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Dan Kovenock
Chapman University, kovenock@chapman.edu

Brian Roberson
Purdue University

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Dan Kovenock† and Brian Roberson‡

Abstract

In this paper, we generalize the General Lotto game (budget constraints satisfied in expectation) and the Colonel Blotto game (budget constraints hold with probability one) to allow for battlefield valuations that are heterogeneous across battlefields and asymmetric across players, and for the players to have asymmetric resource constraints. We completely characterize Nash equilibrium in the generalized version of the General Lotto game and then show how this characterization can be applied to identify equilibria in the Colonel Blotto version of the game. In both games, we find that there exist sets of non-pathological parameter configurations of positive Lebesgue measure with multiple payoff nonequivalent equilibria.

Keywords: Colonel Blotto game, General Lotto game, Multi-battle contest, Redistributive politics, All-pay auction

JEL Classification: C72, D72, D74

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†Dan Kovenock, Economic Science Institute, Argyros School of Business and Economics, Chapman University, One University Drive, Orange, CA 92866 USA t:714-628-7226 E-mail: kovenock@chapman.edu

‡Brian Roberson, Purdue University, Department of Economics, Krannert School of Management, 403 W. State Street, West Lafayette, IN 47907 USA t: 765-494-4531 E-mail: brobers@purdue.edu (Correspondent)

1 Introduction

The Colonel Blotto game is a two-player resource allocation game in which each player is endowed with a level of a resource to allocate across a set of battlefields, within each battlefield the player that allocates the higher level of the resource wins the battlefield, and each player's payoff is the sum of the valuations of the battlefields won. This simple game, which originates with Borel (1921), illustrates some of the fundamental strategic considerations that arise in conflicts or competition involving multi-dimensional resource allocation such as political campaigns, research and development competition (where innovation may involve obtaining a collection of interrelated patents), and military and systems defense.

In this paper, we examine generalized formulations of the General Lotto (budget constraints hold in expectation) and Colonel Blotto (budget constraints hold with probability one) games in which battlefield (or component contest) valuations may be heterogeneous across battlefields and asymmetric across players, and the players may face asymmetric resource constraints.[1] We completely characterize Nash equilibrium in the General Lotto game. This generalizes the symmetric resource constraint versions of the General Lotto game examined in Bell and Cover (1980), Myerson (1993), Sahuguet and Persico (2006), and Hart (2008), where the valuation in the single battlefield[2] is symmetric across players, as well as Kovenock and Roberson (2008) and Washburn (2013), in which battlefield valuations are heterogeneous across battlefields but symmetric across players.[3] We then show how this characterization can be applied over a subset of the parameter space to identify equilibria in generalizations of the Colonel Blotto game (in which budget constraints hold with probability one) that have hitherto been unexplored.

In contrast to existing constant-sum formulations of the General Lotto and Colonel Blotto games, we find that uniqueness of the Nash equilibrium sets of univariate marginal distri-

[1]Other notable formulations of Blotto-type games include Friedman (1958), which introduces a version of the game with the lottery contest success function (see also Robson (2005)), and Hart (2008) which introduces a version of the game in which resource allocations are restricted to be nonnegative integers (see also Hortala-Vallve and Llorente-Saguer (2012) and Dziubiński (2013)).

[2]The models in these papers may either be interpreted as having a single battlefield, where each player's allocation of the resource to this battlefield is drawn from his univariate distribution function that is budget balancing on average, or as a continuum of homogeneous battlefields, where each point in the support of a player's univariate distribution function represents an allocation of the resource to a battlefield and the budget constraint is on the average resource allocation. In this paper, we focus on the first interpretation.

[3]Following Myerson (1993), the General Lotto game has become a benchmark model of redistributive politics. Related political economy applications include Lizzeri and Persico (2001, 2005), Sahuguet and Persico (2006), Roberson (2008), and Crutzen and Sahuguet (2009). See also Laslier and Picard (2002) for a similar application of the Colonel Blotto game.

butions does not extend to the generalized (non-constant-sum) versions of the games examined here. We also show that in the generalized versions of both the General Lotto and Colonel Blotto games there exist sets of non-pathological parameter configurations of positive Lebesgue measure with multiple payoff nonequivalent equilibria.

To provide intuition for why multiple payoff nonequivalent equilibria arise, consider a single all-pay auction in which one player, A, has a low valuation v_A and one player B has a high valuation v_B, i.e. $v_B > v_A$. It is well known that in the unique Nash equilibrium[4] B's expected payoff is $v_B - v_A$ and A's expected payoff is 0. Now suppose v_B decreases and v_A increases maintaining the inequality $v_B \geq v_A$. In this case, the contest becomes more competitive, which results in higher equilibrium expenditures for both players and a lower equilibrium expected payoff for the high valuation player (B). In the General Lotto game, the two players A and B each have a (normalized) value for each battlefield j, $v_{i,j} > 0$, where $\sum_{j=1}^{n} v_{i,j} = 1$ for each player $i = A, B$. The players are also resource constrained, where the expectation of player i's total expenditure across battlefields must be less than or equal to X_i, for $i = A, B$, and player B is assumed to have a resource advantage, $X_B \geq X_A > 0$. Given an equilibrium mixed strategy of the rival player, the budget constrained optimization problem of player i yields a Lagrange multiplier λ_i – the shadow value of an increment to i's budget – that serves as a unit cost to an incremental allocation to each battlefield j. Player i maximizes his payoff by acting as if he is playing in an all-pay auction in each battlefield with constant unit cost equal to the multiplier λ_i. Because of invariance of behavior with respect to affine transformations of utility, this implies that, in an equilibrium generating the pair of player multipliers (λ_A, λ_B), the two players A and B behave in battlefield j as as if they are engaged in an all-pay auction with constant unit cost equal to one and valuations $\frac{v_{A,j}}{\lambda_A}$ and $\frac{v_{B,j}}{\lambda_B}$, respectively. In this all-pay auction, player A has a higher valuation than player B if $\frac{v_{A,j}}{v_{B,j}} > \frac{\lambda_A}{\lambda_B} \equiv \gamma$ and B has a higher value than A if the inequality is reversed. This implies that in each equilibrium of the General Lotto game there exists a cut point, γ, equal to the ratio of the two multipliers λ_A and λ_B induced by the equilibrium, such that for each battlefield j with $\gamma > \frac{v_{A,j}}{v_{B,j}}$ player B utilizes a strategy similar to that of the high valuation player in the all-pay auction and player A utilizes a strategy similar to that of the low valuation player in the all-pay auction. Similarly, for each battlefield j with $\frac{v_{A,j}}{v_{B,j}} > \gamma$, the roles are reversed with player A being the high valuation player and B being the low valuation player. Multiple equilibria arise in this setting because there exist multiple pairs of shadow values (λ_A, λ_B), or alternatively cut points γ, that generate budget-balancing strategies that are mutual best

[4]See Baye, Kovenock and de Vries (1996) for further details.

responses and therefore form an equilibrium. In moving across equilibria, as γ increases the set of battlefields for which $\frac{v_{A,j}}{v_{B,j}} > \gamma$ (weakly) shrinks. Furthermore, in the remaining set of battlefields for which $\frac{v_{A,j}}{v_{B,j}} > \gamma$ each battlefield becomes more competitive, thereby increasing both players' expected expenditures in those battlefields and lowering player A's expected payoff for those battlefields. Conversely, each battlefield j with $\gamma > \frac{v_{A,j}}{v_{B,j}}$ becomes less competitive as γ increases, thereby decreasing both players' expected expenditures in those battlefields and increasing player B's expected payoff across such battlefields. Therefore, in moving from an equilibrium in which $\frac{\lambda_A}{\lambda_B} = \gamma$ to an equilibrium in which $\frac{\lambda_A}{\lambda_B} = \gamma' > \gamma$, the increased allocation to battlefields j with higher values of $\frac{v_{A,j}}{v_{B,j}}$ exactly offsets the reduced allocation to battlefields with lower values of $\frac{v_{A,j}}{v_{B,j}}$, so that budget balance holds.

When battlefield valuations are homogeneous across battlefields and symmetric across players, it is known that for sufficiently asymmetric resource endowments the relationship between the equilibria in the General Lotto and Colonel Blotto games breaks down.[5] In Figure 1, the dashed line illustrates the (resource endowment) weak player A's maximal expected proportion of battlefields won in the General Lotto game with n battlefields, and the solid curve is the weak player's expected proportion of battlefields won in the corresponding Colonel Blotto game, both as a function of the ratio of the weak player's and the strong player's budgets ($\frac{X_A}{X_B}$). Note that the point of departure between the weak player's expected payoffs in the General Lotto game and those arising in the Colonel Blotto game occurs at a ratio $\frac{X_A}{X_B} = \frac{2}{n}$. As the ratio of the weak player's budget to the strong player's budget ($\frac{X_A}{X_B}$) decreases the weak player focuses his resources in smaller and smaller random subsets of battlefields. When the ratio $\frac{X_A}{X_B}$ is less than or equal to $\frac{2}{n}$, the fact that budgets bind with certainty in the Colonel Blotto game results in a situation in which the weak player has exhausted his ability to shrink the size of the subset of battlefields in which he focuses his resources. This binding constraint is what causes the weak player's expected payoffs to decrease below the dashed line. Furthermore, once $\frac{X_A}{X_B}$ is less than $\frac{1}{n}$, the strong player has the ability to outbid the weak player on each and every battlefield. This issue arises because, with sufficiently asymmetric resource endowments, the Colonel Blotto game's binding budget constraint makes it infeasible for the weak player to play a multi-dimensional mixed strategy that is budget-balancing with certainty and that provides the same set of univariate marginals as in the corresponding General Lotto game in which the budget only holds in expectation.

For the generalized version of the Colonel Blotto game examined in this paper, we provide

[5] For asymmetric resource endowments, the characterization of the equilibrium payoffs in the General Lotto game is due Sahuguet and Persico (2006) and for the Colonel Blotto game is due to Roberson (2006).

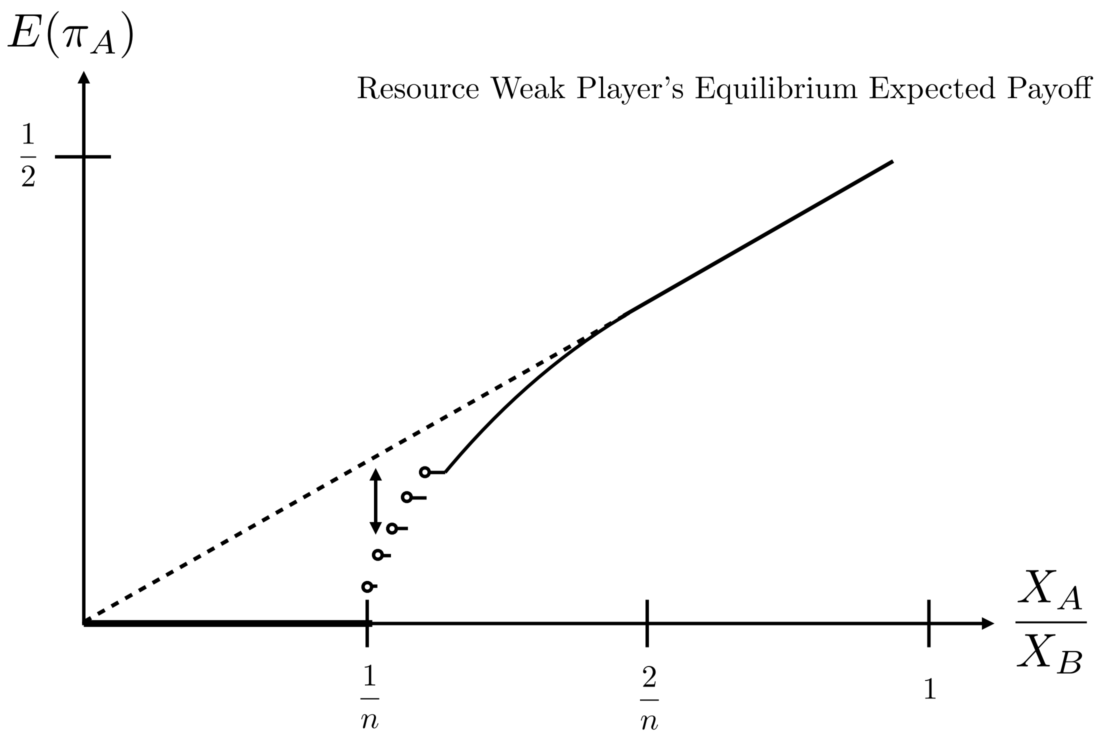

Figure 1: The weak player A's equilibrium expected payoff in the General Lotto (dashed line) and Colonel Blotto (solid line) games as a function of the ratio of the weak player's resource endowment (X_A) to the strong player's resource endowment (X_B)

a sufficient condition for the existence of an equilibrium in which each player's univariate marginal distributions coincide with those of an equilibrium in the corresponding General Lotto version of the game and a sufficient condition for the equilibrium sets of univariate marginal distributions in the two versions of the game to differ across all equilibria. Because the battlefield valuations may be heterogeneous across battlefields and asymmetric across players, this condition involves the players' resource endowments, as well as their n-tuples of battlefield valuations. Furthermore, equilibrium univariate marginal distributions in the Colonel Blotto and General Lotto games may differ in the case of symmetric resource endowments, if the players have sufficiently different battlefield valuation n-tuples. That is, asymmetries in players' valuations alone may be sufficient for the equilibria in the Colonel Blotto and General Lotto games to differ.

Table 1 summarizes the branch of the Colonel Blotto literature that assumes an auction contest success function, a finite number of battlefields, resource endowments that are continuously divisible and are use-it-or-lose-it in the sense that unused resources have no value (i.e. the per unit cost of allocating the resource is 0 up to the budget constraint),[6] and that each player's payoff is the sum of the battlefield valuations in the battlefields won.[7] The type of player objective varies across rows and the cost structure varies across columns. In describing the type of player objective, linear pure count refers to games in which each player's payoff is the sum of the battlefield valuations in the battlefields won, where battlefield valuations are homogeneous across battlefields and symmetric across players, so that each player's payoff is linear in the number, or pure count, of battlefields won. Linear heterogeneous symmetric (asymmetric) is similar, except that battlefield valuations are now heterogeneous across battlefields but symmetric (asymmetric) across players, and each player's payoff is equal to the sum of the player's valuations in the battlefields won.

As shown in Table 1, this paper provides a partial result for each of the three checked cells: linear heterogeneous symmetric objective with asymmetric budget constraints, and linear heterogeneous asymmetric objective with both symmetric and asymmetric budget constraints. Section 4 provides a detailed discussion of the literature in Table 1, including a discussion of the relationship between the equilibrium joint distributions that have been constructed for those formulations and the distributions that we utilize in our analysis of the Generalized Colonel Blotto game. As a point of reference, Table 1 also includes the

[6]For alternative cost functions see Kvasov (2007) and Roberson and Kvasov (2012).

[7]For alternative definitions of success see Szentes and Rosenthal (2003a, 2003b), Golman and Page (2009), Kovenock and Roberson (2010), Tang, Shoham, and Lin (2010), Rinott, Scarsini, and Yu (2012), and Barelli, Govindan, and Wilson (2014).

<table>
<tr><th>Costs →
Objective ↓</th><th>Symmetric Budget
Use-it-or-lose-it Resources</th><th>Asymmetric Budget
Use-it-or-lose-it Resources</th></tr>
<tr><td>Linear Pure-Count</td><td><u>Continuous</u>
Borel & Ville (1938) [$n = 3$]
Gross & Wagner (1950) [$n \geq 2$]
Weinstein (2012) [$n \geq 3$]
<u>Discrete</u>
Hart (2008)</td><td><u>Continuous</u>
Gross & Wagner (1950) [$n = 2$]
Roberson (2006) [$n \geq 3$]
Macdonell & Mastronardi (2015) [$n = 2$]
<u>Discrete</u>
Hart (2008) [partial result]</td></tr>
<tr><td>Linear Heterogeneous
Symmetric</td><td><u>Continuous</u>
Gross (1950)
Laslier (2002)
Thomas (2012)
<u>Discrete</u>
Hortala-Vallve &
Llorente-Saguer (2012)
[partial result]</td><td><u>Continuous</u>
Gross & Wagner (1950) [$n = 2$]
✓
[partial result]
Schwartz et al. (2014) [$n \geq 3$]
[partial result]
Macdonell & Mastronardi (2015) [$n = 2$]</td></tr>
<tr><td>Linear Heterogeneous
Asymmetric</td><td><u>Continuous</u>
✓
[partial result]
<u>Discrete</u>
Hortala-Vallve &
Llorente-Saguer (2012)
[partial result]</td><td><u>Continuous</u>
✓
[partial result]</td></tr>
</table>

Table 1: Blotto game variations with linear count objectives and use-it-or-lose-it resources

corresponding results for the discrete version of the Colonel Blotto game (in which the feasible sets of bids of the players are discrete). Lastly, the case of $n = 2$ places a severe restriction on the set of available joint distributions, which leads to a distinct set of strategic considerations. For both symmetric and asymmetric budgets and both homogeneous and heterogeneous (symmetric) battlefield valuations, the case of $n = 2$ was first examined in Gross and Wagner (1950). Macdonell and Mastronardi (2015) complete the characterization for the case of n = 2 and provide a characterization of equilibrium in a version of the heterogeneous (symmetric) battlefield valuation game with nonlinear asymmetric budgets.

The rest of the paper is organized as follows. In section 2 we provide a description of the General Lotto model. Section 3 provides the results for the General Lotto game and an example of multiple payoff nonequivalent equilibria. Section 4 examines the Generalized Colonel Blotto game and the relationship between our results and the existing literature. Section 5 concludes.

2 The Model

Two players, A and B, simultaneously allocate a resource across a finite number, $n \geq 3$, of independent battlefields. Battlefield j has a (normalized) value of $v_{i,j} > 0$, where $\sum_{j=1}^{n} v_{i,j} = 1$, for player $i = A, B$. Each player has a fixed level of the available resource (or budget), X_i for $i = A, B$. Let $X_B \geq X_A > 0$, and let $\mathbf{x}_i$ denote player i's allocation of the resource $(x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,n})$ across the n-battlefields. In each battlefield the player with the higher resource expenditure wins, and in the event of a tie[8] each player wins the battlefield with probability $\frac{1}{2}$.

In each battlefield j the payoff to player i for a resource expenditure of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} v_{i,j} & \text{if } \; x_{i,j} > x_{-i,j} \\ \frac{v_{i,j}}{2} & \text{if } \; x_{i,j} = x_{-i,j} \\ 0 & \text{if } \; x_{i,j} < x_{-i,j} \end{cases}$$

Each player's payoff across all n battlefields is the sum of the payoffs across the individual

[8] The choice of tie-breaking rule is not critical for any of our results. This is generally true in the General Lotto game and is true for the corresponding parameter ranges covered in our treatment of the Colonel Blotto game. More generally, in the Colonel Blotto game the choice of a tie-breaking rule is important for the parameter range in which the correspondence between General Lotto and Colonel Blotto breaks down. In this range, the tie-breaking rule in the Colonel Blotto game must be chosen judiciously in order to avoid the need for ϵ-equilibrium arguments. See Roberson (2006).

battlefields.

We now define the Generalized Colonel Blotto and General Lotto games.

The Generalized Colonel Blotto Game

The level of the resource allocated to each battlefield must be nonnegative. For player i, the set of feasible resource allocations across the n battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}^n_+ \middle| \sum_{j=1}^{n} x_{i,j} \leq X_i \right\}.$$

A mixed strategy, which we term a *distribution of resources*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \to [0,1]$ with support (denoted $\mathrm{Supp}(P_i)$) contained in player i's set of feasible bids $\mathfrak{B}_i$ and with one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each battlefield j. Player i's allocation of the resource across the n battlefields is a random n-tuple drawn from the n-variate distribution function P_i.

The Generalized Colonel Blotto game, which we label

$$CB\big\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\big\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of the resource subject to their budget constraints, each battlefield is won by the player that allocates the higher level of the resource to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and each player's payoff is the sum of the values of the individual battlefields that he wins.

The Generalized General Lotto Game

In the Generalized General Lotto game, a mixed strategy for player i is still an n-variate distribution function $P_i : \mathbb{R}^n_+ \to [0,1]$ with one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each battlefield j. And, the level of the resource allocated to each battlefield must be nonnegative, $F_{i,j}(x) = 0$ for all $x < 0$. The General Lotto game differs from the Colonel Blotto game in that each player i's budget must hold in expectation, $\sum_{j=1}^n E_{F_{i,j}}(x) \leq X_i$.

The Generalized General Lotto game, which we label

$$GL\big\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\big\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of the resource subject to their budget constraints, each battlefield is won by the player that allocates the higher level of the resource to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and each player's payoff is the sum of the values of the individual battlefields that he wins.

3 Generalized General Lotto Results

In order to provide intuition for our main results, we begin this section with a few informal insights regarding the necessary conditions for equilibrium in the Generalized General Lotto game. First, note that any joint distribution may be broken into a set of univariate marginal distribution functions and an n-copula, the function that maps the univariate marginal distribution functions into a joint distribution function.[9] Given that player $-i$'s strategy is given by the n-variate distribution function P_{-i} with the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^n$, note that player i's expected payoff[10] for any feasible n-variate distribution function P_i with the set of univariate marginal distribution functions $\{F_{i,j}\}_{j=1}^n$ is

$$\pi_i\left(\{F_{i,j}, F_{-i,j}\}_{j=1}^n\right) = \sum_{j=1}^n \left[\int_0^\infty v_{i,j} F_{-i,j}\left(x_{i,j}\right) dF_{i,j}\right]. \tag{1}$$

Recalling that the budget constraint holds in expectation, player i's constrained optimization problem may be written as

$$\max_{\left\{\{F_{i,j}\}_{j=1}^n\right\}} \sum_{j=1}^n \left[\int_0^\infty \left[v_{i,j} F_{-i,j}\left(x_{i,j}\right) - \lambda_i x_{i,j}\right] dF_{i,j}\right] + \lambda_i X_i, \tag{2}$$

where λ_i is the multiplier on player i's expected resource expenditure constraint. Note that for the Generalized General Lotto game both the expected payoff in (1) and the budget constraint depend on only the sets of univariate marginal distribution functions and not the joint distribution function. That is, in the Generalized General Lotto game, any n-copula

[9] See Nelsen (1999) or Schweizer and Sklar (1983) for an introduction to copulas.

[10] This expression is for the case in which none of player $-i$'s univariate marginal distributions contains a mass point.

may be used to map a set of equilibrium univariate marginal distribution functions into an equilibrium joint distribution function. However, because the budget constraint in the Generalized Colonel Blotto game holds with probability one, the choice of a set of univariate marginal distribution functions is constrained in the sense that there must exist an n-copula for which the resulting joint distribution is budget-balancing with probability one. We will return to this issue in Section 4.

For each $j = 1, \ldots, n$ the corresponding first-order condition provides a necessary condition for equilibrium and is given by

$$\frac{d}{dx_{i,j}} \left[v_{i,j} F_{-i,j}\left(x_{i,j}\right) - \lambda_i x_{i,j} \right] = 0. \tag{3}$$

Dividing both sides of (3) by $\lambda_i > 0$, we see that (3) is equivalent to the necessary condition for a single all-pay auction, without a budget constraint, and in which player i's value for the prize is $\frac{v_{i,j}}{\lambda_i}$. In such an all-pay auction, the unique equilibrium[11] is described as follows. If $\frac{v_{i,j}}{\lambda_i} \geq \frac{v_{-i,j}}{\lambda_{-i}}$, then

$$\begin{array}{ll} F_{-i,j}\left(x\right) = \left(\frac{\frac{v_{i,j}}{\lambda_i} - \frac{v_{-i,j}}{\lambda_{-i}}}{\frac{v_{i,j}}{\lambda_i}}\right) + \frac{x}{\frac{v_{i,j}}{\lambda_i}} & x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}}\right] \\ F_{i,j}\left(x\right) = \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}}} & x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}}\right]. \end{array} \tag{4}$$

Next, to solve for the multipliers (λ_A, λ_B), let $\gamma \equiv \frac{\lambda_A}{\lambda_B}$ and let $\Omega_A(\gamma)$ denote the set of battlefields in which $\frac{v_{A,j}}{v_{B,j}} > \gamma$, or equivalently $\frac{v_{A,j}}{\lambda_A} > \frac{v_{B,j}}{\lambda_B}$. The combination of (4) and budget-balance implies the following system of equations, which we refer to as $(\star)$:

$$\sum_{j \in \Omega_A(\gamma)} \frac{v_{B,j}}{2\lambda_B} + \sum_{j \notin \Omega_A(\gamma)} \frac{\left(\frac{v_{A,j}}{\lambda_A}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B}\right)} = X_A \tag{5}$$

$$\sum_{j \in \Omega_A(\gamma)} \frac{\left(\frac{v_{B,j}}{\lambda_B}\right)^2}{2\left(\frac{v_{A,j}}{\lambda_A}\right)} + \sum_{j \notin \Omega_A(\gamma)} \frac{v_{A,j}}{2\lambda_A} = X_B. \tag{6}$$

λ_A^* and λ_B^* are implicitly defined by equations (5) and (6), henceforth referred to as a solution to system $(\star)$.

Our first result is that, for every feasible configuration of battlefield values $\{v_{A,j}, v_{B,j}\}_{j=1}^n$

[11] For more details see Baye, Kovenock, and de Vries (1996).

and resource endowments $\{X_A, X_B\}$, there exists at least one solution to system ($\star$).

Proposition 1. *For any feasible configuration of battlefield values $\{v_{A,j}, v_{B,j}\}_{j=1}^n$ and resource endowments $\{X_A, X_B\}$ there exists a solution to system ($\star$). If $v_{A,j} = v_{B,j}$ for all j, then there exists a unique solution to system ($\star$).*

Proof. We begin with the proof that there exists a solution to system ($\star$), and then examine the issue of uniqueness in constant-sum versions of the game. Recall that $\gamma = \frac{\lambda_A}{\lambda_B}$ and that $\Omega_A(\gamma)$ denotes the set of battlefields in which $\frac{v_{A,j}}{v_{B,j}} > \gamma$. Let $\partial\Omega_A(\gamma)$ denote the (possibly empty) set of battlefields for which $\frac{v_{A,j}}{v_{B,j}} = \gamma$ and let $\widehat{\Gamma} \subset \mathbb{R}_+$ denote the set of γ such that $\partial\Omega_A(\gamma) \neq \emptyset$ — that is the set of γ that satisfy $\gamma = \frac{v_{A,j}}{v_{B,j}}$ for some $j \in \{1, \ldots, n\}$. $\widehat{\Gamma}$ corresponds to the set of γ at which $\Omega_A(\gamma)$ 'changes.'

Let

$$\underline{\gamma} \equiv \min\left\{\min_j\left\{\frac{v_{A,j}}{v_{B,j}}\right\}, \frac{X_B}{X_A}\left(\sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1}\right\} > 0$$

and let

$$\overline{\gamma} \equiv \max\left\{\frac{X_B}{X_A}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}, \max_j\left\{\frac{v_{A,j}}{v_{B,j}}\right\}\right\} < \infty.$$

We first bound the set of potential solutions by showing that if there exists a solution to system ($\star$), then $\gamma \in \left[\underline{\gamma}, \overline{\gamma}\right]$.

Suppose that $\gamma \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$. Because $\Omega_A(\gamma)$ is the set of j for which $\frac{v_{A,j}}{v_{B,j}} > \gamma$, it follows that if $\gamma \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then there exists no j such that $\frac{v_{A,j}}{v_{B,j}} > \gamma$ and $\Omega_A(\gamma) = \emptyset$. Then, because $\Omega_A(\gamma) = \emptyset$ equations (5) and (6) may be written as

$$\frac{\lambda_B}{2\lambda_A^2}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} = X_A \quad \text{and} \quad \frac{1}{2\lambda_A} = X_B. \tag{7}$$

The unique solution to the system in (7) is $\lambda_A^* = \frac{1}{2X_B}$, $\lambda_B^* = \frac{X_A}{2X_B^2}\left(\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}\right)^{-1}$, and $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}$. If $\frac{X_B}{X_A}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, then there exists a unique solution to system ($\star$) for $\gamma \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, $\gamma^* = \frac{X_B}{X_A}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}$. If $\frac{X_B}{X_A}\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} < \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, then there exists no solution to system ($\star$) with $\gamma \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, and so, in any solution to system ($\star$) $\gamma < \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$.

Now, suppose that $\gamma < \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$. Because $\Omega_A(\gamma)$ is the set of j for which $\frac{v_{A,j}}{v_{B,j}} > \gamma$, it follows that if $\gamma < \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then $\frac{v_{A,j}}{v_{B,j}} > \gamma$ for all j and $\Omega_A(\gamma) = \{1, 2, \ldots, n\}$. Then, because $\Omega_A(\gamma) = \{1, 2, \ldots, n\}$ equations (5) and (6) may be written as

$$\frac{1}{2\lambda_B} = X_A \quad \text{and} \quad \frac{\lambda_A}{2\lambda_B^2} \sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} = X_B \tag{8}$$

The unique solution to the system in (8) is $\lambda_A^* = \frac{X_B}{2X_A^2} \left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$, $\lambda_B^* = \frac{1}{2X_A}$, and $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A} \left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$. If $\frac{X_B}{X_A} \left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} < \min_j \{ \frac{v_{A,j}}{v_{B,j}} \}$, then there exists a unique solution to system $(\star)$ for $\gamma < \min_j \{ \frac{v_{A,j}}{v_{B,j}} \}$, $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A} \left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$. If $\frac{X_B}{X_A} \left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} \geq \min_j \{ \frac{v_{A,j}}{v_{B,j}} \}$, then there exists no solution to system $(\star)$ with $\gamma < \min_j \{ \frac{v_{A,j}}{v_{B,j}} \}$, and so, in any solution to system $(\star)$ $\gamma \geq \min_j \{ \frac{v_{A,j}}{v_{B,j}} \}$. This completes the proof that if there exists a solution to system $(\star)$, then $\gamma \in \left[\underline{\gamma}, \overline{\gamma}\right]$.

We now show that for any feasible configuration of battlefield values $\{v_{A,j}, v_{B,j}\}_{j=1}^n$ and resource endowments $\{X_A, X_B\}$ there exists a solution to system $(\star)$ with $\gamma \in \left[\underline{\gamma}, \overline{\gamma}\right]$. Because $\gamma \in \left[\underline{\gamma}, \overline{\gamma}\right]$, it follows directly that $\lambda_A, \lambda_B \in (0, \infty)$. Multiplying both sides of (5) by λ_B and both sides of (6) by λ_A yields

$$\lambda_B X_A = \frac{1}{2} \sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{2\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{\left(v_{A,j}\right)^2}{v_{B,j}} \tag{9}$$

$$\lambda_A X_B = \frac{\gamma^2}{2} \sum_{j \in \Omega_A(\gamma)} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \frac{1}{2} \sum_{j \notin \Omega_A(\gamma)} v_{A,j}. \tag{10}$$

Then dividing (10) by (9), we have:

$$\frac{X_B \gamma}{X_A} = \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}}. \tag{11}$$

The right-hand side of (11) is continuous with respect to γ. In particular, for each $\widehat{\gamma} \in \widehat{\Gamma}$, $\widehat{\gamma} v_{B,k} = v_{A,k}$ for each $k \in \partial\Omega_A(\widehat{\gamma})$. Thus, for each $\widehat{\gamma} \in \widehat{\Gamma}$

$$\lim_{\gamma \to \widehat{\gamma}^+} \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}} = \lim_{\gamma \to \widehat{\gamma}^-} \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}}.$$

Next, note that if $\frac{X_B}{X_A}\sum_{j=1}^{n}\frac{(v_{A,j})^2}{v_{B,j}} \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then $\gamma^* = \frac{X_B}{X_A}\sum_{j=1}^{n}\frac{(v_{A,j})^2}{v_{B,j}}$ is a solution to (11) in which $\Omega_A(\gamma^*) = \emptyset$, and the result follows directly. Similarly, if $\frac{X_B}{X_A}\left(\sum_{j=1}^{n}\frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} < \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then $\gamma^* = \frac{X_B}{X_A}\left(\sum_{j=1}^{n}\frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1}$ is a solution to (11) in which $\Omega_A(\gamma^*) = \{1,\ldots,n\}$ and the result follows directly.

We now examine the case in which $\frac{X_B}{X_A}\sum_{j=1}^{n}\frac{(v_{A,j})^2}{v_{B,j}} < \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$ and $\frac{X_B}{X_A}\left(\sum_{j=1}^{n}\frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} \geq \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$. Note first that if $\min_j\{\frac{v_{A,j}}{v_{B,j}}\} = \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, then $\frac{v_{A,j}}{v_{B,j}} = 1$ for all j and the first inequality becomes $\frac{X_B}{X_A} < 1$, which is violated by the assumption that $\frac{X_B}{X_A} \geq 1$. Hence, in this case $\min_j\{\frac{v_{A,j}}{v_{B,j}}\} < \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$. To verify that a solution in γ to (11) exists multiply both sides of (11) by $\frac{X_A}{X_B}$. The left-hand side of (11) then equals γ and the right-hand side equals the following continuous and increasing function:

$$f(\gamma) = \left(\frac{X_A}{X_B}\right)\left(\frac{\gamma^2\sum_{j\in\Omega_A(\gamma)}\frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j\notin\Omega_A(\gamma)} v_{A,j}}{\sum_{j\in\Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2}\sum_{j\notin\Omega_A(\gamma)}\frac{\left(v_{A,j}\right)^2}{v_{B,j}}}\right).$$

Because, by assumption, $\frac{X_B}{X_A}\sum_{j=1}^{n}\frac{(v_{A,j})^2}{v_{B,j}} < \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, it follows that

$$f\left(\max_j\{\tfrac{v_{A,j}}{v_{B,j}}\}\right) = \left(\frac{X_A}{X_B}\right)\left(\frac{\left(\max_j\{\frac{v_{A,j}}{v_{B,j}}\}\right)^2}{\sum_{j=1}^{n}\frac{\left(v_{A,j}\right)^2}{v_{B,j}}}\right) > \max_j\{\tfrac{v_{A,j}}{v_{B,j}}\} \tag{12}$$

and, as $\frac{X_B}{X_A}\left(\sum_{j=1}^{n}\frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} \geq \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$, it follows that

$$f\left(\min_j\{\tfrac{v_{A,j}}{v_{B,j}}\}\right) = \left(\frac{X_A}{X_B}\right)\left(\min_j\{\tfrac{v_{A,j}}{v_{B,j}}\}\right)^2\left(\sum_{j=1}^{n}\frac{\left(v_{B,j}\right)^2}{v_{A,j}}\right) \leq \min_j\{\tfrac{v_{A,j}}{v_{B,j}}\} \tag{13}$$

Combining (12), (13), with the continuity of $f(\gamma)$, it follows that there exists at least one point $\gamma^* \in \left[\underline{\gamma}, \overline{\gamma}\right]$ such that $f(\gamma^*) = \gamma^*$. This completes the proof of the existence of a γ^* that solves (11), and then given a solution γ^*, (9) and (10) can be used to solve for λ_B and λ_A (a solution to system $(\star)$), respectively.

For uniqueness in the constant-sum game, note that when $v_{A,j} = v_{B,j}$ for all j then $\max_j\{\frac{v_{A,j}}{v_{B,j}}\} = \min_j\{\frac{v_{A,j}}{v_{B,j}}\} = 1$ for all j and $\underline{\gamma} = 1 \leq \overline{\gamma} = \frac{X_B}{X_A}$. Consequently, $\Omega_A(\gamma^*) = \emptyset$ and (11) becomes $\gamma^* = \frac{X_B}{X_A}$. □

Although there exists a unique solution to system $(\star)$ when the game is constant-sum, there may exist multiple solutions to system ($\star$) in non-constant-sum versions of the game, and these multiple solutions give rise to multiple equilibria. Following the statement and proof of Theorem 1, we provide an example in which there are multiple payoff nonequivalent equilibria.

We now examine equilibrium in the general case of the linear heterogeneous asymmetric objective and, then, discuss the special case of the linear heterogeneous symmetric objective.

Theorem 1. *For each solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$), each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginals. If $v_{i,j}/\lambda_i^* \geq v_{-i,j}/\lambda_{-i}^*$, then*

$$\begin{aligned} F_{-i,j}(x) &= \left(\frac{\frac{v_{i,j}}{\lambda_i^*} - \frac{v_{-i,j}}{\lambda_{-i}^*}}{\frac{v_{i,j}}{\lambda_i^*}}\right) + \frac{x}{\frac{v_{i,j}}{\lambda_i^*}} \quad & x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}^*}\right] \\ F_{i,j}(x) &= \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}^*}} & x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}^*}\right]. \end{aligned}$$

Conversely, for each equilibrium of the Generalized General Lotto game, there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$). For each solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$), the expected payoff for player A is $\sum_{j\in\Omega_A(\gamma^*)}\left(v_{A,j} - \frac{\gamma^* v_{B,j}}{2}\right) + \sum_{j\notin\Omega_A(\gamma^*)}\left(\frac{v_{A,j}^2}{2\gamma^* v_{B,j}}\right)$ and the expected payoff for player B is $\sum_{j\notin\Omega_A(\gamma^*)}\left(v_{B,j} - \frac{v_{A,j}}{2\gamma^*}\right) + \sum_{j\in\Omega_A(\gamma^*)}\left(\frac{\gamma^* v_{B,j}^2}{2v_{A,j}}\right)$.

Proof. We now show that for each solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$) any pair of joint distributions (P_A, P_B) with the sets of univariate marginals specified in Theorem 1 is a Nash equilibrium of the Generalized General Lotto game. In the Appendix, we show that: (i) for each equilibrium of the Generalized General Lotto game, there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$) and (ii) for each solution $(\lambda_A^*, \lambda_B^*)$ each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginals.

For the proof that for each solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$) any pair of joint distributions (P_A, P_B) with the sets of univariate marginals specified in Theorem 1 is a Nash equilibrium of the Generalized General Lotto game, we focus on player A, and note that the argument for player B is symmetric. First, observe that because (λ_A, λ_B) is a solution to ($\star$), this is a feasible strategy for player A:

$$\sum_{j=1}^{n}\int_0^\infty x dF_{A,j} = \sum_{j\in\Omega_A(\gamma^*)} \frac{v_{B,j}}{2\lambda_B^*} + \sum_{j\notin\Omega_A(\gamma^*)} \frac{\left(\frac{v_{A,j}}{\lambda_A^*}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B^*}\right)} = X_A.$$

Then, given that player B is following the equilibrium strategy, player A's payoff from an arbitrary strategy with the set of univariate marginals $\left\{\bar{F}_{A,j}\right\}_{j=1}^{n}$ is:

$$\pi_A\left(\left\{\bar{F}_{A,j},F_{B,j}\right\}_{j=1}^{n}\right)=\sum_{j=1}^{n}\int_0^{\infty}v_{A,j}F_{B,j}\left(x\right)d\bar{F}_{A,j}\left(x\right).$$

Because it is never a best response for player A to place strictly positive mass at zero in any battlefield $j\in\Omega_A(\gamma^*)$ nor to provide offers outside of the support of any of player B's univariate marginal distributions, we have:

$$\pi_A\left(\left\{\bar{F}_{A,j},F_{B,j}\right\}_{j=1}^{n}\right)=\sum_{j\in\Omega_A(\gamma^*)}\left[\left(v_{A,j}-\frac{v_{B,j}\lambda_A^*}{\lambda_B^*}\right)+\int_0^{\frac{v_{B,j}}{\lambda_B^*}}x\lambda_A^*d\bar{F}_{A,j}\left(x\right)\right]+\sum_{j\notin\Omega_A(\gamma^*)}\int_0^{\frac{v_{A,j}}{\lambda_A^*}}x\lambda_A^*d\bar{F}_{A,j}\left(x\right).$$

But from the budget constraint, it follows that

$$\pi_A\left(\left\{\bar{F}_{A,j},F_{B,j}\right\}_{j=1}^{n}\right)\leq\sum_{j\in\Omega_A(\gamma^*)}\left(v_{A,j}-\frac{\lambda_A^*v_{B,j}}{\lambda_B^*}\right)+\lambda_A^*X_A$$

which, together with (5), yields

$$\pi_A\left(\left\{\bar{F}_{A,j},F_{B,j}\right\}_{j=1}^{n}\right)\leq\sum_{j\in\Omega_A(\gamma^*)}\left(v_{A,j}-\frac{\gamma^*v_{B,j}}{2}\right)+\sum_{j\notin\Omega_A(\gamma^*)}\left(\frac{v_{A,j}^2}{2\gamma^*v_{B,j}}\right),$$

which holds with equality if $\{\bar{F}_{A,j}\}_{j=1}^{n}$ is the equilibrium strategy. This completes the proof that there are no payoff increasing deviations for player A. A symmetric argument applies to player B, and thus any pair of joint distributions (P_A,P_B) providing the sets of univariate marginal distributions $\left(\{F_{A,j},F_{B,j}\}_{j=1}^{n}\right)$ is an equilibrium.

The Appendix contains the two remaining parts of the proof: (i) for each equilibrium of the Generalized General Lotto game, there exists a corresponding solution $(\lambda_A^*,\lambda_B^*)$ to system $(\star)$ and (ii) for each solution $(\lambda_A^*,\lambda_B^*)$ each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginals. □

Proposition 1 guarantees at least one solution to system $(\star)$ and Theorem 1 demonstrates that corresponding to every such solution there is a unique set of Nash equilibrium univariate marginal distributions in the General Lotto game. If the game is constant-sum (i.e. the players' battlefield valuations are symmetric for all battlefields), then each player has a unique set of equilibrium univariate marginal distribution functions. We now examine a simple example in which player valuations are asymmetric and multiple payoff nonequivalent equilibria arise. For such equilibria to arise there must exist a set of battlefields, termed the disagreement set, in which $v_{A,j} \neq v_{B,j}$. Example 1 is a special case in which the configuration of players' valuations in the disagreement set takes a simple parametric form yielding only two distinct values. Even in this simple case, we find that there are five payoff nonequivalent sets of Nash equilibrium univariate marginal distributions. The parametric form used for battlefield valuations is useful in that it makes the calculation of the set $\Omega_A(\gamma)$ easier, thereby simplifying the problem of solving system $(\star)$. In moving from this example to an arbitrary configuration of battlefield valuations the calculation of the set $\Omega_A(\gamma)$ becomes more involved.

Example 1. *Consider a Generalized General Lotto game in which $X_A = X_B = 1$, and the battlefields may be partitioned into an agreement set, denoted $\mathcal{A}$, in which $v_{A,j} = v_{B,j}$ for each $j \in \mathcal{A}$ and $\sum_{j \in \mathcal{A}} v_{A,j} = \frac{n_{\mathcal{A}}}{n}$, where $n_{\mathcal{A}}$ is the number of battlefields in the agreement set, and a disagreement set, denoted $\mathcal{D}$, with an even number $n_{\mathcal{D}}$ of battlefields, where for the first $\frac{n_{\mathcal{D}}}{2}$ battlefields $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$ and for the last $\frac{n_{\mathcal{D}}}{2}$ battlefields $v_{A,j} = \frac{2\epsilon}{n}$ and $v_{B,j} = \frac{2(1-\epsilon)}{n}$, with $\epsilon \in (0, .5)$. This configuration of battlefield values is illustrated in Figure 2 below.*

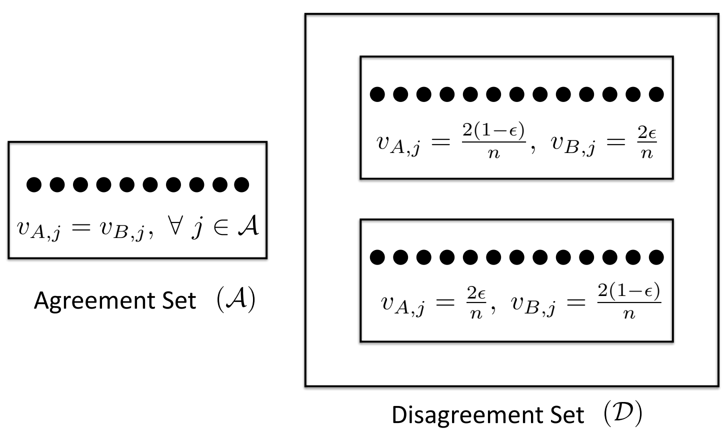

Figure 2: Example 1 battlefield configuration [$\epsilon \in (0, 0.5)$]

For all $\epsilon \in (0, .5)$, $n_D \geq 0$, and $n_A \geq 0$ equation (11) has a solution at $\gamma = 1$, but depending on the values of ϵ, n_D, and n_A there may exist multiple solutions, and thus multiple equilibria. To solve for all possible solutions to system ($\star$) note that (11) may be written as

$$\gamma^3 \sum_{j\in\Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} - \frac{X_B\gamma^2}{X_A} \sum_{j\in\Omega_A(\gamma)} v_{B,j} + \gamma \sum_{j\notin\Omega_A(\gamma)} v_{A,j} - \frac{X_B}{X_A} \sum_{j\notin\Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}} = 0. \qquad (14)$$

Next, note that with symmetric resource constraints it must be the case that either $\frac{1-\epsilon}{\epsilon} > \gamma \geq 1$ or $1 > \gamma \geq \frac{\epsilon}{1-\epsilon}$.[12] If $1 > \gamma \geq \frac{\epsilon}{1-\epsilon}$, then $\Omega_A(\gamma)$ includes $\mathcal{A}$ and the portion of $\mathcal{D}$ in which $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$, and (14) may be written as

$$\gamma^3 \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_\mathcal{D}}{n} + \frac{n_\mathcal{A}}{n}\right) - \gamma^2 \left(\epsilon \cdot \frac{n_\mathcal{D}}{n} + \frac{n_\mathcal{A}}{n}\right) + \gamma \left(\epsilon \cdot \frac{n_\mathcal{D}}{n}\right) - \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_\mathcal{D}}{n}\right) = 0. \qquad (15)$$

Similarly, if $\frac{1-\epsilon}{\epsilon} > \gamma \geq 1$, then $\Omega_A(\gamma)$ includes only the portion of $\mathcal{D}$ in which $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$ and (14) may be written as

$$\gamma^3 \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_\mathcal{D}}{n}\right) - \gamma^2 \left(\epsilon \cdot \frac{n_\mathcal{D}}{n}\right) + \gamma \left(\epsilon \cdot \frac{n_\mathcal{D}}{n} + \frac{n_\mathcal{A}}{n}\right) - \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_\mathcal{D}}{n} + \frac{n_\mathcal{A}}{n}\right) = 0. \qquad (16)$$

If, for example, we let $\epsilon = 0.10$, $(n_\mathcal{A}/n) = 0.1$, and $(n_\mathcal{D}/n) = 0.9$, then there are five solutions to system ($\star$) — equation (15) has two real roots for $1 > \gamma \geq \frac{\epsilon}{1-\epsilon} = \frac{1}{9}$ and equation (16) has three real roots for $9 = \frac{1-\epsilon}{\epsilon} > \gamma \geq 1$ — and Theorem 1 provides the equilibrium expected payoffs and unique sets of equilibrium univariate marginal distributions. These five equilibria are summarized in Table 2 below.

For the two solutions with $1 > \gamma \geq \frac{1}{9}$ equilibrium is described as follows: for all battlefields $j \in \mathcal{A}$ let $v_j \equiv v_{A,j} = v_{B,j}$

$$\begin{array}{ll} F_{B,j}(x) = \left(1 - \frac{\lambda_A}{\lambda_B}\right) + \frac{\lambda_A x}{v_j} & x \in \left[0, \frac{v_j}{\lambda_B}\right] \\ F_{A,j}(x) = \frac{\lambda_B x}{v_j} & x \in \left[0, \frac{v_j}{\lambda_B}\right], \end{array}$$

[12]If $\gamma < \min_j\{\frac{v_{A,j}}{v_{B,j}}\} = \frac{\epsilon}{1-\epsilon}$ then $\Omega_A(\gamma) = \{1, \ldots, n\}$, and if $\gamma \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\} = \frac{1-\epsilon}{\epsilon}$ then $\Omega_A(\gamma) = \emptyset$. In either case, one player has a weakly higher expected expenditure of the resource in every battlefield and a strictly higher expenditure in a nonempty subset of battlefields. With symmetric budget constraints it is clear that this is not possible.

γ^*	λ_A^*	λ_B^*	π_A^*	π_B^*
0.1604	0.0464	0.2893	0.9259	0.5383
0.5669	0.0627	0.1106	0.8650	0.7618
1.00	0.10	0.10	0.82	0.82
1.7640	0.1106	0.0627	0.7618	0.8650
6.2362	0.2893	0.0464	0.5383	0.9259

Table 2: Multiple Equilibria in Example 1

for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{9}{5n}$ and $v_{B,j} = \frac{1}{5n}$

$$\begin{array}{rl} F_{B,j}(x) = \left(1 - \frac{\lambda_A}{9\lambda_B}\right) + \frac{\lambda_A x}{(9/5n)} & x \in \left[0, \frac{1}{5n\lambda_B}\right] \\ F_{A,j}(x) = \frac{\lambda_B x}{(1/5n)} & x \in \left[0, \frac{1}{5n\lambda_B}\right], \end{array}$$

and for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{1}{5n}$ and $v_{B,j} = \frac{9}{5n}$

$$\begin{array}{rl} F_{A,j}(x) = \left(1 - \frac{\lambda_B}{9\lambda_A}\right) + \frac{\lambda_B x}{(9/5n)} & x \in \left[0, \frac{1}{5n\lambda_A}\right] \\ F_{B,j}(x) = \frac{\lambda_A x}{(1/5n)} & x \in \left[0, \frac{1}{5n\lambda_A}\right]. \end{array}$$

The expected payoff for player A is $\frac{91}{100} - \frac{19\gamma}{200} + \frac{1}{200\gamma}$ and the expected payoff for player B is $\frac{81}{100} - \frac{9}{200\gamma} + \frac{11\gamma}{200}$. Similarly, for the three solutions with $9 > \gamma \geq 1$ equilibrium is described as follows: for all battlefields $j \in \mathcal{A}$

$$\begin{array}{rl} F_{A,j}(x) = \left(1 - \frac{\lambda_B}{\lambda_A}\right) + \frac{\lambda_B x}{v_j} & x \in \left[0, \frac{v_j}{\lambda_A}\right] \\ F_{B,j}(x) = \frac{\lambda_A x}{v_j} & x \in \left[0, \frac{v_j}{\lambda_A}\right], \end{array}$$

for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{9}{5n}$ and $v_{B,j} = \frac{1}{5n}$

$$\begin{array}{rl} F_{B,j}(x) = \left(1 - \frac{\lambda_A}{9\lambda_B}\right) + \frac{\lambda_A x}{(9/5n)} & x \in \left[0, \frac{1}{5n\lambda_B}\right] \\ F_{A,j}(x) = \frac{\lambda_B x}{(1/5n)} & x \in \left[0, \frac{1}{5n\lambda_B}\right], \end{array}$$

and for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{1}{5n}$ and $v_{B,j} = \frac{9}{5n}$

$$\begin{array}{rl} F_{A,j}(x) = \left(1 - \frac{\lambda_B}{9\lambda_A}\right) + \frac{\lambda_B x}{(9/5n)} & x \in \left[0, \frac{1}{5n\lambda_A}\right] \\ F_{B,j}(x) = \frac{\lambda_A x}{(1/5n)} & x \in \left[0, \frac{1}{5n\lambda_A}\right]. \end{array}$$

The expected payoff for player A is $\frac{81}{100} - \frac{9\gamma}{200} + \frac{11}{200\gamma}$ and the expected payoff for player B is $\frac{91}{100} - \frac{19}{200\gamma} + \frac{\gamma}{200}$.

Although this example is a simple one in which only three values of the ratio $\frac{v_{A,j}}{v_{B,j}}$ arise, there continues to exist a multiplicity of payoff nonequivalent equilibria even when all of the parameters of the example are slightly perturbed, so that the ratio $\frac{v_{A,j}}{v_{B,j}}$ may be distinct for every battlefield j. In fact, fixing n and taking the relevant space of parameters to be $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n) \in \mathbb{R}^2_{++} \times [\mathrm{Int}(S^{n-1})]^2$, where $\mathrm{Int}(S^{n-1})$ is the interior of the $n-1$ dimensional unit simplex containing the values $\{v_{i,j}\}_{j=1}^n, i = A, B$, there is a set of positive Lebesgue measure in $\mathbb{R}^2_{++} \times [\mathrm{Int}(S^{n-1})]^2$ which contains the parameters in our example and in which such a multiplicity exists.[13] That is, a multiplicity of payoff nonequivalent equilibria should not be viewed as an anomaly.

We conclude the discussion of our results on the (continuous) Generalized General Lotto game by noting that in the special case of the linear heterogeneous symmetric objective, $v_{A,j} = v_{B,j} \equiv v_j$ for all j, the unique solution to system $(\star)$[14] is $\lambda_A^* = \frac{1}{2X_B}$ and $\lambda_B^* = \frac{X_A}{2X_B^2}$. We then have the following corollary, which appears in a closely related form in Bell and Cover (1980), Sahuguet and Persico (2006), and Washburn (2013).

Corollary 1. *If $v_{A,j} = v_{B,j} \equiv v_j$ for all j, then the unique set of Nash equilibrium univariate*

[13]For a fixed number of battlefields n, the equilibrium values γ^* are the solutions in γ to equation (14). If the set of indices $\Omega_A(\gamma)$ is invariant over an interval of γ's, the left-hand side of (14) is a cubic in γ over that interval. In our specific numerical example with $\epsilon = 0.1$, $\frac{n_A}{n} = 0.1$, and $\frac{n_D}{n} = 0.9$, the sets of indices $\Omega_A(\gamma)$ are invariant in each of two adjacent domains of γ, $1 > \gamma \geq \frac{1}{9}$ and $9 > \gamma \geq 1$, but differ across the domains (represented, respectively, by equations (15) and (16)). More generally, because the set of indices $\Omega_A(\gamma)$ changes only at values of γ for which $\gamma = \frac{v_{A,j}}{v_{B,j}}$ for some j, the coefficients of γ in the cubic are fixed over distinct intervals between adjacent values of $\frac{v_{A,j}}{v_{B,j}}$ and the left-hand side of (14) is, in fact, continuous in γ over $[\underline{\gamma}, \overline{\gamma}]$, including at values of γ at which the set of indices $\Omega_A(\gamma)$ changes. Moreover, the left hand side of (14) is also continuous in the $2n+2$-tuple of parameters $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n)$ over the relevant domain. In the numerical example, two of the five solutions γ^* to (14) identified in Table 2 are interior to $[\frac{1}{9}, 1)$ and two are interior to $[1, 9)$. (The remaining solution $\gamma^* = 1$ is on the boundary of the two sets). It is easily verified that none of the four solutions to (14) that are interior to $[\frac{1}{9}, 1)$ or $[1, 9)$ are multiple roots of the polynomial in γ (for the fixed set of indices $\Omega_A(\gamma)$ applicable over the interval). Therefore, they cannot represent tangencies to the γ-axis of the applicable polynomial, but rather represent values of γ where the left-hand side of (14) cuts the origin. As a consequence, for sufficiently small perturbations of the $2n+2$-tuple of parameters chosen in the example, for each of these four values of γ^* there exists a neighborhood about γ^*such that the set of indices contained in $\Omega_A(\gamma^*)$ coincides with the set of indices in the example and, for that fixed set $\Omega_A(\gamma^*)$, the polynomial in γ given by the left-hand side of (14) has a root within the neighborhood. That is, there is an open set of parameters $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n)$ containing those in the example for which there are solutions to (14) "close" to the four values of γ^* identified in the interior of $[\frac{1}{9}, 1)$ and $[1, 9)$.

[14]As $\frac{X_A}{X_B} \leq 1$ it must be the case that $\lambda_B \leq \lambda_A$.

marginal distributions of the Generalized General Lotto game are, for all $j \in \{1, \ldots, n\}$:

$$\begin{array}{ll} F_{A,j}(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x}{2v_j X_B}\left(\frac{X_A}{X_B}\right) & x \in [0, 2v_j X_B] \\ F_{B,j}(x) = \frac{x}{2v_j X_B} & x \in [0, 2v_j X_B] \end{array}$$

The expected payoff for player A is $\frac{X_A}{2X_B}$ and the expected payoff for player B is $1 - \frac{X_A}{2X_B}$.

4 Generalized Colonel Blotto Results

As pointed out in Hart (2008, 2014), the General Lotto game can be used as an intermediate step in solving the Colonel Blotto game. In moving from the Generalized General Lotto game to the Generalized Colonel Blotto game, we face the added requirement that for each player a joint distribution exists that satisfies the player's budget constraint with probability one, and not just in expectation. From this, it is clear that if a pair of joint distributions is found that yields for each player the set of univariate marginal distributions corresponding to an equilibrium in the Generalized General Lotto game and each joint distribution satisfies the constraint that the budget holds with probability one, then this pair will also be an equilibrium of the Colonel Blotto version of the game.

The following proposition provides a sufficient condition for the set of univariate marginal distributions corresponding to an equilibrium in the Generalized General Lotto game given in Theorem 1 to be generated by a pair of joint distributions that balance the players' respective budgets with probability one. That is, it provides a sufficient condition for an equilibrium set of univariate marginal distributions in the Generalized General Lotto game to be attainable in equilibrium in the Generalized Colonel Blotto game. In the analysis that follows, consider a partition of the battlefields into subsets based on distinct pairs of valuations $v_{A,j}$ and $v_{B,j}$ so that two battlefields h and m, $h, m \in \{1, \ldots, n\}$, are in the same set in the partition if and only if $v_{A,h} = v_{A,m}$ and $v_{B,h} = v_{B,m}$. Let $k \leq n$ denote the number of subsets in this partition, $j \in \{1, \ldots, k\}$ index the distinct pairs of battlefield valuations $(v_{A,j}, v_{B,j})$, and $n_j \geq 1$ denote the number of battlefields with the distinct pair of valuations $(v_{A,j}, v_{B,j})$.

Proposition 2. *Given a solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$, if for each distinct pair of battlefield valuations $(v_{A,j}, v_{B,j})$ with $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} \leq 1$, for some $i \in \{A, B\}$, it is the case that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$, then there exists a Nash equilibrium of the Generalized Colonel Blotto game with the same set of univariate marginal distributions and expected payoffs as in the corresponding equilibrium* *in Theorem 1.*

Given the $k \leq n$ distinct pairs of battlefield valuations, we can form independent n_j-variate marginal distributions on each of the $j = 1, \ldots, k$ subsets of battlefields with a distinct pair of valuations, where the budget constraint for each subset j is equal to the expected expenditure from the Theorem 1 set of univariate marginal distribution functions on that subset of battlefields. For example, if $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$, then from Theorem 1 it follows that player $-i$'s expected expenditure on the jth set of battlefields is $\frac{n_j}{2}\left(\frac{v_{-i,j}}{\lambda_{-i}^*}\right)^2 / \left(\frac{v_{i,j}}{\lambda_i^*}\right)$ and i's expected expenditure on the jth set of battlefields is $n_j\left(\frac{v_{-i,j}}{2\lambda_{-i}^*}\right)$. Then, the problem of constructing an equilibrium n-variate joint distribution P_i, for each player i, that is budget balancing with probability one and that provides the univariate marginals given in Theorem 1 is replaced by the problem of constructing k n_j-variate marginal distributions, denoted $P_{i,j}$, one for each of the k sets of battlefields with distinct valuations, and then letting each player i's joint distribution be defined as $P_i(x) = \prod_{j=1}^k P_{i,j}(x_j)$ where x_j is the restriction of x to the battlefields in set j. For each of the sets of battlefields $j \in \{1, \ldots, k\}$ if $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$ and it is the case that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$, then player i's n_j-variate marginal distribution $P_{i,j}$ may be formed by deterministically allocating $n_j\left(\frac{v_{-i,j}}{2\lambda_{-i}^*}\right)$ of i's budget to the subset j of battlefields and then constructing $P_{i,j}$ using the existing construction methods in Gross and Wagner (1950), Roberson (2006), or Weinstein (2012). Similarly, player $-i$'s n_j-variate marginal distribution $P_{-i,j}$ may be formed by deterministically allocating $\frac{n_j}{2}\left(\frac{v_{-i,j}}{\lambda_{-i}^*}\right)^2 / \left(\frac{v_{i,j}}{\lambda_i^*}\right)$ of $-i$'s budget to the subset j of battlefields and then constructing $P_{-i,j}$ using the distribution from Roberson (2006, Theorem 4 p.9).[15] Any such construction provides the necessary univariate marginals characterized in Theorem 1 and the resulting joint distributions P_i and P_{-i} are budget-balancing with probability one.

To fix ideas regarding the multi-variate marginal distributions that are utilized in the construction method summarized above, it is instructive to briefly examine an example of such a multi-variate marginal. For the case of $n_j = 3$ and $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} > \frac{2}{n_j}$, the support of $P_{-i,j}$ is given in Figure 3 below. Note that the support of $P_{-i,j}$ lies on the budget hyperplane $\sum_{i=1}^3 x_i = 3\left(\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}\right)\left(\frac{v_{-i,j}}{2\lambda_{-i}^*}\right)$ and, as is shown in Roberson (2006), there exists a distribution of mass across this support that provides the set of univariate marginals specified by Theorem

[15] In Roberson (2006) the construction is carried out with respect to the players' aggregate resource endowments $X_A \leq X_B$. Note that in this paper's subset j of battlefields player $-i$'s budget is $X_{-i,j} \equiv \frac{n_j}{2}\left(\frac{v_{-i,j}}{\lambda_{-i}^*}\right)^2 / \left(\frac{v_{i,j}}{\lambda_i^*}\right)$ and player i's budget is $X_{i,j} \equiv n_j\left(\frac{v_{-i,j}}{2\lambda_{-i}^*}\right)$, where $X_{-i,j} \leq X_{i,j}$. To apply the construction in Roberson (2006) to an n_j-variate marginal distribution in this paper, substitute player $-i$ and $X_{-i,j}$ for player A and X_A, respectively, and player i and $X_{i,j}$ for player B and X_B respectively.

1. The role of the condition that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ for each $j = 1, \ldots, k$ can be seen in Figure 3, where the condition implies that that budget hyperplane cuts through the n_j-box, or hypercube, above its intercepts and thus the upper bound of the support of each of the univariate marginals is feasible. Furthermore, Roberson (2006) shows that, as long as the hyperplane cuts through the hypercube above its intercepts, the constraint that the support of the joint distribution satisfies the budget constraint with probability one is satisfied and the results in Theorem 1 extend directly to the Colonel Blotto version of the game. Thus, the conditions of Proposition 2 are sufficient for the existence of budget-balancing joint distributions, one for each player, that provide the sets of equilibrium univariate marginal distributions in Theorem 1. It follows directly that a necessary condition for the existence of such a joint distribution is $X_i \geq \max_j \left\{\min\left\{\frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*}\right\}\right\}$ for each player i. This necessary condition states that each player's budget-balancing hyperplane cuts through the n-box formed by the supports of each of the n univariate marginals specified by Theorem 1 above its intercepts.

The conditions in Proposition 2 provide a sufficient condition for the existence of a Nash equilibrium of the Generalized Colonel Blotto game with the same set of equilibrium univariate marginals as an equilibrium of the corresponding General Lotto game. If $X_B > nX_A$ then, clearly, this relationship breaks down and in any equilibrium of the Generalized Colonel Blotto game player B wins every battlefield with certainty. But, it turns out that there exists a stronger condition that can be invoked. Recall that given a solution to system ($\star$) a necessary condition for the existence of a budget-balancing joint distribution that provides the Theorem 1 sets of univariate marginals is $X_i \geq \max_j \left\{\min\left\{\frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*}\right\}\right\}$ for each player i. Thus, if in a Generalized Colonel Blotto game $CB\big\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\big\}$, it is the case that for each solution to system ($\star$) there exists a player i such that $X_i < \max_j \left\{\min\left\{\frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*}\right\}\right\}$, then there exists no equilibrium in which both players utilize joint distributions providing the Theorem 1 sets of univariate marginals. That is, the constraint on the joint distribution function is binding. This result is summarized in the following corollary.

Corollary 2. *If for each solution $(\lambda_A^*, \lambda_B^*)$ to system ($\star$) there exists a player $i \in \{A, B\}$ such that $X_i < \max_j \left\{\min\left\{\frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*}\right\}\right\}$, then there exists no equilibrium of the Colonel Blotto game with the same set of univariate marginal distributions as in Theorem 1.*

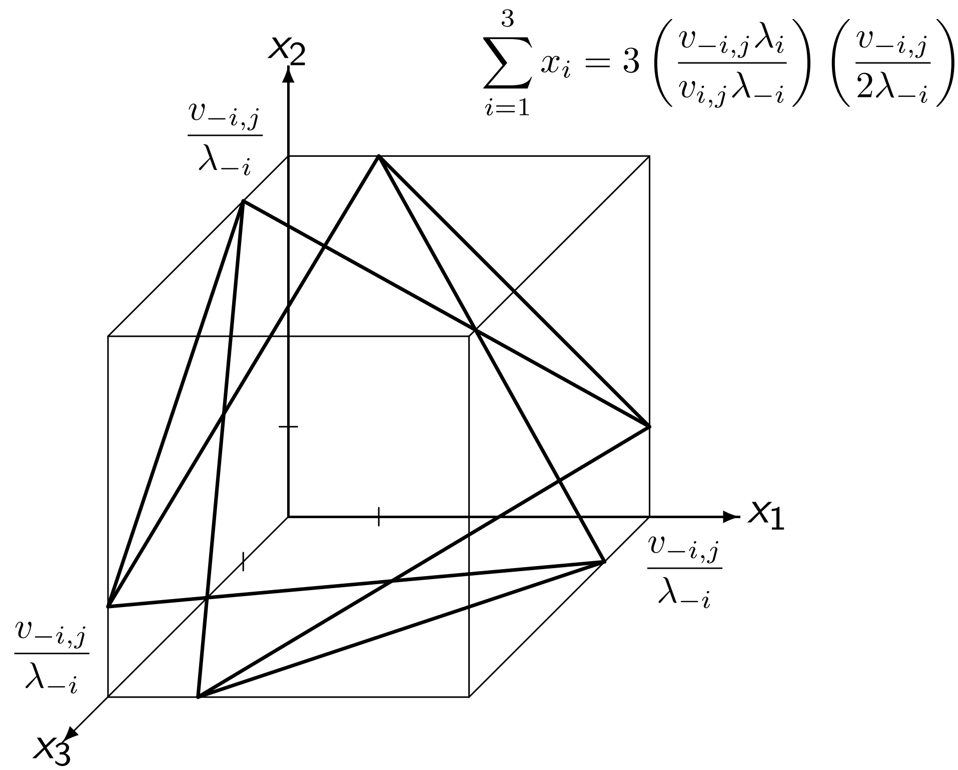

Figure 3: Support of $P_{-i,j}$ for the case of $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$

4.1 Relationship to the Colonel Blotto Literature

As indicated in Table 1 of Section 1, Proposition 2 provides a partial characterization of two variations of the (continuous) Colonel Blotto game that have not previously been examined. We now briefly summarize how our results relate to the literature on the four variations that have been previously examined (liner-pure count objective with symmetric and asymmetric budgets, and the linear heterogeneous symmetric objective with symmetric and asymmetric budgets), and focus, in particular, on the equilibrium joint distribution functions previously identified in the literature.

Because all of the constructions of the equilibrium joint distributions in the first two rows of the symmetric budget column of Table 1 — with the exception of Weinstein (2012), which we will return to below — involve randomizing on the surface of an n-gon, the two following properties of regular n-gons are worth noting: (1) the sum of the perpendiculars from any point in a regular n-gon to the sides of the regular n-gon is equal to n times the inradius, i.e. the radius of the incircle (the largest circle that can be inscribed in the n-gon) and (2) if each side of the regular n-gon has length $(2/n)\tan(\pi/n)$, then the inradius is equal to $(1/n)$. Normalizing the symmetric budget to one unit of a (use-it-or-lose-it) resource, these two properties of regular n-gons imply that any arbitrary point in a regular n-gon with side length of $(2/n)\tan(\pi/n)$ is budget balancing in that the perpendiculars sum to one, and, for the case of $n = 3$, this is illustrated in panel A of Figure 4 below where $x_1 + x_2 + x_3 = 1$. For $n = 3$ any distribution on the surface of a regular 3-gon with side lengths $(2/3)\tan(\pi/3)$ that generates uniform marginal distributions on $[0, 2/3]$ for each of the three battlefields is an equilibrium joint distribution, and Borel and Ville (1938) provide two such equilibrium joint distributions.[16] Gross and Wagner (1950), making use of the two properties of regular n-gons listed above, show that both types of equilibria in Borel and Ville (1938) for the linear pure-count objective game with symmetric budgets and $n = 3$ can be directly extended to $n > 3$. They also provide a new fractal equilibrium.

For the case of symmetric budgets, the regular n-gon approach can be modified to allow for battlefield valuations to be symmetric across players, but heterogeneous across battlefields, i.e. the linear heterogeneous symmetric objective game with $n \geq 3$ as in the second row of the first column of Table 1. This is exactly what is demonstrated in Gross (1950) and Laslier (2002),[17] where the modification involves partitioning the n battlefields into three

[16] Borel (1921), a paper on mixed strategies in zero-sum games, introduces the Colonel Blotto game as an example, but does not provide a solution.

[17] See also Thomas (2012), who provides a new construction method for the linear heterogeneous symmetric objective game with symmetric budgets and $n \geq 3$. Thomas's approach also involves irregular n-gons, but

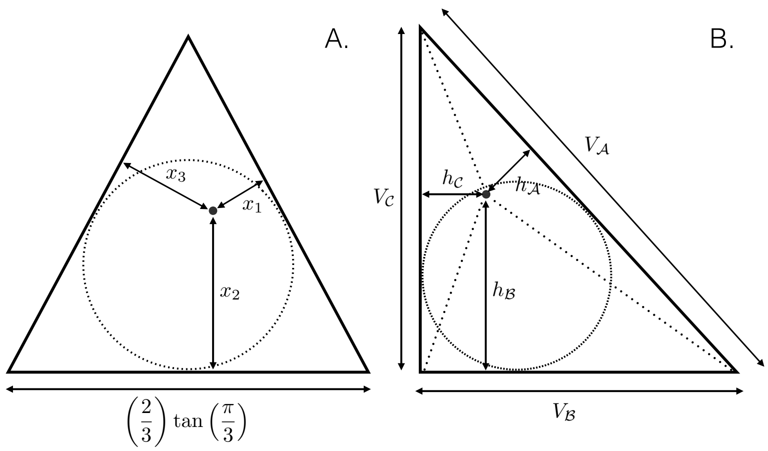

Figure 4: Arbitrary points in a regular and an irregular 3-gon

sets, denoted $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and then randomizing on the surface of the irregular triangle with the three side lengths equal to the total valuations of each of the three sets of battlefields, henceforth denoted $V_{\mathcal{A}}$, $V_{\mathcal{B}}$, and $V_{\mathcal{C}}$, respectively.[18] Then, as illustrated in panel B of Figure 4, for each point on the surface of this irregular triangle the sum across the three sides of the product of each perpendicular and the length of its corresponding side is equal to a constant. That is, $h_{\mathcal{A}}V_{\mathcal{A}} + h_{\mathcal{B}}V_{\mathcal{B}} + h_{\mathcal{C}}V_{\mathcal{C}}$ is equal to twice the surface area of the triangle which, with $V_{\mathcal{A}} + V_{\mathcal{B}} + V_{\mathcal{C}} = 1$, is equal to the inradius. Furthermore, note that $h_i \leq 2r$ for all i, where r denotes the inradius. Thus, for any tri-variate distribution on the incircle the random variable $\tilde{h}_i$ is contained in the interval $[0, 2r]$ for each $i = \mathcal{A}, \mathcal{B}, \mathcal{C}$. Thus, we can construct an n-variate distribution function where the random variable $\tilde{h}_{\mathcal{A}}$ is transformed into $\tilde{x}_j \equiv \frac{\tilde{h}_{\mathcal{A}} v_j}{r}$ for each $j \in \mathcal{A}$, and a similar transformation is carried out for each $j \in \mathcal{B}$ and $j \in \mathcal{C}$. The resulting n-variate distribution function is budget-balancing with probability one ($\sum_{j=1}^{n} x_j = 1$) and for each $j = 1, \ldots, n$, the random variable $\tilde{x}_j$ is contained in $[0, 2v_j]$. Lastly, as shown in Gross (1950) and Laslier (2002), one of the Borel and Ville (1938) so-

the method differs in that it does not involve merging the battlefields into three groups, but instead utilizes an irregular n-gon in which the number of sides equals the number battlefields.

[18]This construction, and the following discussion, is for the case in which no battlefield has a value that is over half of the total value of all battlefields and for which it is not possible to combine battlefields into four groups with equal sums of valuations. For more details on the remaining two special cases see Laslier (2002).

lutions can be used for the tri-variate distribution of the $\tilde{h}_i$ variables. In this case, each $\tilde{h}_i$ is uniformly distributed on the interval $[0, 2r]$ — so that each $\tilde{x}_j$ is uniformly distributed on the interval $[0, 2v_j]$ for each battlefield j where v_j is the value of battlefield j — and with symmetric budgets, equilibrium in the linear heterogeneous symmetric objective game requires, utilizing a similar argument as the linear-pure count game, that the univariate marginal distribution functions are uniform on $[0, 2v_j]$ for each battlefield j.

A drawback of using n-gons to construct budget-balancing joint distribution functions is that this method reduces the dimensionality of the set of points that can be used to form the support of the joint distribution function. With symmetric budgets and symmetric battlefield valuations, this reduction in dimensionality does not preclude the construction of equilibrium joint distribution functions. However, with asymmetric budgets and/or asymmetric battlefield valuations it is easier, if not necessary, to work directly with the budget hyperplane in $\mathbb{R}^n$, as in Roberson (2006) and Weinstein (2012). In this paper, we utilized this full dimensionality approach to examine a subset of possible parameter configurations for each of the three checked cells in Table 1.

For the first two rows of the asymmetric budget column of Table 1, the case of $n = 2$ — where the Blotto game's binding budget constraint implies that an increase in the allocation of the resource in one battlefield necessarily implies a corresponding decrease in the allocation of the resource to the remaining battlefield — leads to a substantively different set of strategic considerations than those arising in the case of $n \geq 3$. For $n = 2$, Gross and Wagner (1950) provide an equilibrium for all feasible parameter configurations in the first two rows of the asymmetric budget column of Table 1. Macdonell and Mastronardi (2015) complete the characterization of equilibrium and examine the case of non-linear budgets. For the case of the linear heterogeneous symmetric objective with $n \geq 3$, Schwartz et al. (2014)[19] show how in this constant-sum case where battlefield valuations are heterogeneous across battlefields but symmetric across players, the construction utilized in Roberson (2006) can be extended, along the lines described above, to construct a Nash equilibrium of the Generalized Colonel Blotto game with the same set of equilibrium univariate marginal distributions and expected payoffs as in the corresponding equilibrium in Corollary 1.

[19]Following the first circulated version of our paper, Schwartz et al. (2014) independently derived the special case of our construction for the constant-sum game with the linear heterogeneous symmetric objective and asymmetric budgets.

5 Conclusion

In this paper we provide a complete characterization of the set of Nash equilibria in the Generalized General Lotto game in which battlefield valuations may be heterogeneous across battlefields and asymmetric across players, and in which players' budgets may be asymmetric. We demonstrate that there exist non-pathological parameter configurations for which multiple payoff nonequivalent equilibria exist.

We then show that this characterization may be applied to extend the existing analysis of equilibrium in the Colonel Blotto game to incorporate a range of parameter configurations with heterogeneous battlefield valuations and asymmetric valuations and budgets across players. For the Generalized Colonel Blotto game we provide sufficient conditions for the existence of an equilibrium pair of joint distributions with univariate marginal distributions that coincide with those of an equilibrium in the Generalized General Lotto game. Characterization of Colonel Blotto equilibria for the remaining subset of parameter configurations remains an open question but, for this region, we provide a sufficient condition for the sets of equilibrium univariate marginal distributions to differ from those arising in any equilibrium of the General Lotto game.

6 Appendix

This Appendix contains the remaining two parts of the proof of Theorem 1: (i) for each equilibrium of the Generalized General Lotto game there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$ and (ii) for each solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$ each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginal distributions. We begin with the proof of part (i), and then conclude with the proof of part (ii).

The proof of the converse claim in Theorem 1, that for each equilibrium of the Generalized General Lotto game there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$, extends the arguments in Hart (2008) on the continuous General Lotto game and Hart (2014) on the relationship between the all-pay auction and the continuous General Lotto game. We begin by noting that the standard constant-sum continuous General Lotto game, denoted $L\{X_A, X_B\}$, is a special case of the Generalized General Lotto game in which $n = 1$, $v_A = v_B = 1$, and a strategy is a univariate distribution function denoted F_i, for $i = A, B$, with $E_{F_i}(x) \leq X_i$. Let $\widetilde{x}_i$ denote the realization of a random variable distributed according to the distribution function F_i.[20] Player A's expected payoff in the General Lotto game is given by

$$\pi_A(F_A, F_B) = \Pr(\widetilde{x}_A > \widetilde{x}_B) + \tfrac{1}{2}\Pr(\widetilde{x}_A = \widetilde{x}_B)$$

and player B's expected payoff is given by

$$\pi_B(F_B, F_A) = \Pr(\widetilde{x}_B > \widetilde{x}_A) + \tfrac{1}{2}\Pr(\widetilde{x}_B = \widetilde{x}_A) = 1 - \Pr(\widetilde{x}_A > \widetilde{x}_B) - \tfrac{1}{2}\Pr(\widetilde{x}_A = \widetilde{x}_B).$$

In this constant-sum game player A chooses F_A to maximize $\Pr(\widetilde{x}_A > \widetilde{x}_B) + \frac{1}{2}\Pr(\widetilde{x}_A = \widetilde{x}_B)$ and player B chooses F_B to minimize $\Pr(\widetilde{x}_A > \widetilde{x}_B) + \frac{1}{2}\Pr(\widetilde{x}_A = \widetilde{x}_B)$.

Equilibrium in the (continuous) General Lotto game with strictly positive budgets is characterized by Sahuguet and Persico (2006) and Hart (2008). The following theorem extends that characterization to allow for one or both of the players to have a budget of 0. Unlike the case of $X_B \geq X_A > 0$, if either $X_A = 0$ and $X_B > 0$ or $X_A > 0$ and $X_B = 0$, then there are multiple equilibria.[21] However, because the game is constant sum, the equilibrium

[20] Here and in the remainder of the Appendix, whenever we introduce a random variable that is distributed according to a player's joint or univariate marginal distribution we assume that it is independent of the random variable distributed according to the opponent's corresponding distribution.

[21] For the player i with $X_i = 0$, the unique equilibrium strategy is $F_{i,j}^*(0) = 1$, but for player $-i$ with $X_{-i} > 0$ any distribution function with $F_{-i,j}^*(0) = 0$ and $E_{F_{-i,j}}(x) \leq X_{-i}$ is an equilibrium strategy.

expected payoffs are unique for all possible resource endowments (X_A, X_B).

Theorem 2. *For the General Lotto game $L\{X_A, X_B\}$ with $X_B \geq X_A > 0$, the unique equilibrium strategies are*

$$F_A(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x \cdot X_A}{2X_B^2} \quad \text{for } x \in [0, 2X_B]$$

$$F_B(x) = \frac{x}{2X_B} \quad \text{for } x \in [0, 2X_B]$$

and the equilibrium expected payoffs are $\frac{X_A}{2X_B}$ for player A and $1 - \frac{X_A}{2X_B}$ for player B.

For the General Lotto game $L\{X_A, X_B\}$ with $X_B = 0$ and/or $X_A = 0$:

1. *If $X_A = 0$ and $X_B > 0$, then the unique equilibrium expected payoffs are 0 for player A and 1 for player B.*

2. *If $X_A > 0$ and $X_B = 0$, then the unique equilibrium expected payoffs are 1 for player A and 0 for player B.*

3. *If $X_A = 0$ and $X_B = 0$, then the unique equilibrium strategies are $F^*_{A,j}(0) = F^*_{B,j}(0) = 1$ and the equilibrium expected payoffs are $\frac{1}{2}$ for player A and $\frac{1}{2}$ for player B.*

In moving from the General Lotto game $L\{X_A, X_B\}$ to the Generalized General Lotto game $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$, recall that in the Generalized General Lotto game a strategy is an n-variate distribution function, P_i for $i = A, B$, that satisfies the constraint that $\sum_{j=1}^n E_{F_{i,j}}(x) \leq X_i$, where $F_{i,j}$ is the univariate marginal distribution of P_i for battlefield j. Let $\widetilde{x}_{i,j}$ denote the realization of a random variable distributed according to the univariate marginal distribution $F_{i,j}$. Then, given the strategy profile (P_A, P_B), player A's expected payoff is given by

$$\pi_A(P_A, P_B) = \sum_{j=1}^n v_{A,j} \left(\Pr(\widetilde{x}_{A,j} > \widetilde{x}_{B,j}) + \frac{1}{2}\Pr(\widetilde{x}_{A,j} = \widetilde{x}_{B,j})\right)$$

and player B's expected payoff is given by

$$\pi_B(P_B, P_A) = \sum_{j=1}^n v_{B,j} \left(1 - \Pr(\widetilde{x}_{A,j} > \widetilde{x}_{B,j}) - \frac{1}{2}\Pr(\widetilde{x}_{A,j} = \widetilde{x}_{B,j})\right).$$

Given an equilibrium (P_A^*, P_B^*), let $X_{i,j}^* \equiv E_{F_{i,j}^*}(x)$ for $i = A, B$ denote player i's expected allocation of the resource to battlefield j under the strategy P_i^*.

Lemma 1. *If* (P_A^*, P_B^*) *is an equilibrium of* $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$*, then within each battlefield* j*,* $(F_{A,j}^*, F_{B,j}^*)$ *is an equilibrium of* $L(X_{A,j}^*, X_{B,j}^*)$*.*

Proof. If (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$, then there are no payoff-increasing deviations for either player. But one feasible type of deviation for player i is to hold constant $X_{i,j}^*$ on each battlefield j and choose a feasible deviation $\widehat{P}_i$ with the set of univariate marginals $\{\widehat{F}_{i,j}\}_{j=1}^n$ with $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$ for all j. Let $\widehat{x}_{i,j}$ denote the realization of a random variable distributed according to the univariate marginal distribution function $\widehat{F}_{i,j}$. Because in battlefield j each player i does not have a payoff increasing deviation $\widehat{F}_{i,j}$ with $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$, it follows that

$$v_{i,j}\left(\Pr(\widetilde{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2}\Pr(\widetilde{x}_{i,j} = \widetilde{x}_{-i,j})\right) \geq v_{i,j}\left(\Pr(\widehat{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2}\Pr(\widehat{x}_{i,j} = \widetilde{x}_{-i,j})\right) \tag{17}$$

for all possible univariate marginal distributions $\widehat{F}_{i,j}$ with $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$. But it follows directly from (17) that

$$\left(\Pr(\widetilde{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2}\Pr(\widetilde{x}_{i,j} = \widetilde{x}_{-i,j})\right) \geq \left(\Pr(\widehat{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2}\Pr(\widehat{x}_{i,j} = \widetilde{x}_{-i,j})\right) \tag{18}$$

for all possible deviations $\widehat{F}_{i,j}$ with $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$, and, thus, $(F_{A,j}^*, F_{B,j}^*)$ is an equilibrium of $L(X_{A,j}^*, X_{B,j}^*)$. □

To complete the proof of the claim that if (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$, then there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$, Lemmas 2-4 collectively establish that in any equilibrium (P_A^*, P_B^*) of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ it must be the case that $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$ for all j. Because $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$ for all j, it follows from Lemma 1 and Theorem 2 that the equilibrium univariate marginal distributions are uniquely determined. Using the unique equilibrium univariate marginal distributions, Lemma 5 completes the proof that there exists a corresponding solution $(\lambda_A^*, \lambda_B^*)$ to system $(\star)$.

Lemma 2. *If* (P_A^*, P_B^*) *is an equilibrium of* $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$*, then*

$$\max\{X_{A,j}^*, X_{B,j}^*\} > 0 \quad \text{for all } j.$$

Proof. By way of contradiction, suppose that there exists an equilibrium (P_A^*, P_B^*) in which for some battlefield k $\max\{X_{A,k}^*, X_{B,k}^*\} = 0$, which implies that $F_{A,k}^*(0) = F_{B,k}^*(0) = 1$. We begin with the case in which $\sum_{j=1}^{n} E_{F_{A,j}^*}(x) < X_A$, and then examine the case in which $\sum_{j=1}^{n} E_{F_{A,j}^*}(x) = X_A$. If $\sum_{j=1}^{n} E_{F_{A,j}^*}(x) < X_A$, then player A can increase his payoff by $\frac{v_{A,k}}{2}$ by allocating a strictly positive level of the resource $X_{A,k} \leq X_A - \sum_{j=1}^{n} E_{F_{A,j}^*}(x)$ to battlefield k and setting $F_{A,k}(0) = 0$, a contradiction.

For $\sum_{j=1}^{n} E_{F_{A,j}^*}(x) = X_A > 0$, there exists at least one battlefield j' in which $X_{A,j'}^* > 0$ and there are two cases to consider: (i) $X_{B,j'}^* = 0$ and (ii) $X_{B,j'}^* > 0$. In case (i), because $\sum_{j=1}^{n} E_{F_{A,j}^*}(x) = X_A > 0$ and in battlefield j' $X_{A,j'}^* > 0$ and $X_{B,j'}^* = 0$, player A can increase his payoff by $\frac{v_{A,k}}{2}$ by shifting $X_{A,k} < X_{A,j'}^*$ of the resource from battlefield j' to battlefield k and setting $F_{A,k}(0) = F_{A,j'}(0) = 0$, a contradiction.

In case (ii), $X_{A,j'}^* > 0$ and $X_{B,j'}^* > 0$, and it follows from Lemma 1 and Theorem 2 that $F_{B,j'}^*(x)$ is the unique equilibrium strategy in the General Lotto game $L(X_{A,j'}^*, X_{B,j'}^*)$, where the support of $F_{B,j'}^*(x)$, denoted $\mathrm{supp}(F_{B,j'}^*(x))$, is $[0, 2\max\{X_{A,j'}^*, X_{B,j'}^*\}]$. Thus, player A can increase his total expected payoff by an amount arbitrarily close to $\frac{v_{A,k}}{2}$ by shifting, for a sufficiently small $\epsilon > 0$, ϵ of the resource from battlefield j' to battlefield k, in battlefield k choosing a distribution function $F_{A,k}(x)$ with $F_{A,k}(0) = 0$ and $E_{F_{A,k}}(x) = \epsilon$, and in battlefield j' choosing a distribution function $F_{A,j'}(x)$ with $F_{A,j'}(0) = 0$, $E_{F_{A,j'}}(x) = X_{A,j'}^* - \epsilon$, and $\mathrm{supp}(F_{A,j'}) \subseteq \mathrm{supp}(F_{B,j'}^*(x))$. In battlefield j' player A's expected payoff from the distribution function $F_{A,j'}(x)$ when player B's distribution function is $F_{B,j'}^*(x)$ is given by

$$v_{A,j'} \int_0^\infty F_{B,j'}^*(x) dF_{A,j'}(x) = \begin{cases} v_{A,j'} \left(\left(1 - \frac{X_{B,j'}^*}{X_{A,j'}^*}\right) + \frac{\left(X_{A,j'}^* - \epsilon\right) X_{B,j'}^*}{2\left(X_{A,j'}^*\right)^2} \right) & \text{if } X_{A,j'}^* > X_{B,j'}^* \\ v_{A,j'} \left(\frac{\left(X_{A,j'}^* - \epsilon\right)}{2X_{B,j'}^*} \right) & \text{if } X_{A,j'}^* \leq X_{B,j'}^* \end{cases}$$

Thus, the loss in player A's payoff in battlefield j' approaches 0 as ϵ approaches 0, but the gain on battlefield k is $\frac{v_{A,k}}{2}$ for all $\epsilon > 0$. This is a contradiction to the assumption that (P_A^*, P_B^*) is an equilibrium and completes the proof that if (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ then $\max\{X_{A,j}^*, X_{B,j}^*\} > 0$ for all j. □

Lemma 3. *If (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$, then $\sum_{j=1}^{n} E_{F_{i,j}^*}(x) > 0$ for each player $i = A, B$.*

Proof. By way of contradiction, suppose that there exists an equilibrium (P_A^*, P_B^*) in which $\sum_{j=1}^{n} E_{F_{i,j}^*}(x) = 0$ for some player i. From Lemma 2, it follows that for player $-i$, $X_{-i,j}^* > 0$ for all j, which from Lemma 1 and Theorem 2 implies that player i earns an equilibrium expected payoff of 0. If $i = B$, then because $X_B \geq X_A > 0$, it is clear that player B has a payoff increasing deviation that involves mimicking player A's strategy, which yields B an expected payoff of $\frac{1}{2}\sum_{j=1}^{n} v_{B,j}$. Hence a contradiction. If $i = A$, then player A can mimic player B's strategy with probability $\frac{X_A}{X_B}$ and bid 0 in every battlefield with probability $(1 - \frac{X_A}{X_B})$, which similarly yields A an expected payoff of $\frac{X_A}{2X_B}\sum_{j=1}^{n} v_{A,j}$. This yields a contradiction and completes the proof. □

Lemma 4. *If (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^{n})$, then*

$$\min\{X_{A,j}^*, X_{B,j}^*\} > 0 \quad \text{for all } j.$$

Proof. By way of contradiction, suppose that there exists an equilibrium (P_A^*, P_B^*) in which there is at least one battlefield k with $\min\{X_{A,k}^*, X_{B,k}^*\} = 0$. There are two cases to consider: (i) $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$ for all j or (ii) $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$ for at least one, but not all j. Beginning with case (i), because $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$ for all j, from Lemma 3 $\sum_{j=1}^{n} E_{F_{i,j}^*}(x) > 0$ for each player i, and from Lemma 2 $\max\{X_{A,j}^*, X_{B,j}^*\} > 0$ for all j, there exists at least one battlefield j' with $X_{A,j'}^* > 0$ and $X_{B,j'}^* = 0$ and at least one battlefield j'' with $X_{A,j''}^* = 0$ and $X_{B,j''}^* > 0$. But, player B can strictly increase his total expected payoff by decreasing $X_{B,j''}^*$ by an $\epsilon \in (0, \min\{X_{B,j''}^*, X_{A,j'}^*\})$, allocating ϵ to battlefield j', and utilizing a univariate marginal distribution on battlefield j' that places mass $\left(1 - \frac{\epsilon}{X_{A,j'}^*}\right)$ on 0 and randomizes according to $F_{A,j'}^*$ with probability $\frac{\epsilon}{X_{A,j'}^*}$. Such a deviation would increase player B's expected payoff on battlefield j' by $\frac{\epsilon v_{B,j'}}{2X_{A,j'}^*}$ with no decrease in the expected payoff on battlefield j'', a contradiction.

For case (ii), if $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$ for at least one, but not all j, then there exists at least one battlefield j' with $\min\{X_{A,j'}^*, X_{B,j'}^*\} > 0$ and at least one battlefield k with $\min\{X_{A,k}^*, X_{B,k}^*\} = 0$. Because from Lemma 2 $\max\{X_{A,k}^*, X_{B,k}^*\} > 0$, there exists a player i with $X_{i,k} > 0$ and a player $-i$ with $X_{-i,k} = 0$. Then, because from Lemma 1, each player's unique equilibrium univariate marginal distribution in battlefield j' is given by Theorem 2, player i has a payoff increasing deviation that involves shifting $\epsilon \in (0, X_{i,k}^*)$ of the resource from battlefield k to battlefield j', in battlefield k choosing a distribution function $F_{i,k}(x)$ with $F_{i,k}(0) = 0$ and $E_{F_{i,k}}(x) = X_{i,k}^* - \epsilon$, in battlefield j' choosing a distribution function $F_{i,j'}(x)$ with $F_{i,j'}(0) = 0$, $E_{F_{i,j'}}(x) = X_{i,j'}^* + \epsilon$, and $\mathrm{supp}(F_{i,j'}) \subseteq \mathrm{supp}(F_{-i,j'}^*(x))$. Such a deviation results in no loss to player i's expected payoff in battlefield k. In battlefield j', player i's expected payoff from the distribution function $F_{i,j'}(x)$ when player $-i$'s distribution function is $F^*_{-i,j'}(x)$ is given by

$$v_{i,j'}\int_0^\infty F^*_{-i,j'}(x)dF_{i,j'}(x)=\begin{cases} v_{i,j'}\left(\left(1-\frac{X^*_{-i,j'}}{X^*_{i,j'}}\right)+\frac{\left(X^*_{i,j'}+\epsilon\right)X^*_{-i,j'}}{2\left(X^*_{i,j'}\right)^2}\right) & \text{if } X^*_{i,j'}>X^*_{-i,j'} \\ v_{i,j'}\left(\frac{\left(X^*_{i,j'}+\epsilon\right)}{2X^*_{-i,j'}}\right) & \text{if } X^*_{i,j'}\leq X^*_{-i,j'} \end{cases}.$$

Thus, for all $\epsilon\in(0,X^*_{i,k})$ player i's expected payoff in battlefield j' is strictly higher under the deviation, and there is no loss to player i's expected payoff in battlefield k. This is a contradiction and completes the proof that if (P^*_A,P^*_B) is an equilibrium of $GL(X_A,X_B,n,\{v_{A,j},v_{B,j}\}_{j=1}^n)$ then $\min\{X^*_{A,j},X^*_{B,j}\}>0$ for all j. □

Lemma 5. *If (P^*_A,P^*_B) is an equilibrium of $GL(X_A,X_B,n,\{v_{A,j},v_{B,j}\}_{j=1}^n)$, then there exists a corresponding solution $(\lambda^*_A,\lambda^*_B)$ to system $(\star)$.*

Proof. From Lemma 4, $\min\{X^*_{A,j},X^*_{B,j}\}>0$ for all j. Then, because (P^*_A,P^*_B) is an equilibrium of $GL(X_A,X_B,n,\{v_{A,j},v_{B,j}\}_{j=1}^n)$ and $\min\{X^*_{A,j},X^*_{B,j}\}>0$ for all j, it follows from Lemma 1 that in each battlefield j the players' unique equilibrium univariate marginal distributions are given by Theorem 2. Because, the unique equilibrium univariate marginal distributions given by Theorem 2 are linear, it follows that for player A any deviation P_A that satisfies the following two conditions is payoff maximizing and feasible: (i) in each battlefield j the associated univariate marginal distribution function $F_{A,j}(x)$ satisfies $F_{A,j}(0)=0$ if $X^*_{A,j}>X^*_{B,j}$ and $\text{supp}(F_{A,j})\subseteq\text{supp}(F^*_{B,j}(x))$, and (ii) across battlefields $\sum_{j=1}^n E_{F_{A,j}}(x)=X_A$. Letting $X_{A,j}=E_{F_{A,j}}(x)$, player A's total expected payoff from such a joint distribution function P_A, given that player B is using the joint distribution function P^*_B, is given by

$$\begin{aligned}\pi_A(P_A,P^*_B)&=\sum_{j=1}^n v_{A,j}\int_0^\infty F^*_{B,j}(x)dF_{A,j}(x)\\&=\sum_{j|X^*_{A,j}>X^*_{B,j}} v_{A,j}\left[\left(1-\frac{X^*_{B,j}}{X^*_{A,j}}\right)+\frac{X_{A,j}X^*_{B,j}}{2(X^*_{A,j})^2}\right]+\sum_{j|X^*_{A,j}\leq X^*_{B,j}} v_{A,j}\left(\frac{X_{A,j}}{2X^*_{B,j}}\right).\end{aligned}\tag{19}$$

Similarly, for player B the maximum achievable total expected payoff from a feasible deviation P_B with $\{X_{B,j}\}_{j=1}^n$ is given by

$$\begin{aligned}\pi_B(P_B, P_A^*) &= \sum_{j=1}^n v_{B,j} \int_0^\infty F_{A,j}^*(x) dF_{B,j}(x) \\ &= \sum_{j|X_{A,j}^* \geq X_{B,j}^*} v_{B,j} \left(\frac{X_{B,j}}{2X_{A,j}^*} \right) + \sum_{j|X_{A,j}^* < X_{B,j}^*} v_{B,j} \left[\left(1 - \frac{X_{A,j}^*}{X_{B,j}^*} \right) + \frac{X_{B,j} X_{A,j}^*}{2(X_{B,j}^*)^2} \right]\end{aligned} \tag{20}$$

Because (P_A^*, P_B^*) is an equilibrium of $GL(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$, it must be the case that player A is maximizing equation (19) with respect to $\{X_{A,j}\}_{j=1}^n$ and player B is maximizing equation (20) with respect to $\{X_{B,j}\}_{j=1}^n$. Then, because equations (19) and (20) are concave and continuously differentiable with respect to $\{X_{i,j}\}_{j=1}^n \in \mathbb{R}_+^n$, $i = A, B$ respectively, it follows that there exists a Lagrange multiplier $\lambda_i^* \geq 0$ such that the Kuhn-Tucker first-order conditions hold:[22]

$$\begin{cases} v_{i,j} \left(\frac{1}{2X_{-i,j}^*} \right) - \lambda_i^* = 0 & \text{in each battlefield } j \text{ with } X_{i,j}^* \leq X_{-i,j}^* \\ v_{i,j} \left(\frac{X_{-i,j}^*}{2(X_{i,j}^*)^2} \right) - \lambda_i^* = 0 & \text{in each battlefield } j \text{ with } X_{i,j}^* > X_{-i,j}^* \end{cases}, \tag{21}$$

with complementary slackness condition $\lambda_i^* \geq 0$, $\sum_{j=1}^n X_{i,j}^* \leq X_i$, and $\lambda_i^* \left(\sum_{j=1}^n X_{i,j}^* - X_i \right) = 0$. Complementary slackness is clearly satisfied because from (19) and (20) it is clearly suboptimal to set $\sum_{j=1}^n X_{i,j} < X_i$.

From the first-order conditions in (21) we see that in each battlefield j with $X_{i,j}^* > X_{-i,j}^*$: (i) $X_{i,j}^* = \frac{v_{-i,j}}{2\lambda_{-i}^*}$ and (ii) $X_{i,j}^* = \left(\frac{v_{i,j} X_{-i,j}^*}{2\lambda_i^*} \right)^{1/2}$ or equivalently $X_{-i,j}^* = \frac{\left(v_{-i,j}/2\lambda_{-i}^* \right)^2}{\left(v_{i,j}/2\lambda_i^* \right)}$. Combining (i) and (ii), it follows from budget balance that λ_A^* and λ_B^* solve

$$\sum_{j|X_{A,j}^* > X_{B,j}^*} \frac{v_{B,j}}{2\lambda_B^*} + \sum_{j|X_{A,j}^* \leq X_{B,j}^*} \frac{\left(\frac{v_{A,j}}{\lambda_A^*} \right)^2}{2 \left(\frac{v_{B,j}}{\lambda_B^*} \right)} = X_A \tag{22}$$

and

$$\sum_{j|X_{A,j}^* \geq X_{B,j}^*} \frac{\left(\frac{v_{B,j}}{\lambda_B^*} \right)^2}{2 \left(\frac{v_{A,j}}{\lambda_A^*} \right)} + \sum_{j|X_{A,j}^* < X_{B,j}^*} \frac{v_{A,j}}{2\lambda_A^*} = X_B \tag{23}$$

Because $X_{A,j}^* = \frac{v_{B,j}}{2\lambda_B^*}$ and $X_{B,j}^* = \frac{\left(v_{B,j}/2\lambda_B^* \right)^2}{\left(v_{A,j}/2\lambda_A^* \right)}$ when $X_{A,j}^* > X_{B,j}^*$, and $X_{A,j}^* = \frac{\left(v_{A,j}/2\lambda_A^* \right)^2}{\left(v_{B,j}/2\lambda_B^* \right)}$ and

[22]For further details see p. 187 of Sundaram (1996).

$X^*_{B,j} = \frac{v_{A,j}}{2\lambda^*_A}$ when $X^*_{A,j} \leq X^*_{B,j}$, it follows that $\frac{v_{A,j}}{\lambda^*_A} > \frac{v_{B,j}}{\lambda^*_B}$ if and only if $X^*_{A,j} > X^*_{B,j}$. Thus, the system (22) and (23) is equivalent to system ($\star$). $\square$

This completes the proof of part (i), for each equilibrium of the Generalized General Lotto game, there exists a corresponding solution $(\lambda^*_A, \lambda^*_B)$ to system ($\star$).

We now conclude with the proof of part (ii), for each solution $(\lambda^*_A, \lambda^*_B)$ to system ($\star$) each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginal distributions. From the argument utilized in the proof of Lemma 5, it follows that for each solution $(\lambda^*_A, \lambda^*_B)$ each player i's n-tuple of the expected allocation of the resource to each of the n battlefields, $\{X^*_{i,j}\}^n_{j=1}$, is uniquely determined. Namely, $X^*_{A,j} = \frac{v_{B,j}}{2\lambda^*_B}$ and $X^*_{B,j} = \frac{(v_{B,j}/2\lambda^*_B)^2}{(v_{A,j}/2\lambda^*_A)}$ when $X^*_{A,j} > X^*_{B,j}$, and $X^*_{A,j} = \frac{(v_{A,j}/2\lambda^*_A)^2}{(v_{B,j}/2\lambda^*_B)}$ and $X^*_{B,j} = \frac{v_{A,j}}{2\lambda^*_A}$ when $X^*_{A,j} \leq X^*_{B,j}$. From Lemma 4, each player's expected allocation of the resource to each battlefield is strictly positive, $\min\{X^*_{A,j}, X^*_{B,j}\} > 0$ for all j. Then, because $\{X^*_{A,j}, X^*_{B,j}\}^n_{j=1}$ is uniquely determined by $(\lambda^*_A, \lambda^*_B)$ and $\min\{X^*_{A,j}, X^*_{B,j}\} > 0$ for all j, it follows from Lemma 1 that in each battlefield j the players' unique equilibrium univariate marginal distributions are given by Theorem 2. That is, for each solution $(\lambda^*_A, \lambda^*_B)$ each player in the Generalized General Lotto game has a unique set of Nash equilibrium univariate marginals, and this completes the proof of Theorem 1.

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