D_Q(\psi, \phi) = 1 - \sqrt{1 - |\langle \phi|\psi\rangle|^2}, \tag{8}$$

which is called the quantum overlap.

In general, $L_C(\psi, \phi) \leq L_Q(\psi, \phi)$. This is because the response functions representing measurements provide only coarse-grained information about λ, so even the optimal quantum measurement may render $|\psi\rangle$ and $|\phi\rangle$ less distinguishable than they would be if you knew λ exactly. Naively, one might expect that the ψ-epistemic explanation of quantum indistinguishability requires that

$L_Q(\psi,\phi) = L_C(\psi,\phi)$, since then the indistinguishability of $|\psi\rangle$ and $|\phi\rangle$ would be entirely accounted for by the classical indistinguishability of μ_ψ and μ_ϕ. However, a certain amount of coarse graining of measurements should be expected in an ontological model. For example, if the theory is deterministic, i.e., if λ determines the outcomes of all measurements uniquely, then quantum measurements must only reveal coarse grained information about λ on pain of violating the uncertainty principle. Therefore, both the overlap of probability measures and the coarse-grained nature of measurements play a role in explaining quantum indistinguishability, and one should expect a balance between these two effects in a viable ψ-epistemic theory. It is only if $L_C(\psi,\phi) \ll L_Q(\psi,\phi)$ that the ψ-epistemic explanation is in trouble, since then the overlap plays almost no role in explaining indistinguishability. For this reason, the scaling of the ratio $k(\psi,\phi) = L_C(\psi,\phi)/L_Q(\psi,\phi)$, i.e., how quickly it tends to zero in Hilbert space dimension, is of more interest than its precise value.

The following proposition is the main tool used to bound the classical overlap.

Proposition 1: Let $\Gamma \in \Sigma$ be a set that is measure one according to μ_ϕ. Then $L_C(\psi,\phi) \leq \mu_\psi(\Gamma)$.

Proof.— Since $\mu_\phi(\Gamma) = 1$, $\mu_\phi(\Lambda\backslash\Gamma) = 0$. Hence, $L_C(\psi,\phi) = \inf_{\Omega\in\Sigma}[\mu_\psi(\Omega) - \mu_\phi(\Lambda\backslash\Omega)] \leq \mu_\psi(\Gamma) - \mu_\phi(\Lambda\backslash\Gamma) = \mu_\psi(\Gamma)$. ■

A few more definitions are required before proving the main results. Let V be a finite set of pure states in $\mathbb{C}^d$. Its orthogonality graph $G = (V,E)$ has the states as its vertices and there is an edge $(|a\rangle, |b\rangle) \in E$ iff $\langle a|b\rangle = 0$. For every such edge, there exists an orthonormal basis M such that $|a\rangle, |b\rangle \in M$. A covering set $\mathcal{M}$ is a finite set of orthonormal bases such that, for every $(|a\rangle, |b\rangle) \in E$, there exists an $M \in \mathcal{M}$ such that $|a\rangle, |b\rangle \in M$. Finally, the independence number $\alpha(G)$ of a graph $G = (V,E)$ is the cardinality of the largest subset $U \subseteq V$ of vertices such that if $u \in U$ and $(u,v) \in E$ then $v \notin U$; i.e., U contains no pairs of vertices that are connected by an edge.

Theorem 2: Let V be a finite set of pure states in $\mathbb{C}^d$ and let $G = (V,E)$ be its orthogonality graph. Then, for any pure state $|\psi\rangle \in \mathbb{C}^d$, in any ontological model,

$$\sum_{|a\rangle\in V} L_C(\psi,a) \leq \alpha(G). \tag{9}$$

Proof.— Let $\mathcal{M}$ be a covering set of bases. Then, for $|a\rangle \in V$, define the sets $\Gamma^a_{\mathcal{M}} = \cap_{\{M\in\mathcal{M} \| a\rangle\in M\}} \Gamma^a_M$. Now, $\Gamma^a_{\mathcal{M}}$ is a measure one set according to μ_a because it is the intersection of a finite collection of measure one sets. Proposition 1 then implies that $L_C(\psi,a) \leq \mu_\psi(\Gamma^a_{\mathcal{M}})$ for any $|\psi\rangle$. Hence, $\sum_{|a\rangle\in V} L_C(\psi,a) \leq \sum_{|a\rangle\in V} \mu_\psi(\Gamma^a_{\mathcal{M}})$. Now, let χ_a be the indicator function of $\Gamma^a_{\mathcal{M}}$, i.e., $\chi_a(\lambda) = 1$ if $\lambda \in \Gamma^a_{\mathcal{M}}$ and is zero otherwise. Then,

$$\sum_{|a\rangle\in V} \mu_\psi(\Gamma^a_{\mathcal{M}}) = \sum_{|a\rangle\in V} \int_\Lambda \chi_a(\lambda) d\mu_\psi \tag{10}$$

$$= \int_\Lambda \left[\sum_{|a\rangle\in V} \chi_a(\lambda)\right] d\mu_\psi \tag{11}$$

$$\leq \sup_{\lambda\in\Lambda} \left[\sum_{|a\rangle\in V} \chi_a(\lambda)\right], \tag{12}$$

where the last line follows from convexity. The last line is upper bounded by the maximum number of sets $\Gamma^a_{\mathcal{M}}$ that any given λ can be in as $|a\rangle$ varies over V. However, $\Gamma^a_{\mathcal{M}}$ and $\Gamma^b_{\mathcal{M}}$ are disjoint whenever $(|a\rangle, |b\rangle) \in E$ because they are subsets of Γ^a_M and Γ^a_M for some basis M and the latter are disjoint. Therefore, λ can only be in one of $\Gamma^a_{\mathcal{M}}$ or $\Gamma^b_{\mathcal{M}}$ whenever $|a\rangle$ and $|b\rangle$ are connected by an edge in the orthogonality graph. Therefore, the maximum number of such sets that λ can be in is upper bounded by the independence number of G. ■

Readers familiar with the literature on noncontextuality inequalities will note a similarity between theorem 2 and a result of [18,19], which shows that the maximal noncontextual value of a class of noncontextuality inequalities is bounded by the independence number of the orthogonality graph. This is not accidental as, up to the removal of measure zero sets, a model is Kochen-Specker noncontextual if and only if $\int_\Lambda \xi_M(a|\lambda) d\mu_\psi = \mu_\psi(\Gamma^a_{\mathcal{M}})$, where now $\mathcal{M}$ is the set of measurement bases involved in the noncontextuality inequality [10]. Thus, in a noncontextual model $\mu_\psi(\Gamma^a_{\mathcal{M}})$ is the total probability of obtaining outcome $|a\rangle$ when measuring a system prepared in the state $|\psi\rangle$, so the sum of such probabilities is bounded in the same way.

Corollary 3: Let V be a finite set of pure states in $\mathbb{C}^d$ and let $G = (V,E)$ be its orthogonality graph. For any pure state $|\psi\rangle \in \mathbb{C}^d$ let

$$\bar{k}(\psi) = \frac{1}{|V|} \sum_{|a\rangle\in V} \frac{L_C(\psi,a)}{L_Q(\psi,a)}, \tag{13}$$

be the average ratio of classical to quantum overlaps in an ontological model. Then,

$$\bar{k}(\psi) \leq \frac{2\alpha(G)}{|V| \min_{|a\rangle\in V} |\langle a|\psi\rangle|^2}, \tag{14}$$

where $\alpha(G)$ is the independence number of G.

Proof.— For $0 \leq x \leq 1$, note that $1 - \sqrt{1-x} \geq x/2$, and, hence, $L_Q(\psi,\phi) \geq \frac{1}{2}|\langle\psi|\phi\rangle|^2$. Thus,

$$\bar{k}(\psi) \leq \frac{2}{|V|} \sum_{|a\rangle\in V} \frac{L_C(\psi,a)}{|\langle\phi|a\rangle|^2} \leq \frac{2\sum_{|a\rangle\in V} L_C(\psi,a)}{|V| \min_{|a\rangle\in V} |\langle a|\psi\rangle|^2}.$$

The result then follows from Theorem 2. ■

Theorem 4: When d is divisible by 4, there exists a set of pure states V in $\mathbb{C}^d$ and a state $|\psi\rangle \in \mathbb{C}^d$ such that, in any ontological model,

$$\bar{k}(\psi) = \sum_{|a\rangle \in V} \frac{L_C(\psi, a)}{L_Q(\psi, a)} \leq de^{-cd}, \tag{15}$$

where c is a positive constant.

Proof.— The construction is based on the Hadamard states and the Frankl-Rödl theorem, which are commonly used in quantum information theory (see, e.g., [20–22]).

Let V be the set of Hadamard states, i.e., the set of vectors of the form $1/\sqrt{d}(\pm 1, \pm 1, \ldots, \pm 1)^T$ and let $|\psi\rangle = (1, 0, 0, \ldots, 0)^T$. There are 2^d vectors in V and each vector $|a\rangle \in V$ satisfies $|\langle a|\psi\rangle|^2 = 1/d$, so this is also the minimum. The orthogonality graph $G = (V, E)$ of V is known in the literature as a Hadamard graph [22]. It follows from the Frankl-Rödl theorem (theorem 1.11 in [23]) that, for d divisible by 4, there exists an $\epsilon > 0$ such that $\alpha(G) \leq (2 - \epsilon)^d$ and thus corollary 3 implies

$$\bar{k}(\psi) \leq \frac{2(2-\epsilon)^d}{2^d \frac{1}{d}} = 2de^{-cd}, \tag{16}$$

where $c = \ln 2 - \ln(2 - \epsilon)$ is a positive constant. ∎

Remarks.—When d is not divisible by 4, the Hadamard states of dimension $\tilde{d} = 4\lfloor d/4 \rfloor$ can be embedded in $\mathbb{C}^d$ by setting the remaining components of the vectors to zero. This yields $\bar{k}(\psi) \leq 2\tilde{d}e^{-c\tilde{d}}$ for all Hilbert space dimensions.

Since $\bar{k}(\psi)$ is an average over V, there must exist at least one $|a\rangle \in V$ such that $k(\psi, a) \leq \bar{k}(\psi) \leq 2\tilde{d}e^{-c\tilde{d}}$. It is fairly reasonable to assume that $k(\psi, a)$ only depends on $|\langle a|\psi\rangle|^2$, in which case the bound $k(\psi, a) \leq 2\tilde{d}e^{-c\tilde{d}}$ would apply to all $|a\rangle \in V$, since $|\langle a|\psi\rangle|^2 = 1/d$ is the same for the states used in this construction.

Conclusions.—In this Letter, I have exhibited a family of states for which the ratio of classical to quantum overlaps must be $\leq 2e^{-cd}$ in Hilbert space dimensions d that are divisible by 4, and where c is a positive constant. This represents an exponential improvement in asymptotic scaling over the previous result of $4/(d-1)$ [16]. This presents a severe problem for the ψ-epistemic explanation of quantum indistinguishability, as the portion of the indistinguishability that can be accounted for by the overlap of probability measures decreases rapidly in large Hilbert space dimension. It would be interesting to further pin down the value of c to see which result gives the best bound for small Hilbert space dimension, for which the results may be amenable to experimental test. Similarly, the connection to contextuality could be further exploited to derive additional bounds.

Finally, Montina has derived an upper bound on the classical communication complexity of simulating a qubit identity channel using the existence of a ψ-epistemic ontological model for $d = 2$ [24]. It would be interesting to determine if lower bounds on this task in higher dimensions could be derived from overlap bounds.

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