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REPRESENTATION FORMULAS FOR HARDY SPACE FUNCTIONS THROUGH THE CUNTZ RELATIONS AND NEW INTERPOLATION PROBLEMS

DANIEL ALPAY, PALLE JORGENSEN, IZCHAK LEWKOWICZ, AND ITZIK MARZIANO

ABSTRACT. We introduce connections between the Cuntz relations and the Hardy space $\mathbf{H}_2$ of the open unit disk $\mathbb{D}$. We then use them to solve a new kind of multipoint interpolation problem in $\mathbf{H}_2$, where for instance, only a linear combination of the values of a function at given points is preassigned, rather than the values at the points themselves.

CONTENTS

1. INTRODUCTION

One motivation for studying representations of the Cuntz relations comes from signal processing, sub-band filters, and their applications to wavelets. This falls within a larger context of multiscale problems,

see for example [17]. In this work we study the Cuntz relations in a different context, and introduce connections between them and the Hardy space $\mathbf{H}_2$ of the open unit disk $\mathbb{D}$. We prove in particular the following results: Let b be a finite Blaschke product of degree M, and let $e_1, \ldots, e_M$ be an orthonormal basis of $\mathbf{H}_2 \ominus b\mathbf{H}_2$. A function f belongs to $\mathbf{H}_2$ and has norm less or equal to 1 if and only if it can be written as

$$(1.1) \qquad f(z) = \sum_{j=1}^{M} e_j(z) f_j(b(z)),$$

where $f_1, \ldots, f_M \in \mathbf{H}_2$, are uniquely defined, and are such that

$$(1.2) \qquad \|f\|_{\mathbf{H}_2}^2 = \sum_{j=1}^{M} \|f_j\|_{\mathbf{H}_2}^2.$$

From now on, we denote by $\|f\|_2$ the norm of an element of $\mathbf{H}_2$. Using Leech's factorization theorem (see Section 2 below), we prove that, equivalently, f belongs to $\mathbf{H}_2$ and has norm less or equal to 1 if and only if it can be written as

$$(1.3) \qquad f(z) = \frac{\sum_{j=1}^{M} e_j(z) \sigma_{1j}(b(z))}{1 - b(z) \sigma_2(b(z))},$$

where

$$(1.4) \qquad \sigma = \begin{pmatrix} \sigma_{11} \\ \vdots \\ \sigma_{1M} \\ \sigma_2 \end{pmatrix}$$

is a Schur function, i.e. analytic and contractive in $\mathbb{D}$.

Representation (1.1) allows us to solve various interpolation problems in $\mathbf{H}_2$ by translating them into tangential interpolation problems at one point (in fact at the origin) in $\mathbf{H}_2^M$. The solution of this latter problem, or more generally, of the bitangential interpolation problem in $\mathbf{H}_2^{p \times q}$ is well known. See for instance [1, 2].

Similarly, the representation (1.3) allows us to solve various interpolation problems in $\mathbf{H}_2$ by translating them into tangential interpolation problems at *one* point (here too, in fact at the origin), for $\mathbb{C}^{M+1}$-valued Schur functions, whose solution is well known. See for instance [12] for the general bitangential interpolation problem for matrix-valued Schur

functions.

We now illustrate these points. First note that, for b the Blaschke product with zeroes the points $a_1, \ldots, a_M$, (1.1) leads to

$$(1.5) \qquad f(a_\ell) = \sum_{j=1}^{M} e_j(a_\ell) f_j(0), \quad \ell = 1, \ldots M.$$

For preassigned values of $f(a_\ell)$, $\ell = 1, \ldots, M$, this reduces the Nevanlinna-Pick interpolation problem for M points in $\mathbf{H}_2$ to a tangential interpolation problem at the origin for functions in $\mathbf{H}_2^M$, whose solution, as already mentioned, is well known. The novelty in the present paper is by exploiting the above reduction scheme to solve multipoint interpolation problems in $\mathbf{H}_2$. For example, consider the following problem:

Problem: *Given M points $a_1, \ldots, a_M$ in $\mathbb{D}$ and $u = \begin{pmatrix} u_1 & u_2 & \cdots & u_M \end{pmatrix} \in \mathbb{C}^{1 \times M}$ and $\gamma \in \mathbb{C}$, find all $f \in \mathbf{H}_2$ such that*

$$(1.6) \qquad \sum_{\ell=1}^{M} u_\ell f(a_\ell) = \gamma.$$

Solution using (1.1)**:** It follows from (1.5) that

$$(1.7) \qquad \begin{aligned} \sum_{\ell=1}^{M} u_\ell f(a_\ell) &= \sum_{j=1}^{M} \left(\sum_{\ell=1}^{M} u_\ell e_j(a_\ell) \right) f_j(0) \\ &= \sum_{j=1}^{M} v_j f_j(0), \end{aligned}$$

with

$$(1.8) \qquad v_j = \sum_{\ell=1}^{M} u_\ell e_j(a_\ell), \quad j = 1, \ldots, M.$$

For preassigned value of the left side of (1.7) this is a classical tangential interpolation problem for $\mathbb{C}^M$-valued functions with entries in the Hardy space. Let $v = \begin{pmatrix} v_1 & v_2 & \cdots & v_M \end{pmatrix} \in \mathbb{C}^{1 \times M}$. Assuming $vv^* \neq 0$ we have that the set of solutions is given by

$$\begin{pmatrix} f_1(z) \\ f_2(z) \\ \vdots \\ f_M(z) \end{pmatrix} = \gamma \frac{v^*}{vv^*} + \left(I_M + (z-1) \frac{v^* v}{vv^*} \right) \begin{pmatrix} g_1(z) \\ g_2(z) \\ \vdots \\ g_M(z) \end{pmatrix},$$

where $g_1, \ldots, g_M \in \mathbf{H}_2$ and

$$\sum_{\ell=1}^{M} \|f_j\|_2^2 = \frac{|\gamma|^2}{vv^*} + \sum_{\ell=1}^{M} \|g_j\|_2^2.$$

It follows from (1.1) that a function $f \in \mathbf{H}_2$ satisfies (1.6) if and only if it can be written as

$$f(z) = \frac{\gamma}{vv^*} \sum_{j=1}^{M} e_j(z)v_j^* + \begin{pmatrix} e_1(z) & e_2(z) & \cdots & e_M(z) \end{pmatrix} B(z) \begin{pmatrix} g_1(z) \\ g_2(z) \\ \vdots \\ g_M(z) \end{pmatrix},$$

where we have denoted $B(z) = \left(I_M + (z-1)\frac{v^*v}{vv^*}\right)$. Note that B is an elementary Blaschke factor, with zero at the origin.

Solution using (1.3): In the case of representation (1.3), we have similarly

(1.9) $$f(a_\ell) = \sum_{j=1}^{M} e_j(a_\ell)\sigma_{1j}(0), \quad \ell = 1, \ldots, M.$$

For preassigned values of $f(a_\ell)$, $\ell = 1, \ldots, M$, this reduces the Nevanlinna-Pick interpolation problem for M points in $\mathbf{H}_2$ to a tangential interpolation problem at the origin for matrix-valued Schur functions (1.4). As in the previous discussion we exploit the above reduction scheme to interpolation problem in the Schur class for multipoint interpolation problems. For example, in the case of the interpolation constraint (1.6), it follows from (1.9) that

(1.10)
$$\sum_{\ell=1}^{M} u_\ell f(a_\ell) = \sum_{\ell=1}^{M} u_\ell \left(\sum_{j=1}^{M} e_j(a_\ell)\right)\sigma_{1j}(0)$$
$$= \sum_{j=1}^{M} v_j \sigma_{1j}(0),$$

with $v_1, \ldots, v_M$ as in (1.8). For preassigned value of the left side of (1.10) this is a classical tangential interpolation problem for $\mathbb{C}^{M+1}$-valued Schur functions.

Problems of the form (1.9) have been studied for $M = 2$, under the name multipoint interpolation problem, in [6]. In that paper, an involution φ of the open unit disk which maps a_1 into a_2 is used. Then one

notes that the function

$$F(z) = \begin{pmatrix} f(z) \\ f(\varphi(z)) \end{pmatrix}$$

satisfies the symmetry

$$F(\varphi(z)) = JF(z), \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This reduces the interpolation problem in $\mathbf{H}_2$ to an interpolation problem with symmetries in $\mathbf{H}_2^2$. Unfortunately this method does not extend to the case $M > 2$. For a related interpolation problem (for Nevanlinna functions), see also [14], where the n-th composition of the map φ is equal to the identity map: $\varphi^{\circ n}(z) = z$.

In the present paper, we use a decomposition of elements in $\mathbf{H}_2$ associated with isometries defined from b and which satisfy the Cuntz relation. The representation of Hardy functions, proved in [5], plays a major role in the reduction to interpolation problems in the setting of Schur functions. To ease the notation, we set the discussion in the framework of scalar-valued functions, but the paper itself (as well as [5]) is developed for matrix-valued functions. Besides being a key player in complex analysis, the Hardy space $\mathbf{H}_2$ of the open unit disk plays an important role in signal processing and in the theory of linear dynamical systems. An element f in $\mathbf{H}_2$ can be described in (at least) three different ways. In terms of: (1) power series, (2) integral conditions, or (3), a positive definite kernel. More precisely, in case (1) one sees f as the z-transform of a discrete signal with finite energy, that is the z-transform of a sequence $(f_n)_{n\in\mathbb{N}_0} \in \ell_2$:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad \|f\|_{\mathbf{H}_2}^2 \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} |f_n|^2 < \infty,$$

In case (2), one expresses the norm (in the equivalent way) as

$$\|f\|_2^2 = \frac{1}{2\pi} \sup_{r\in(0,1)} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty,$$

and sees f as the transfer function (filter) of a ℓ_1-ℓ_2 stable linear system. See [20]. In case (3) we use the fact that $\mathbf{H}_2$ is the reproducing kernel Hilbert space with reproducing kernel $\frac{1}{1-zw^*}$. From the characterization of elements in a reproducing kernel Hilbert space, a function f defined in $\mathbb{D}$ belongs to $\mathbf{H}_2$ if and only if for some $M > 0$, the kernel

(1.11)
$$\frac{1}{1 - zw^*} - \frac{f(z)f(w)^*}{M}$$

is positive definite there. The smallest such M is $\|f\|_2^2$. For $M = 1$, rewriting (1.11) as

$$\frac{a(z)a(w)^* - h(z)h(w)^*}{1 - zw^*}, \quad \text{with} \quad a(z) = \begin{pmatrix} 1 & -zf(z) \end{pmatrix},$$

and using Leech's factorization theorem (see next section), it was proved in [5] that f admits a (in general not unique) representation of the form

$$(1.12) \qquad\qquad f(z) = \frac{\sigma_1(z)}{1 - z\sigma_2(z)},$$

where $\sigma(z) = \begin{pmatrix} \sigma_1(z) \\ \sigma_2(z) \end{pmatrix}$ is analytic and contractive in the open unit disk.

Let now $a \in \mathbb{D}$, and

$$b_a(z) = \frac{z - a}{1 - za^*}.$$

In [10] was proved that the map

$$(1.13) \qquad\qquad T_a f(z) = \frac{\sqrt{1 - |a|^2}}{1 - za^*} f(b_a(z))$$

is from $\mathbf{H}_2$ onto itself and unitary. In the present paper we replace b_a by an arbitrary finite Blaschke product, and define a counterpart of the operator T_a. If $M = \deg b$, we now have, instead of the unitary map, T_a a set of isometries $S_1, \ldots, S_M$ in $\mathbf{H}_2$, defined as follows: Take $e_1, \ldots, e_M$ be an orthonormal basis of the space $\mathbf{H}_2 \ominus b\mathbf{H}_2$. Then,

$$(1.14) \qquad\qquad (S_j h)(z) = e_j(z)h(b(z)), \quad h \in \mathbf{H}_2,$$

with $S_1, \ldots, S_M$ satisfying the Cuntz relations:

$$(1.15) \qquad\qquad \sum_{j=1}^{M} S_j S_j^* = I_{\mathbf{H}_2},$$

$$(1.16) \qquad\qquad S_j^* S_k = \begin{cases} I_{\mathbf{H}_2}, & \text{if} \quad j = k, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from these relations that every element $f \in \mathbf{H}_2$ can be written in a unique way as (1.1):

$$f(z) = \sum_{j=1}^{M} e_j(z) f_j(b(z)),$$

where the $f_j \in \mathbf{H}_2$ and satisfy (1.2)

$$\|f\|_2^2 = \sum_{j=1}^{M} \|f_j\|_2^2.$$

We note that $S_1, \ldots, S_M$ in (1.14) form a finite system of M isometries with orthogonal ranges in $\mathbf{H}_2$, with the sum of the ranges equal to all of $\mathbf{H}_2$. Thus they define a representation of the Cuntz relations. This is a special case of a result of Courtney, Muhly and Schmidt, see [18, Theorem 3.3]. We send the reader to [18] for a survey of the relevant literature and in particular for a discussion of the related papers [28, 27]. For completeness, we provide a proof, in the matrix-valued case, using reproducing kernel spaces techniques (see Section 4). As already mentioned, one motivation for studying representations of the Cuntz relations comes from signal processing. Our present application of the Cuntz relations to Leech's problem from harmonic analysis is entirely new. The immediate relevance to sub-band filters is a careful selecting of the Cuntz isometries, one for each frequency sub-band, see [26, Chapter 9]. In the case of wavelet applications, the number M is the scaling number characterizing the particular family of wavelets under discussion.

We note that relations with the Cuntz relations in the indefinite inner product case have been considered in [9], in the setting of de Branges Rovnyak spaces; see [15, 16]. This suggests connections with interpolation in these spaces (see [2, Section 11], [13]), which will be considered elsewhere. We briefly discuss some of these aspects in Section 7.

Our paper is interdisciplinary, a mix of pure and applied, and we are motivated by several prior developments and work by other authors. This we discuss in section 2, 3 and 4 below. For the readers convenience, we mention here briefly some of these connections. One motivation comes from earlier work [1] on two-sided, and tangential, interpolation for matrix functions; see also references [2] through [5], and [14]. We make additional connections also to to interpolation in de Branges-Rovnyak spaces [13], to wavelet filters, see e.g., [17], and to iterated function systems, see [18] by Courtney, Muhly, and Schmidt, and [28] by Rochberg, to Hardy classes [29, 30] and to classical harmonic analysis; see e.g., [32, 33, 34, 35].

The paper consists of six sections besides the introduction. Sections 2 and 3 are of a review nature: In the second section we discuss Leech's theorem and in the third section we discuss the realization result of [5]. In Section 4 we consider, in the matrix-valued case, a set of operators which satisfy the Cuntz relations, and were considered earlier in [18] in the scalar case. Section 5 is devoted to the proof of the matrix version of (1.3). We use the representation theorem of Section 5 to solve in

Section 6 new types of multipoint interpolation problems. Finally we outline in the last section how some of the results extend to the case of de Branges Rovnyak spaces.

2. LEECH'S THEOREM

As already mentioned in the introduction, we set the paper in the framework of matrix-valued functions. When the Taylor coefficients f_n are $\mathbb{C}^{p \times q}$-valued, one defines a $\mathbb{C}^{q \times q}$-valued quadratic form by

$$[f, f] \overset{\text{def.}}{=} \frac{1}{2\pi} \int_0^{2\pi} (f(e^{it}))^* f(e^{it}) dt = \sum_{n=0}^{\infty} f_n^* f_n.$$

The space $\mathbf{H}^{p \times q}$ consists of the functions for which $\text{Tr}\,[f, f] < \infty$ In [5] a representation theorem for elements $f \in \mathbf{H}_2^{p \times q}$ such that $[f, f] \leq I_q$ in terms of Schur functions was presented. See Theorem 3.1 below. Recall first that a $\mathbb{C}^{p \times q}$-valued function σ *defined* in the open unit disk is analytic and contractive in the open unit disk if and only if the kernel

$$(2.1) \qquad K_\sigma(z, w) = \frac{I_p - \sigma(z)\sigma(w)^*}{1 - zw^*}$$

is positive definite in the open unit disk. Such functions are called Schur functions, denoted by $\mathscr{S}^{p \times q}$. Given $\sigma \in \mathscr{S}^{p \times q}$ and a $\mathbb{C}^{k \times p}$-valued function A analytic in the open unit disk, the kernel

$$A(z)K_\sigma(z, w)A(w)^* = \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - zw^*}$$

where $B = A\sigma$, is positive in $\mathbb{D}$. Leech's theorem asserts that the converse holds: if A and B are respectively $\mathbb{C}^{k \times p}$-valued and $\mathbb{C}^{k \times q}$-valued functions *defined* in $\mathbb{D}$ and such that the kernel

$$(2.2) \qquad \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - zw^*}$$

is positive definite in $\mathbb{D}$, then there exists $\sigma \in \mathscr{S}^{p \times q}$ such that $B = A\sigma$. Two proofs of this theorem hold. The first assumes that A and B are bounded in the open unit disk, and uses a commutant lifting result of M. Rosenblum. See [29] for Rosenblum's result and [30, Example 1, p. 107], [7] for Leech's theorem. The other proof requires only analyticity of A and B in $\mathbb{D}$, and uses tangential interpolation theory for Schur functions, together with the normal family theorem. One can extend these arguments to functions of bounded type in $\mathbb{D}$, or even further weaken these hypothesis. For completeness, we now outline a proof of Leech's theorem for continuous functions A and B. We first recall the following: Let $N \in \mathbb{N}$ and let $w_1, \ldots, w_N \in \mathbb{D}$, $\xi_1, \ldots, \xi_N \in \mathbb{C}^p$

and $\eta_1, \ldots, \eta_N \in \mathbb{C}^q$. The tangential Nevanlinna-Pick interpolation problem consists in finding all Schur functions $\sigma \in \mathscr{S}^{p \times q}$ such that

$$\sigma(w_j)^* \xi_j = \eta_j, \quad j = 1, \ldots, N.$$

The fact that the function $K_\sigma(z, w)$ defined by (2.1) is positive definite in $\mathbb{D}$ implies that a necessary condition for the tangential Nevanlinna-Pick interpolation problem to have a solution is that the $N \times N$ Hermitian matrix P (known as the Pick matrix) with ℓj entry

$$(2.3) \qquad P_{\ell,j} = \frac{\xi_\ell^* \xi_j - \eta_\ell^* \eta_j}{1 - w_\ell w_j^*}$$

is non negative. This condition is in fact also sufficient, and there are various methods to describe all solutions in terms of a linear fractional transformation. See for instance [12, 19, 21, 22]. With this result at hand we can outline a proof of Leech's theorem as follows: We assume given two functions A and B, respectively $\mathbb{C}^{k \times q}$-valued and $\mathbb{C}^{k \times q}$-valued, continuous in $\mathbb{D}$ and such that the kernel

$$\frac{A(z)A(w)^* - B(z)B(w)^*}{1 - zw^*}$$

is positive definite there. Consider $w_1, w_2 \ldots$ a countable set of points dense in the open unit disk. The Hermitian block matrix with ℓj entry

$$\frac{A(w_\ell)A(w_j)^* - B(w_\ell)B(w_j)^*}{1 - w_\ell w_j^*}, \quad \ell, j = 1, \ldots, N,$$

is non negative, and therefore, by the above mentioned result on Nevanlinna-Pick interpolation, there exists a Schur function $\sigma_N \in \mathscr{S}^{p \times q}$ such that

$$A(w_\ell)\sigma_N(w_\ell) = B(w_\ell), \quad \ell = 1, \ldots N.$$

To conclude the proof, one uses the normal family theorem to find a function $\sigma \in \mathscr{S}^{p \times q}$ such that $A(w_\ell)\sigma(w_\ell) = B(w_\ell)$ for $\ell \in \mathbb{N}$. By continuity, this equality extends then to all of $\mathbb{D}$.

3. A REPRESENTATION OF $\mathbf{H}_2^{p \times 2}$ FUNCTIONS

Leech's theorem can be used to find a representation of elements of $\mathbf{H}_2$ in terms of Schur functions, as we now recall. See [5, 4]. The following result is proved in [5]. In the scalar case, it was proved earlier by Sarason using different methods. See [35, p. 50], [32, 33, 34]. In the discussion we shall find it convenient to partition $\sigma \in \mathscr{S}^{(p+q) \times q}$ as

$$(3.1) \qquad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix},$$

with σ_1 being $\mathbb{C}^{p \times q}$-valued and σ_2 being $\mathbb{C}^{q \times q}$-valued.

Theorem 3.1. *Let* $H \in \mathbf{H}_2^{p \times q}$. *Then the following are equivalent:*
(1) *It holds that:*

(3.2) $$[H, H] \leq I_q,$$

(2) *The kernel*

(3.3) $$\frac{I_p}{1 - zw^*} - H(z)H(w)^*$$

is positive definite in $\mathbb{D}$.
(3) *There is a Schur function* $\sigma \in \mathscr{S}^{(p+q) \times q}$ *(see* (3.1)*) so that*

(3.4) $$H(z) = \sigma_1(z)(I_q - z\sigma_2(z))^{-1}.$$

The key to proof of this theorem is to note that the kernel (3.3) can be rewritten in the form (2.2) with

$$A(z) = \begin{pmatrix} I_p & zH(z) \end{pmatrix} \quad \text{and} \quad B(z) = H(z),$$

and apply Leech's theorem: There exists $\sigma \in \mathscr{S}^{(p+q) \times q}$ as in (3.1) such that $A(z)\sigma(z) = B(z)$, that is,

$$\sigma_1(z) + zH(z)\sigma_2(z) = H(z).$$

Equation (3.1) follows.

We note that an extension of the previous theorem to elements in the Arveson space was given in [3, Theorem 10.3, p. 182].

4. THE CUNTZ RELATIONS IN $\mathbf{H}_2^p$

Let b be a finite Blaschke product of degree M, and let

$$\mathscr{H}(b) = \mathbf{H}_2 \ominus b\mathbf{H}_2.$$

It is well known that this space is finite dimensional and R_0-invariant, where

$$R_0 f(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathscr{H}(b).$$

Let

$$\begin{pmatrix} f_1(z) & \dots & f_M(z) \end{pmatrix} = C(I_M - zA)^{-1}$$

denote a basis of $\mathscr{H}(b)$, where $(C, A) \in \mathbb{C}^{1 \times M} \times \mathbb{C}^{M \times M}$ is an observable pair, namely:

$$\cap_{n=0}^{\infty} \ker CA^n = \{0\}.$$

Since the spectrum of A is inside the open unit disk, the series

$$P = \sum_{\ell=0}^{\infty} A^{\ell*} C^* C A^{\ell}$$

converges and $P > 0$. The matrix P is the Gram matrix (observability Gramian in control terminology) of the basis $f_1, \ldots, f_M$, and satisfies:

$$P = \frac{1}{2\pi} \int_0^{2\pi} \left(f_1(e^{it}) \quad \ldots \quad f_M(e^{it}) \right)^* \left(f_1(e^{it}) \quad \ldots \quad f_M(e^{it}) \right) dt.$$

This matrix turns to be identical to the Pick matrix defined in (2.3). We denote by $\mathbf{H}_2^p$ the Hilbert space of $\mathbb{C}^p$-valued functions with entries in $\mathbf{H}_2$, and with norm

$$\| \begin{pmatrix} h_1 \\ \vdots \\ h_p \end{pmatrix} \|_{\mathbf{H}_2^p}^2 = \sum_{n=1}^p \|h_n\|_{\mathbf{H}_2}^2.$$

We note that $\mathbf{H}_2^p$ is the reproducing kernel Hilbert space with reproducing kernel $\frac{I_p}{1-zw^*}$.

We now introduce the Cuntz relations into this framework. This was treated earlier in [18] using different methods.

Theorem 4.1. *Let $e_1, \ldots, e_M$ be an orthonormal basis of $\mathscr{H}(b)$ and, for $j = 1, \ldots M$*

$$(4.1) \qquad\qquad (S_j h)(z) = e_j(z) h(b(z)), \quad h \in \mathbf{H}_2^p.$$

Then the S_j satisfy the Cuntz relations:

$$(4.2) \qquad\qquad \sum_{j=1}^M S_j S_j^* = I_{\mathbf{H}_2^p},$$

$$(4.3) \qquad\qquad S_j^* S_k = \begin{cases} I_{\mathbf{H}_2^p}, & if \quad j = k, \\ 0, & otherwise. \end{cases}$$

Proof: We proceed in a number of steps. The proof of the Cuntz identities (4.2)-(4.3) is given in STEPS 4 and 5, respectively.

STEP 1: *The set $\mathbf{H}_2^p(b)$ of functions of the form*

$$F(z) = f(b(z)), \quad f \in \mathbf{H}_2^p,$$

with norm

$$\|F\|_{\mathbf{H}_2^p(b)} = \|f\|_{\mathbf{H}_2^p}.$$

is the reproducing kernel Hilbert space with reproducing kernel $\frac{I_p}{1-b(z)b(w)^*}$.

This can be checked directly, but is also a special case of [8, Theorem 3.1, p. 109].

STEP 2: *The operator M_{e_j} of multiplication by e_j is an isometry from $\mathbf{H}_2^p(b)$ into $\mathbf{H}_2^p$. Furthermore, the range of M_{e_j} and M_{e_k} are orthogonal for $j \neq k$.*

Indeed, let $u, v \in \mathbb{C}^p$. It holds that

$$\langle e_j b^n u, e_k b^m v \rangle_{\mathbf{H}_2^p} = \begin{cases} v^* u & j = k \quad \text{and} \quad m = n, \\ 0, & \text{otherwise.} \end{cases}$$

We use that multiplication by b is an isometry from $\mathbf{H}_2$ into itself. If $n = m$ and $j = k$, the claim is clear. If $n = m$ and $j \neq k$, this is just the orthogonality of e_j and e_k. If $n > m$, we have

$$\langle e_j b^n u, e_k b^m v \rangle_{\mathbf{H}_2^p} = \langle e_j b^{n-m} u, e_k v \rangle_{\mathbf{H}_2^p} = 0,$$

since $e_j b^{n-m} u \in b\mathbf{H}_2^p$ is orthogonal to $e_k v$ whose components belong to $\mathscr{H}(b) = \mathbf{H}_2 \ominus b\mathbf{H}_2$. The case $n < m$ is obtained by interchanging the role of j and k.

STEP 3: *Let $e_1, \ldots, e_M$ denote an orthonormal basis of $\mathscr{H}(b)$. Then,*

$$(4.4) \qquad\qquad \mathbf{H}_2^p = \oplus_{j=1}^M e_j \mathbf{H}_2^p(b).$$

Indeed, the reproducing kernel is written in terms of the orthonormal basis as (see for instance [11, (6) p. 346], [31])

$$(4.5) \qquad\qquad K_b(z,w) = \sum_{j=1}^M e_j(z)(e_j(w))^*.$$

Thus

$$(4.6) \qquad \frac{I_p}{1-zw^*} = \frac{I_p}{1-b(z)b(w)^*}\frac{1-b(z)b(w)^*}{1-zw^*} = \sum_{j=1}^M k_j(z,w),$$

with

$$k_j(z,w) = \frac{e_j(z)(e_j(w))^* I_p}{1-b(z)b(w)^*}.$$

Equality (4.6) expresses the positive definite kernel $\frac{I_p}{1-zw^*}$ as a sum of positive definite kernels. The reproducing kernel space associated to k_j is $e_j \mathbf{H}_2(b)$. Therefore, (4.4) holds as a sum of vector spaces; see [11, p. 352]. Since, by STEP 2, $e_j \mathbf{H}_2^p(b)$ is isometrically included into $\mathbf{H}_2^p$, the

sum is orthogonal.

STEP 4: S_j and S_k are isometries, with orthogonal ranges when $j \neq k$.

The fact that S_j is an isometry follows from STEPS 1 and 2. Indeed the range of S_j is in $\mathbf{H}_2^p$ by STEP 2 and

$$\|S_j h\|_{\mathbf{H}_2^p}^2 = \|h(b)\|_{\mathbf{H}_2^p(b)}^2 \quad \text{(by STEP 2)}$$
$$= \|h\|_{\mathbf{H}_2^p}^2 \quad \text{(by STEP 1)}.$$

Furthermore, for $f, g \in \mathbf{H}_2^p$ and $j \neq k$,

$$\langle S_j f, S_k g \rangle_{\mathbf{H}_2^p} = \langle M_{e_j} f(b), M_{e_k} g(b) \rangle_{\mathbf{H}_2^p} = 0$$

by STEP 2.

STEP 5. It holds that $\sum_{j=1}^M S_j S_j^* = I_{\mathbf{H}_2^p}$.

Indeed, by the properties of multiplication and composition operators in reproducing kernel Hilbert spaces, we have that, with $\rho_w(z) = 1 - zw^*$, and $u \in \mathbb{C}^p$:

$$(S_j^* \frac{u}{\rho_w})(z) = \frac{u}{\rho_{b(w)}(z)}(e_j(w))^*.$$

Thus

$$(\sum_{j=1}^M S_j S_j^*)(\frac{u}{\rho_w})(z) = \sum_{j=1}^M \frac{1}{1 - b(z)b(w)^*} e_j(z)(e_j(w))^* = \frac{u}{\rho_w(z)},$$

where we have used (4.5) and (4.6). This ends the proof since the closed linear span of the functions $\frac{1}{\rho_w}$ is all of $\mathbf{H}_2$. $\qquad \square$

Thus we have the following decomposition result for elements in $\mathbf{H}_2$. When $M = 1$, this result reduces to the fact that the operator T_a defined in (1.13) is a unitary map from $\mathbf{H}_2$ into itself.

Theorem 4.2. *Let b be a finite Blaschke product of degree M, and let $e_1, \ldots, e_M$ be an orthonormal basis of $\mathscr{H}(b)$. Then, every element $H \in \mathbf{H}_2^{p \times q}$ can be written in a unique way as*

$$(4.7) \qquad H(z) = \sum_{j=1}^M e_j(z) H_j(b(z)),$$

where the $H_j \in \mathbf{H}_2^{p \times q}$ and

$$(4.8) \qquad [H, H] = \sum_{j=1}^{M} [H_j, H_j]$$

Proof: We define operators $S_1, \ldots, S_M$ as in (4.1). Let $H \in \mathbf{H}_2^{p \times q}$ and $\xi \in \mathbb{C}^q$. It follows from the definition (4.1) of the S_j and from (4.2) that

$$H(z)\xi = \sum_{j=1}^{M} e_j(z) H_j \xi(z),$$

where $H_j \in \mathbf{H}_2^{p \times q}$ is defined by $H_j \xi = S_j^*(H\xi)$. Taking now into account (4.3) we have

$$
\begin{aligned}
\xi^*[H, H]\xi &= \langle H\xi, H\xi \rangle_{\mathbf{H}_2^p} \\
&= \sum_{\ell, j=1}^{M} \langle S_\ell S_\ell^*(H\xi), S_j S_j^*(H\xi) \rangle_{\mathbf{H}_2^p} \\
&= \sum_{j=1}^{M} \langle S_j^*(H\xi), S_j^*(H\xi) \rangle_{\mathbf{H}_2^p} \\
&= \sum_{j=1}^{M} \langle H_j \xi, H_j \xi \rangle_{\mathbf{H}_2^p} \\
&= \xi^* \sum_{j=1}^{M} [H_j, H_j]\xi,
\end{aligned}
$$

and hence (4.8) holds. $\qquad \qquad \square$

5. Representation of elements of $\mathbf{H}_2^{p \times q}$

We now present a generalization of Theorem 3.1. To this end, we generalize (1.4) to a partitioning of a matrix-valued functions $\sigma \in \mathscr{S}^{(Mp+q) \times q}$ as

$$(5.1) \qquad \sigma = \begin{pmatrix} \sigma_{11} \\ \vdots \\ \sigma_{1M} \\ \sigma_2 \end{pmatrix},$$

with $\sigma_{11}, \ldots, \sigma_{1M}$ being $\mathbb{C}^{p \times q}$-valued and σ_2 being $\mathbb{C}^{q \times q}$-valued.

Theorem 5.1. *Let b be a preassigned finite Blaschke product, and let $e_1, \ldots, e_M$ be an orthonormal basis of $\mathscr{H}(b)$. Let $H \in \mathbf{H}_2^{p \times q}$. Then the following are equivalent:*
(1) Condition (3.2) holds: $[H, H] \leq I_q$.
(2) There exists $\sigma \in \mathscr{S}^{(Mp+q) \times q}$ such that

$$(5.2) \qquad H(z) = \left(\sum_{j=1}^{M} e_j(z) \sigma_{1j}(b(z)) \right) (I_q - b(z)\sigma_2(b(z)))^{-1},$$

where $\sigma \in \mathscr{S}^{(Mp+q) \times q}$ is as in (5.1).

Proof: By Theorem 4.2, H subject to (3.2) can be written in a unique way as (4.7), and it follows from (4.8) that

$$\sum_{j=1}^{M} [H_j, H_j] \leq I_q.$$

Using Theorem 3.1 with the function

$$G = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_M \end{pmatrix} \in \mathbf{H}_2^{Mp \times q},$$

we see that there exists $\sigma \in \mathscr{S}^{(Mp+q) \times q}$ (see (5.1)) such that

$$\begin{pmatrix} H_1(z) \\ H_2(z) \\ \vdots \\ H_M(z) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(z) \\ \sigma_{12}(z) \\ \vdots \\ \sigma_{1M}(z) \end{pmatrix} (I_q - z\sigma_2(z))^{-1}.$$

The result follows using (4.7). $\qquad \square$

The results in [5] are a special case of a family of interpolation problems with relaxed constraints. See [25, 24]. We plan in a future publication to consider these results in our new extended setting.

6. New interpolation problems

We have outlined in the introduction the connections between multipoint interpolations and representations (1.1) and (1.3). We now add some details. Interpolation problems whose solutions are outlined in the present section will be considered in full details in a future publication.

The case of (1.1)**:** We consider the following problem: Find all functions $H \in \mathbf{H}_2^{p \times q}$ such that

$$(6.1) \qquad \sum_{j=1}^{M} \xi_j H^{(j-1)}(a) = \gamma,$$

for some pre-assigned matrices $\xi_1, \ldots, \xi_M \in \mathbb{C}^{r \times p}$ and $\gamma \in \mathbb{C}^{r \times q}$. To solve this problem we use (4.7) with

$$b(z) = \left(\frac{z - a}{1 - za^*} \right)^M.$$

A basis of $\mathscr{H}(b)$ is given by

$$\frac{1}{1 - za^*}, \frac{z}{(1 - za^*)^2}, \ldots, \frac{z^{M-1}}{(1 - za^*)^M}.$$

(see for instance [21]). Set

$$(6.2) \qquad E(z) = \left(\frac{1}{1-za^*} I_p \quad \frac{z}{(1-za^*)^2} I_p \quad \cdots \quad \frac{z^{M-1}}{(1-za^*)^M} I_p \right).$$

Since

$$(6.3) \qquad b(a) = b'(a) = \cdots = b^{(M-1)}(a) = 0,$$

and with

$$\mathscr{H}(z) = \begin{pmatrix} H_1(b(z)) \\ H_2(b(z)) \\ \vdots \\ H_M(b(z)) \end{pmatrix} \in \mathbf{H}_2^{Mp \times q},$$

we have that

$$H(a) = E(a)\mathscr{H}(0)$$
$$H'(a) = E'(a)\mathscr{H}(0)$$
$$\vdots$$
$$H^{(M-1)}(0) = E^{(M-1)}(a)\mathscr{H}(0).$$

Therefore, the interpolation problem (6.1) is equivalent to

$$C\mathscr{H}(0) = \gamma,$$

with $C \in \mathbb{C}^{r \times Mp}$ given by

$$C = \sum_{j=0}^{M-1} \xi_j E^{(j)}(a).$$

When $CC^* > 0$, this in turn can be solved using [2, Section 7], or directly, as

$$\mathscr{H}(z) = C^*(CC^*)^{-1}\gamma + \left(I_{Mp} + (z-1)C^*(CC^*)^{-1}C\right)\mathscr{G}(z),$$

where $\mathscr{G} \in \mathbf{H}_2^{Mp \times q}$. The formula for H follows. We note that

$$[\mathscr{H}, \mathscr{H}] = \gamma^*(CC^*)^{-1}\gamma + [\mathscr{G}, \mathscr{G}].$$

The case where CC^* is not invertible is solved using pseudo-inverses.

The case of (1.3)**:** We here assume first that $p = q = 1$ and

$$b(z) = \prod_{\ell=1}^{M} \frac{z - a_\ell}{1 - za_\ell^*},$$

where the a_ℓ are distinct points in $\mathbb{D}$. We have now

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad A = \text{diag}\,(a_1^*, a_2^*, \ldots, a_M^*).$$

Note that the pair (C, A) is observable. Define

(6.4) $$P_{\ell j} = \frac{1}{1 - a_j a_\ell^*}, \quad \ell, j = 1, \ldots, M.$$

Namely we are in the case (2.3) with the $\xi_j = 1$ and the $\eta_j = 0$. In other words, P is the Pick matrix obtained while interpolating all the points a_ℓ to the origin. We mention the papers [36] and [23] for a related discussion.

Proposition 6.1. *Let* $u = \begin{pmatrix} u_1 & u_2 & \cdots & u_M \end{pmatrix} \in \mathbb{C}^{1 \times M}$ *and* $\gamma \in \mathbb{C}$ *be pre-assigned. Then the following are equivalent:*
(1) *It holds that*

$$\sum_{\ell=1}^{M} u_\ell h(a_\ell) = \gamma \quad and \quad \|h\|_{\mathbf{H}_2} \leq 1.$$

(2) h *is of the form*

$$C(I - zA)^{-1}P^{-1/2}\sigma_1(b(z))(1 - b(z)\sigma_2(b(z)))^{-1},$$

where $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \in \mathscr{S}^{M+1}$ *is such that*

(6.5) $$\begin{pmatrix} uP^{1/2} & 0 \end{pmatrix} \sigma(0) = \gamma,$$

where P *is defined by* (6.4).

When $b(z) = (\frac{z-a}{1-za^*})^M$ for some $M \in \mathbb{N}$ and $a \in \mathbb{D}$, one obtains a different kind of interpolation problem, as we now explain. Rewriting (5.2) as

$$(6.6) \qquad H(z)(I_q - b(z)\sigma_2(b(z))) = E(z)\sigma(b(z)),$$

where $E(z)$ is given by (6.2) with $p = 1$, and

$$\sigma_1(z) = \begin{pmatrix} \sigma_{11}(z) \\ \sigma_{12}(z) \\ \vdots \\ \sigma_{1M}(z) \end{pmatrix}.$$

Differentiating (6.6) $M - 1$ times and taking into account (6.3), we obtain that

$$(6.7) \qquad \begin{pmatrix} H(a) \\ H'(a) \\ \vdots \\ H^{(M-1)}(a) \end{pmatrix} = \begin{pmatrix} E(a) & 0_{p \times p} \\ E'(a) & 0_{p \times p} \\ & \vdots \\ E^{(M-1)}(a) & 0_{p \times p} \end{pmatrix} \begin{pmatrix} \sigma_1(0) \\ \sigma_2(0) \end{pmatrix}.$$

This allows to reduce (6.1) to a standard tangential interpolation problem for Schur functions.

7. The case of de Branges Rovnyak spaces

Let s be a Schur function. The kernel $k_s(z, w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}$ is positive definite in the open unit disk, and the associated reproducing kerenl Hilbert space will be denoted by $\mathscr{H}(s)$. Such spaces were introduced and studied in depth by de Branges and Rovnyak in [16]. When s is an inner function (and in particular when s is finite Blaschke product), we have

$$\mathscr{H}(s) = \mathbf{H}_2 \ominus s\mathbf{H}_2.$$

In general, $\mathscr{H}(s)$ is only contractively included in $\mathbf{H}_2$. Let moreover b be a finite Blaschke product. We have

$$\frac{1 - s(b(z))s(b(w))^*}{1 - zw^*} = \frac{1 - s(b(z))s(b(w))^*}{1 - b(z)b(w)^*} \frac{1 - b(z)b(w)^*}{1 - zw^*}$$

$$(7.1) \qquad\qquad = \sum_{j=1}^M e_j(z)e_j(w)^* \frac{1 - s(b(z))s(b(w))^*}{1 - b(z)b(w)^*},$$

with $e_1, \dots, e_M$ an orthogonal basis of $\mathscr{H}(b)$. This decomposition allows us to define the operators $S_1, \dots, S_M$ as in (4.1), so that the following holds:

Theorem 7.1. *The operators $S_1, \ldots, S_M$ are continuous from $\mathscr{H}(s)$ into $\mathscr{H}(s(b))$ and satisfy the Cuntz relations:*

$$(7.2) \qquad \sum_{j=1}^{M} S_j S_j^* = I_{\mathscr{H}(s(b))},$$

$$(7.3) \qquad S_j^* S_k = \begin{cases} I_{\mathscr{H}(s)}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We proceed in a number of steps.

STEP 1: *The reproducing kernel Hilbert space $\mathscr{M}(s, b)$ with reproducing kernel $\frac{1 - s(b(z))s(b(w))^*}{1 - b(z)b(w)^*}$ consists of the functions of the form $F(z) = f(b(z))$, with $f \in \mathscr{H}(s)$ and norm*

$$\|F\|_{\mathscr{M}(s,b)} = \|f\|_{\mathscr{H}(s)}.$$

This follows from a direct computation.

STEP 2: *The formula*

$$\left(T_j(k_{s(b)}(\cdot, w))\right)(z) = k_s(z, b(w))e_j(w)^*, \quad w \in \mathbb{D}.$$

defines a bounded densely defined operator, which has an extension to all of $\mathscr{H}(s(b))$, and whose adjoint is S_j.

This follows from the decomposition (7.1).

STEP 3: (7.2) *holds.*

Indeed

$$\left(\left(\sum_{j=1}^{M} S_j S_j^*\right) k_{s(b)}(\cdot, w)\right)(z) = \sum_{j=1}^{M} e_j(z)k_s(b(z), b(w))e_j(w)^*$$
$$= k_{s(b)}(z, w),$$

and hence, by continuity, equality (7.2) holds in $\mathscr{H}(s(b))$.

STEP 4: *For $j \neq k$ we have*

$$(7.4) \qquad e_j \mathscr{M}(s, b) \cap e_k \mathscr{M}(s, b) = \{0\}$$

Indeed, let $\mathbf{H}_2(b)$ be as in STEP 1 in the proof of Theorem 4.1. We have $\mathscr{H}(s) \subset \mathbf{H}_2$ and hence

$$\mathscr{M}(s, b) \subset \mathbf{H}_2(b).$$

Thus (7.4) follows from STEP 2 of that same theorem.

STEP 5: *Let M_{e_j} denote the operator of multiplication by e_j. It holds that*

(7.5) $$\mathscr{H}(s(b)) = \oplus_{j=1}^M M_{e_j} \mathscr{M}(s, b).$$

This follows from the decomposition (7.1), which implies that the sum

$$\mathscr{H}(s(b)) = \sum_{j=1}^M M_{e_j} \mathscr{M}(s, b)$$

holds, and from STEP 4, which insures that the sum is direct.

STEP 5: *(7.3) hold.*

Indeed, from STEP 4, the range of the operators M_{e_j} and M_{e_k} are orthogonal for $j \neq k$, and M_{e_j} is an isometry. $\qquad\square$

Finally, we remark that (7.2) leads to decompositions of elements of the space $\mathscr{H}(s(b))$ in terms of elements of the space $\mathscr{H}(s)$ similar to (1.1): Every element $f \in \mathscr{H}(s(b))$ can be written in a unique way as

$$f(z) = \sum_{j=1}^M e_j(z) f_j(b(z)),$$

where $f_1, \ldots, f_M \in \mathscr{H}(s)$. Furthermore

$$\|f\|_{\mathscr{H}(s(b))}^2 = \sum_{j=1}^M \|f_j\|_{\mathscr{H}(s)}^2.$$

Multipoint interpolation problems can be also considered in this setting, building in particular on the recent work of Ball, Bolotnikov and ter Horst [13] on interpolation in de Branges Rovnyak spaces. This will be developed in a separate publication.

Our paper is meant as an interdisciplinary contribution, and it involves an approach to filters and to operators having its genesis in many different fields, both within mathematics and within engineering. We hope that we have succeeded at least partially in communicating across traditional lines of division separating these fields. As a result our listed

references included below is likely to be incomplete. We thank in particular Professor Paul Muhly for improving our reference list.

With apologies to Goethe and to Frenchmen:

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different

Johann Wolfgang von Goethe.

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