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Schur Functions and Their Realizations in the Slice Hyperholomorphic Setting

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SCHUR FUNCTIONS AND THEIR REALIZATIONS IN THE SLICE HYPERHOLOMORPHIC SETTING

DANIEL ALPAY, FABRIZIO COLOMBO, AND IRENE SABADINI

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ABSTRACT. In this paper we start the study of Schur analysis in the quaternionic setting using the theory of slice hyperholomorphic functions. The novelty of our approach is that slice hyperholomorphic functions allows to write realizations in terms of a suitable resolvent, the so called S -resolvent operator and to extend several results that hold in the complex case to the quaternionic case. We discuss reproducing kernels, positive definite functions in this setting and we show how they can be obtained in our setting using the extension operator and the slice regular product. We define Schur multipliers, and find their co-isometric realization in terms of the associated de Branges-Rovnyak space.

1. INTRODUCTION

In this paper we develop Schur analysis, and in particular the Schur algorithm, and a theory of linear systems when the complex numbers are replaced by the skew-field of quaternions. An important tool is the theory of slice hyperholomorphic functions. So there is a combination of a non-commutative setting (since the quaternions lack the commutativity property) and of analyticity (via the slice hyperholomorphic functions). Since the paper is aimed at two different audiences, namely researchers from Clifford analysis and researchers from operator theory and classical linear system theory, we will survey the basic definitions from both fields needed in the paper.

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We denote by \mathcal{S} the set of functions analytic and contractive in the open unit disk $\mathbb{D} \subset \mathbb{C}$. Such functions bear various names, and we will call them *Schur functions* in the present paper. Let $s \in \mathcal{S}$, and assume that $|s(0)| < 1$ (and therefore, by the maximum modulus principle, s is not equal to a unitary constant, but takes strictly contractive values in \mathbb{D}). Then, the function

$$(1.1) \quad s^{(1)}(z) = \begin{cases} \frac{1}{z} \frac{s(z) - s(0)}{1 - \overline{s(0)}s(z)}, & z \neq 0, \\ \frac{s'(0)}{1 - |s(0)|^2}, & z = 0, \end{cases}$$

also belongs to \mathcal{S} . More generally, the recursion

$$(1.2) \quad s^{(n+1)}(z) = \begin{cases} \frac{1}{z} \frac{s^{(n)}(z) - s^{(n)}(0)}{1 - \overline{s^{(n)}(0)}s^{(n)}(z)}, & z \neq 0, \\ \frac{(s^{(n)})'(0)}{1 - |s^{(n)}(0)|^2}, & z = 0, \end{cases}$$

defines a sequence, finite or infinite, of Schur functions $s^{(0)}, s^{(1)}, \dots$. The sequence is infinite if

$$s^{(n)}(0) \in \mathbb{D}, \quad n = 0, 1, \dots,$$

while it stops at rank n if $|s^{(n)}(0)| = 1$. The construction of this sequence is the celebrated *Schur algorithm*, developed by I. Schur in 1917. See [72, 73, 58]. The numbers $\rho_n = s^{(n)}(0)$ are called the Schur coefficients associated to s , and the sequence (finite or infinite) of Schur coefficients uniquely determines s .

Let us already make the following remark at this point. In the Schur algorithm one makes use of Schwarz's lemma and of the elementary fact that if u and v are in the open unit disk, so is

$$(1.3) \quad \frac{u - v}{1 - u\bar{v}}.$$

Let us denote by \mathbb{B} the open unit ball of the quaternions. If one replaces in (1.3) u and v by quaternions of norm strictly less than 1, then both

$$(u - v)(1 - u\bar{v})^{-1} \quad \text{and} \quad (1 - u\bar{v})^{-1}(u - v)$$

are still in \mathbb{B} . But, as we shall see, these transformations will not keep the property of being slice hyperholomorphic, and a different approach will be needed. Note that the Schwarz lemma will not help to develop a Schur algorithm here because of the lack of commutativity. If s is slice hyperholomorphic into \mathbb{B} (and in particular $|s(0)| < 1$), the functions

$$(1 - s(p)\overline{s(0)})^{-1}(s(p) - s(0)),$$

or

$$(s(p) - s(0))(1 - s(p)\overline{s(0)})^{-1}$$

are indeed contractive in \mathbb{B} , but they will not be slice hyperholomorphic. So there is no direct counterpart of (1.1). One needs to use the notion of Schur multipliers and

the slice regular product.

Schur analysis originates with the works of Schur, Herglotz, and others (see [55] for reprints of the original works), and can be seen as a collection of questions pertaining to Schur functions and their various generalizations to other settings. Among the problems we mention in particular:

(1) Classical interpolation problems such as Carathéodory-Fejér and Nevanlinna-Pick problems, and their matrix-valued versions; see for instance [51, 52]. The case of boundary interpolation is of special importance. See for instance [26] for related recent results.

(2) Realization of Schur functions in the form $s(z) = D + zC(I - zA)^{-1}B$, where the operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is subject to various metric constraints, namely coisometric, isometric and unitary. See [28, 29, 5].

(3) Schur functions are closely related to the theory of linear systems. The term *linear system* encompasses a wide range of situations. Here we have in mind input output of the form

$$y_n = \sum_{m=0}^n s_m u_{n-m}, \quad n = 0, 1, \dots,$$

where s_0, s_1, \dots is a sequence of matrices of $\mathbb{C}^{M \times N}$ (the impulse response), $(u_n)_{n \in \mathbb{N}_0}$ is a sequence of vectors of \mathbb{C}^N (the input sequence), and $(y_n)_{n \in \mathbb{N}_0}$ is a sequence of vectors of \mathbb{C}^M (the output sequence). The function $s(z) = \sum_{n=0}^{\infty} s_n z^n$ is a Schur function if and only if the ℓ_2 norm of the output sequence is always less or equal to the ℓ_2 norm of the input sequence. In other words, Schur functions are the transfer functions of time-invariant dissipative linear systems.

(4) Yet another direction of research is related to inverse scattering; see [32, 49, 6, 7].

(5) Last but not least we mention the connection with fast algorithms; see [70, 71].

These various questions make sense in more general settings, of which we mention in particular the several complex variables case, the indefinite case, the time-varying case, the non-commutative case, the case of compact Riemann surfaces, and the case of several complex variables, to name a few. References are given in the last section of the paper. In the present work we present a counterpart of Schur functions and of the Schur algorithm in the quaternionic setting.

The main tool that we use to extend Schur analysis to the quaternionic setting is the theory of slice hyperholomorphic, or slice regular, functions. Some references for this theory of functions are [56, 33, 38]. For the generalization to functions with values in a Clifford algebra, called again slice hyperholomorphic or slice monogenic functions, we refer the reader to [44, 46, 45], and to [57] for functions with values in a real alternative algebra. Finally, we mention that it exists a non constant coefficients differential operator whose kernel contains slice hyperholomorphic functions defined on suitable domains, [35]. The theory of slice regular functions allows to define the quaternionic functional calculus and its associated S -resolvent operator. The importance of the S -resolvent operator is in the definition of the quaternionic version of the operator

$(I - zA)^{-1}$ that appears in the realization function $s(z) = D + zC(I - zA)^{-1}B$. It turns out that when A is a quaternionic matrix and p is a quaternion then $(I - pA)^{-1}$ has to be replaced by $(I - \bar{p}A)(|p|^2A^2 - 2\operatorname{Re}(p)A + I)^{-1}$ which is equal to $p^{-1}S_R^{-1}(p^{-1}, A)$ where $S_R^{-1}(p^{-1}, A)$ is the right S -resolvent operator associated to the quaternionic matrix A . For some results on the quaternionic functional calculus we refer the reader to [34, 37, 39, 41]. Slice monogenic functions admit a functional calculus for n -tuples of operators and for this theory we mention [43, 40, 36]. The book [47] collects some of the main results on the theory of slice hyperholomorphic functions and the related functional calculi.

Finally we mention the paper [15, 16, 14], where Schur multipliers were introduced and studied in the quaternionic setting using the Cauchy-Kovalesvkaya product and series of Fueter polynomials, and the papers [59, 68, 67], which treat various aspects of a theory of linear systems in the quaternionic setting. Our approach is quite different from the methods used there.

The paper consists of eight sections besides the introduction, and its outline is as follows: in Section 2 we review the main aspects of the theory of slice hyperholomorphic functions and the S -resolvent operators. In Section 3 we study the counterpart of state space equations in the slice hyperholomorphic setting. This leads us naturally to the notion of rational function, defined and studied in Section 4. In Section 5 we study certain positive definite functions. This paves the way to the definition of Schur multipliers and to a version of Schwarz' lemma in Section 6. In Section 7 we exhibit a coisometric realization of a Schur multiplier. Section 8 is devoted to the Schur algorithm in the present setting. In the last section we discuss briefly future directions of research.

2. SLICE HYPERHOLOMORPHIC FUNCTIONS AND THE S -RESOLVENT OPERATORS

In this section we introduce the preliminary results related to the theory of slice hyperholomorphic functions and the quaternionic S -resolvent operators. In the next section we will use these tools to define in a suitable way the quaternionic slice hyperholomorphic transfer function associated to quaternionic linear systems.

By \mathbb{H} we denote the algebra of real quaternions $p = x_0 + ix_1 + jx_2 + kx_3$ which can also be written as $p = \operatorname{Re}(p) + \operatorname{Im}(p)$ where $x_0 = \operatorname{Re}(p)$ and $ix_1 + jx_2 + kx_3 = \operatorname{Im}(p)$. By \mathbb{S} we indicate the set of unit purely imaginary quaternions, i.e.

$$\mathbb{S} = \{p = x_1i + x_2j + x_3k : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

In the literature there are several notions of quaternion valued hyperholomorphic functions. In this paper we will consider a notion which includes power series in the quaternionic variable, the so-called slice regular or slice hyperholomorphic functions. The main reference for the material in this section is the book [47].

Definition 2.1. *Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \rightarrow \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on \mathbb{C}_I .*

- (1) We say that f is a left slice regular function (or slice regular or slice hyperholomorphic) if, for every $I \in \mathbb{S}$, we have:

$$\frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0.$$

- (2) We say that f is right slice regular function (or right slice hyperholomorphic) if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I \right) = 0.$$

Definition 2.2. Given $p \in \mathbb{H}$, if q is not real we can define $I_p := \text{Im}(p)/|\text{Im}(p)|$, so $p = \text{Re}(p) + I_p |\text{Im}(p)|$. We denote by $[p]$ the set of all elements of the form $\text{Re}(p) + J |\text{Im}(p)|$ when J varies in \mathbb{S} . We say that $[p]$ is the 2-sphere defined by q .

Definition 2.3. Let Ω be a domain in \mathbb{H} . We say that Ω is a slice domain (*s-domain for short*) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_I$ is a domain in \mathbb{C}_I for all $I \in \mathbb{S}$. We say that Ω is axially symmetric if, for all $p \in \Omega$, the 2-sphere $[p]$ is contained in Ω .

In the sequel we will work mainly on the unit sphere in \mathbb{H} with center at the origin, which is trivially an axially symmetric s-domain.

Theorem 2.4 (Representation Formula). Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain. Let f be a left slice regular function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + I_p y \in \Omega$:

$$(2.1) \quad f(p) = f(x + I_p y) = \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} I_p I [f(\bar{z}) - f(z)],$$

where $z := x + Iy$, $\bar{z} := x - Iy \in \mathbb{H}$. Let f be a right slice regular function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + I_p y \in \Omega$:

$$(2.2) \quad f(x + I_p y) = \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} [f(\bar{z}) - f(z)] I I_p.$$

The Representation Formula allows to extend any function $f : \Omega \subseteq \mathbb{C}_I \rightarrow \mathbb{H}$ defined on an axially symmetric open set Ω intersecting the real axis and in the kernel of the Cauchy-Riemann operator to a function $f : \tilde{\Omega} \subseteq \mathbb{H} \rightarrow \mathbb{H}$ slice regular where $\tilde{\Omega}$ is the smallest axially symmetric open set in \mathbb{H} containing Ω by means of the *extension operator*

$$(2.3) \quad \text{ext}(f)(p) := \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} I_p I [f(\bar{z}) - f(z)], \quad z, \bar{z} \in \mathbb{C}_I.$$

Slice regular functions satisfy an identity principle, specifically, two of them coincide in a domain Ω intersecting the real axis if they coincide on a subset of $\Omega \cap \mathbb{R}$ having an accumulation point. As a consequence we have:

Proposition 2.5. Any quaternion valued real analytic function defined in $(a, b) \subseteq \mathbb{R}$ extends uniquely to a slice regular function defined in a suitable open set containing (a, b) .

Proof. Let $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{H}$ be a real analytic function. Then it can be uniquely extended to a suitable domain D containing (a, b) , by extending its real components, to a function $\tilde{f} : D \subseteq \mathbb{C} \rightarrow \mathbb{H}$ which is in the kernel of the Cauchy-Riemann operator.

By using the extension operator (2.3) \tilde{f} extends to a slice hyperholomorphic function defined on the smallest axially symmetric open set $\tilde{\Omega}$ containing D . The uniqueness follows from the identity principle. \square

Given two slice regular functions their product is not, in general, slice regular. It is possible to introduce a suitable product denoted by \star , see [47, Definition 4.3.5, p. 125] (note that in the literature this product is denoted by $*$ but here we use this symbol in order to avoid confusion with adjoint of operators). Here we describe the \star -product which gives a left slice regular product. It is possible to define an analog product to multiply two right slice regular functions and obtain a function with the same regularity. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \rightarrow \mathbb{H}$ be slice regular functions. Fix $I, J \in \mathbb{S}$, with $I \perp J$. The Splitting Lemma guarantees the existence of four holomorphic functions $F, G, H, K : \Omega \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ such that for all $z = x + Iy \in \Omega \cap \mathbb{C}_I$

$$f_I(z) = F(z) + G(z)J, \quad g_I(z) = H(z) + K(z)J.$$

Define the function $f_I \star g_I : \Omega \cap \mathbb{C}_I \rightarrow \mathbb{H}$ as

$$(2.4) \quad f_I \star g_I(z) = [F(z)H(z) - G(z)\overline{K(\bar{z})}] + [F(z)K(z) + G(z)\overline{H(\bar{z})}]J.$$

Then $f_I \star g_I(z)$ is obviously a holomorphic map and hence we can give the following definition.

Definition 2.6. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \rightarrow \mathbb{H}$ be slice regular. The function*

$$f \star g(p) = \text{ext}(f_I \star g_I)(p)$$

is called the slice regular product of f and g .

Remark 2.7. It is immediate to verify that the \star -product is associative, distributive but, in general, it is not commutative.

When slice regular functions are defined on a ball $\mathbb{B}_R \subseteq \mathbb{H}$ with center at the origin and radius R then they admit power series expansions. Suppose that on \mathbb{B}_R we have $f(p) = \sum_{n=0}^{\infty} p^n a_n$ and $g(p) = \sum_{n=0}^{\infty} p^n b_n$, then the \star -product is given by

$$(2.5) \quad (f \star g)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^n a_r b_{n-r}.$$

In other words, the coefficient sequence $(c_n)_{n \in \mathbb{N}_0}$ is the Cauchy product, or the convolution product, of the sequences of coefficients associated to f and g . See for instance [54, (2) p. 199] for the Cauchy product in a non-commutative setting.

Remark 2.8. From Proposition 2.5, it immediately follows that $f \star g$ is uniquely determined by $f(x)g(x)$, $x \in \mathbb{B}_R \cap \mathbb{R}$.

We observe that another product in Clifford analysis (see [27, 42]), namely the Cauchy-Kovaleskaya product which allows to obtain Cauchy-Fueter regular functions or monogenic functions, can be seen as a (different) convolution. More generally, pointwise product is often best replaced by convolution of underlying coefficient sequences. See [27]. We mention as an example the Wick product in white noise space analysis. See [60], and see [10, 11] for applications to linear system theory.

Remark 2.9. Let $f(p) = \sum_n p^n a_n$ and $g(p) = \sum_n p^n b_n$. If f has real coefficients then $f \star g = g \star f$.

Given a slice regular function f it is possible to construct its slice regular reciprocal, denoted by $f^{-\star}$. We do not provide the general construction, which can be found in [47] since it is sufficient to construct the inverse of a polynomial or a power series with center at the origin. Given $f(q)$ as above, let us introduce the notation

$$f^c(p) = \sum_{n=0}^{\infty} p^n \bar{a}_n, \quad f^s(p) = (f^c \star f)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^n a_r \bar{a}_{n-r}.$$

Note that the series f^s has real coefficients. The slice regular reciprocal is then defined as

$$f^{-\star} := (f^s)^{-1} f^c.$$

In an analogous way one can define the right slice regular reciprocal $f^{-\star} := f^c (f^s)^{-1}$, of a right regular function $f(p) = \sum_n a_n p^n$.

We do not introduce a specific symbol in order to distinguish the left and the right slice regular reciprocal, since the context will clarify the case we are considering. There is however a remarkable case of reciprocal that deserves further explanations. Consider the function

$$S(r, p) := p - q, \quad p, q \in \mathbb{H}.$$

Its slice regular reciprocal can be constructed in four possible ways: we can construct a reciprocal which is left (resp. right) regular with respect to the variable p or left (resp. right) regular with respect to the variable q . Accordingly to these possibilities we obtain the function (see [47])

$$S_L^{-1}(p, q) = -(q^2 - 2q\operatorname{Re}(p) + |p|^2)^{-1}(q - \bar{p})$$

which corresponds to the left slice regular reciprocal in the variable q . The function $S_L^{-1}(p, q)$ is left regular in the variable q by construction and it turns out to be right regular with respect to p . While the other possibility gives

$$S_R^{-1}(p, q) := -(q - \bar{p})(q^2 - 2\operatorname{Re}(p)q + |p|^2)^{-1},$$

which is right slice regular in q and left slice regular with respect to p .

Remark 2.10. When no confusion will arise we will write instead of $(p-q)^{-\star}$ its explicit expression using $S_L^{-1}(p, q)$ or $S_R^{-1}(p, q)$, according to the left or right slice regularity required.

It is possible to show that both kernels can be written in two different ways

$$S_L^{-1}(p, q) = -(q^2 - 2q\operatorname{Re}(p) + |p|^2)^{-1}(q - \bar{p}) = (p - \bar{q})(r^2 - 2\operatorname{Re}(q)p + |q|^2)^{-1},$$

and

$$S_R^{-1}(p, q) = -(q - \bar{p})(q^2 - 2\operatorname{Re}(p)q + |p|^2)^{-1} = (p^2 - 2\operatorname{Re}(q)r + |q|^2)^{-1}(r - \bar{q}),$$

thus we have

$$S_R^{-1}(p, q) = -S_L^{-1}(q, p).$$

Given two polynomials $P(p)$, $Q(p)$ it is possible to define two possible "quotients" as $P^{-\star} \star Q$ or $Q \star P^{-\star}$ and due to the noncommutativity of quaternions, these quotients do not coincide. Thus it is necessary to make a choice between the left and the right quotient of P and Q and this will be one of the various definitions of *rational functions*.

A number of equivalent characterization of rational functions are given in Section 4.

The realization of Schur functions in the form $s(z) = D + zC(I - zA)^{-1}B$, where one wishes to replace the complex variable z by a quaternionic variable, and where the operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is now a quaternionic operator matrix, requires a new concept to replace the classical resolvent operator $(I - zA)^{-1}$. This new object is the so called S -resolvent operator associated to the quaternionic functional calculus. Let V be a two sided quaternionic Banach space, we denote by $\mathcal{B}(V)$ the quaternionic Banach spaces of all bounded linear (left or right) operators endowed with the natural norm. A remarkable fact is that the following theorem holds in the case in which the components of the quaternionic operator A do not commute.

Theorem 2.11. *Let $A \in \mathcal{B}(V)$. Then, for $\|A\| < |r|$, we have*

$$(2.6) \quad \sum_{n=0}^{\infty} A^n r^{-1-n} = -(A^2 - 2\operatorname{Re}(r)A + |r|^2 I)^{-1}(A - \bar{r}I),$$

$$(2.7) \quad \sum_{n=0}^{\infty} r^{-1-n} A^n = -(A - I\bar{r})(A^2 - 2\operatorname{Re}(r)A + |r|^2 I)^{-1},$$

where I denotes the identity operator.

The notion of S -spectrum of a quaternionic operator A is suggested by the definition of S -resolvent operator that is the kernel for the quaternionic functional calculus; see (2.6) and (2.7).

Definition 2.12 (The S -spectrum and the S -resolvent sets of quaternionic operators). *Let $T \in \mathcal{B}(V)$. We define the S -spectrum $\sigma_S(A)$ of T as:*

$$\sigma_S(A) = \{r \in \mathbb{H} : A^2 - 2\operatorname{Re}(r)A + |r|^2 I \text{ is not invertible}\}.$$

The S -resolvent set $\rho_S(A)$ is defined by

$$\rho_S(A) = \mathbb{H} \setminus \sigma_S(A).$$

Remark 2.13. We have proved that the S -spectrum $\sigma_S(A)$ is a compact nonempty set and if $p \in \mathbb{H}$ belongs to $\sigma_S(A)$, then all the elements of the sphere $[p]$ are contained in $\sigma_S(A)$.

Definition 2.14 (The S -resolvent operator). *Let V be a bilateral quaternionic Banach space, $A \in \mathcal{B}(V)$ and $r \in \rho_S(A)$. We define the left S -resolvent operator as*

$$(2.8) \quad S_L^{-1}(r, A) := -(A^2 - 2\operatorname{Re}(r)A + |r|^2 I)^{-1}(A - \bar{r}I).$$

We define the right S -resolvent operator as

$$(2.9) \quad S_R^{-1}(r, A) := -(A - I\bar{r})(A^2 - 2\operatorname{Re}(r)A + |r|^2 I)^{-1}.$$

Theorem 2.15 (The S -resolvent equation). *Let V be a bilateral quaternionic Banach space, $A \in \mathcal{B}(V)$ and $r \in \rho_S(A)$. Then the left S -resolvent operator defined in (2.8) satisfies the equation*

$$(2.10) \quad S_L^{-1}(r, A)r - AS_L^{-1}(r, A) = I.$$

and the right S -resolvent operator defined in (2.9) satisfies the equation

$$(2.11) \quad sS_R^{-1}(s, A) - S_R^{-1}(s, A)A = I.$$

Moreover, $S_L^{-1}(r, \cdot) : \rho_S(A) \rightarrow \mathcal{B}(V)$ is right slice regular in the variable r and $S_R^{-1}(r, \cdot) : \rho_S(A) \rightarrow \mathcal{B}(V)$ is left slice regular in the variable r .

The notation introduced in Remark 2.10 is not necessarily valid when we replacing operators in place of the quaternionic variables. We prove that it is still the case in the following proposition which is crucial and can be proved by suitably modifying the proof of Theorem 2.11.

Proposition 2.16. *Let \mathcal{H} be a two sided quaternionic Hilbert space and let A be a bounded right linear quaternionic operator from \mathcal{H} into itself. Then, for $|p| \|A\| < 1$ we have*

$$(2.12) \quad \sum_{n=0}^{\infty} p^n A^n = p^{-1} S_R^{-1}(p^{-1}, A) = (I - \bar{p}A)(|p|^2 A^2 - 2\operatorname{Re}(p)A + I)^{-1},$$

and

$$(2.13) \quad (I - pA)^{-\star} = \sum_{n=0}^{\infty} p^n A^n.$$

Proof. From (2.7), it is immediate that

$$\sum_{n=0}^{\infty} p^n A^n = p^{-1} S_R^{-1}(p^{-1}, A)$$

and by writing explicitly $S_R^{-1}(p^{-1}, A)$ we obtain

$$p^{-1} S_R^{-1}(p^{-1}, A) = -p^{-1} \left(A - \frac{p}{|p|^2} I \right) \left(A^2 - 2 \frac{\operatorname{Re}(p)}{|p|^2} A + \frac{1}{|p|^2} I \right)^{-1}$$

and with some computations we get the second equality in (2.12). To prove (2.13) we consider the function $p^{-1} - q$ and its slice regular reciprocal with respect to q which is obtained using the formula $f^{-\star} = f^c(f^s)^{-1}$, see Section 2. We obtain $(p^{-1} - q)^{-\star} = S_R^{-1}(p^{-1}, q)$ and, by the functional calculus, we can substitute q by a quaternionic operator A (it is crucial to observe that the components of A do not necessarily commute) and we obtain:

$$(p^{-1}I - A)^{-\star} = -S_R^{-1}(p^{-1}, A) = -(A - \bar{p}^{-1}I)(A^2 - 2\operatorname{Re}(p^{-1})A + |p^{-1}|^2I)^{-1}$$

since

$$p^{-1}(p^{-1}I - A)^{-\star} = [p(p^{-1}I - A)]^{-\star} = (I - pA)^{-\star}.$$

By using (2.12) we obtain (2.13). \square

3. STATE SPACE EQUATIONS AND REALIZATION

We now show that if we consider the quaternionic linear system

$$(3.1) \quad \begin{cases} x_{n+1} = x_n A + u_n B, & n = 0, 1, \dots \\ y_n = x_n C + u_n D, \end{cases}$$

where A, B, C, D are given quaternion matrices, then the quaternionic "transfer function" cannot be defined by simply taking \mathcal{Z} -transform as in the complex case. The following considerations show the problem that arises: Given a sequence of quaternions (or of quaternionic matrices of common size) $U = \{u_n\}_{n \in \mathbb{N}}$, we define the quaternionic \mathcal{Z} -transform as

$$\mathcal{Z}(U) := \mathcal{U}(p) := \sum_{n=0}^{\infty} p^n u_n,$$

so the \mathcal{Z} -transform is right linear

$$\mathcal{Z}(UA) = \mathcal{Z}(U)A.$$

Since $\mathcal{Z}(U)$ is a power series centered at the origin it is slice hyperholomorphic. The main properties of the classical $\mathcal{Z}(U)$ -transform still hold in the quaternionic setting. In particular, if we set

$$\tau_{-1}U := (u_1, u_2, \dots)$$

we have

$$\mathcal{Z}(\tau_{-1}U) = p^{-1}\mathcal{Z}(U) \quad \text{if } u_0 = 0.$$

If we apply the \mathcal{Z} -transform to system (3.1) we get

$$(3.2) \quad \begin{cases} p^{-1}\mathcal{X}(p) = \mathcal{X}(p)A + \mathcal{U}(p)B, & n = 0, 1, \dots \\ \mathcal{Y}(p) = \mathcal{X}(p)C + \mathcal{U}(p)D. \end{cases}$$

The natural definition of the transfer function of the system is

$$\mathcal{H}(p) := (\mathcal{U}(p))^{-1}\mathcal{Y}(p)$$

so we have to solve the quaternionic equation

$$p^{-1}\mathcal{X}(p) - \mathcal{X}(p)A = BU(p)$$

which has the solution

$$\mathcal{X}(p) = \sum_{n=0}^{\infty} p^n BU(p)A^{n+1}.$$

Because of the term B , this expression need not be slice hyperholomorphic. By replacing this solution in the second equation in (3.2), we obtain

$$\mathcal{Y}(p) = \sum_{n=0}^{\infty} p^n BU(p)A^{n+1}C + \mathcal{U}(p)D.$$

It turns out that $\mathcal{H}(p)$ depends on $\mathcal{U}(p)$, in fact

$$\begin{aligned} \mathcal{H}(p) &:= (\mathcal{U}(p))^{-1} \left(\sum_{n=0}^{\infty} p^n BU(p)A^{n+1}C + \mathcal{U}(p)D \right) \\ &= (\mathcal{U}(p))^{-1} \sum_{n=0}^{\infty} p^n BU(p)A^{n+1}C + D, \end{aligned}$$

and need not be slice hyperholomorphic. We now follow an approach based on slice hyperholomorphic to overcome the above difficulty and to give a good definition of transfer function.

Theorem 3.1. *Let A, B, C, D be given matrices of appropriate size with quaternionic entries. Suppose that $\{u_n\}_{n \in \mathbb{N}_0}$ is a given sequence of vectors with quaternionic entries, and of appropriate size. Consider the system*

$$(3.3) \quad \begin{cases} x_{n+1} = Ax_n + Bu_n, & n = 0, 1, \dots \\ y_n = Cx_n + Du_n, \end{cases}$$

and define its slice hyperholomorphic transfer function matrix-valued as

$$\mathcal{H}(p) := \mathcal{Y}(p) \star (\mathcal{U}(p))^{-\star}$$

where $\mathcal{Y}(p)$ and $\mathcal{U}(p)$ are the slice hyperholomorphic extensions of the Z -transforms of y_n and of u_n , respectively. Then we have

$$(3.4) \quad \mathcal{H}(p) = D + pC \star (I - pA)^{-\star} B.$$

Moreover, in terms of the right S -resolvent operator (3.4) can be written as

$$(3.5) \quad \mathcal{H}(p) = C \star S_R^{-1}(p^{-1}, A)B + D.$$

Proof. Let us first consider system (3.3) in the complex plane \mathbb{C}_I , for a fixed $I \in \mathbb{S}$, where A, B, C, D , now denoted by a, b, c, d , are given element in \mathbb{C}_I . Suppose that $\{u_n\}_{n \in \mathbb{N}_0}$ is a given sequence of complex numbers in \mathbb{C}_I :

$$(3.6) \quad \begin{cases} x_{n+1} = ax_n + bu_n, & n = 0, 1, \dots \\ y_n = cx_n + du_n. \end{cases}$$

By setting

$$z := u + Iv \in \mathbb{C}_I$$

and by taking the classical \mathcal{Z} -transform we get

$$(3.7) \quad \begin{cases} \mathcal{X}(z) = za \mathcal{X}(z) + zb \mathcal{U}(z) \\ \mathcal{Y}(z) = c \mathcal{X}(z) + d \mathcal{U}(z). \end{cases}$$

On \mathbb{C}_I all the objects commute, thus we have:

$$(3.8) \quad \begin{cases} \mathcal{X}(z) = (1 - za)^{-1} zb \mathcal{U}(z), \\ \mathcal{Y}(z) = c \mathcal{X}(z) + d \mathcal{U}(z). \end{cases}$$

All the functions in the complex variable z involved in system (3.8) are holomorphic on the plane \mathbb{C}_I , thus we can use the extension operator (2.3) to obtain a function in the quaternionic variable p which is slice hyperholomorphic and since

$$\text{ext}(f(z)g(z)) = \text{ext}(f(z) \star g(z)) = f(p) \star g(p)$$

we get

$$(3.9) \quad \begin{cases} \mathcal{X}(p) = (1 - pa)^{-\star} \star (pb) \star \mathcal{U}(p), \\ \mathcal{Y}(p) = c \star \mathcal{X}(p) + d \star \mathcal{U}(p) \end{cases}$$

from which

$$(3.10) \quad \begin{cases} \mathcal{X}(p) = (1 - pa)^{-\star} \star (pb) \star \mathcal{U}(p), \\ \mathcal{Y}(p) = c \star (1 - pa)^{-\star} \star (pb) \star \mathcal{U}(p) + d \star \mathcal{U}(p). \end{cases}$$

Since we have defined $\mathcal{H}(p)$ as

$$\mathcal{H}(p) := \mathcal{Y}(p) \star (\mathcal{U}(p))^{-\star}$$

we obtain

$$\begin{aligned} \mathcal{H}(p) &= \left(c \star (1 - pa)^{-\star} \star (pb) \star \mathcal{U}(p) + d \star \mathcal{U}(p) \right) \star (\mathcal{U}(p))^{-\star} \\ &= c \star (1 - pa)^{-\star} \star (pb) + d. \end{aligned}$$

This function is slice hyperholomorphic in p with values in the quaternions so it makes sense to consider a, b, c and d as quaternions. In order to get a matrix valued function it is sufficient to replace a, b, c and d , respectively, with the matrices A, B, C, D , of appropriate size and with quaternionic entries. We obtain the quaternionic transfer function

$$\mathcal{H}(p) = D + C \star (I - pA)^{-\star} \star (pB) = D + pC \star (I - pA)^{-\star} B$$

which is a slice regular function. Finally, using (2.12), we obtain the equality (3.5). \square

Remark 3.2. Using the explicit expression for $(1 - pA)^{-\star}$ we can also write

$$\mathcal{H}(p) = D + C \star (I - \bar{p}A)(|p|^2 A^2 - 2\operatorname{Re}(p)A + I)^{-1} \star (pB),$$

and if we remove the \star -product we get

$$\mathcal{H}(p) = D + (pC - |p|^2 CA)(|p|^2 A^2 - 2\operatorname{Re}(p)A + I)^{-1} B.$$

4. RATIONAL FUNCTIONS

Motivated by the discussion in the previous section, we now give various equivalent definitions of rationality. We first consider the case of $\mathbb{H}^{M \times N}$ -valued functions of a real variable, and give a definition and some preliminary lemmas.

Definition 4.1. *A function $f(x)$ of the real variable x and with values in $\mathbb{H}^{M \times N}$ will be said to be rational if it can be written as*

$$(4.1) \quad f(x) = \frac{M(x)}{m(x)},$$

where M is a $\mathbb{H}^{M \times N}$ -valued polynomial and $m \in \mathbb{R}[x]$.

Clearly sums and products of rational functions of appropriate sizes stay rational. The case of the inverse is considered in the next lemma.

Lemma 4.2. *Let f be a rational function from \mathbb{R} into $\mathbb{H}^{N \times N}$, and assume that $f(x_0)$ is invertible for some $x_0 \in \mathbb{R}$. Then, $f(x)$ is invertible for all $x \in \mathbb{R}$, at the possible exception of a finite number of values, and f^{-1} is a rational function.*

Proof. We proceed by induction. When $N = 1$, a polynomial in $\mathbb{H}[x]$ can be seen as a matrix polynomial in $\mathbb{R}^{4 \times 4}[x]$, and this latter is invertible for all but at most a finite number of real values of x . Let now $r(x) = \frac{t_1(x)}{t(x)}$, where $t_1 \in \mathbb{H}[x]$ and $t \in \mathbb{R}[x]$. Assume $t \neq 0$. Then, since x is real, we may write

$$t^{-1}(x) = \frac{\overline{t(x)}}{t(x)\overline{t(x)}} = \frac{\bar{t}(x)}{(\bar{t}t)(x)}.$$

This ends the proof for $N = 1$ since $(t\bar{t})(x) \in \mathbb{R}[x]$.

Assume now the induction proved at rank N and let R be a $\mathbb{H}^{(N+1) \times (N+1)}$ -valued rational function, invertible for at least one $x_0 \in \mathbb{R}$. Then identifying $\mathbb{H}^{(N+1) \times (N+1)}$ with $\mathbb{R}^{4(N+1) \times 4(N+1)}$ we see that R is invertible for all, but at most a finite number of real values of x

$$R(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

Without loss of generality, we can assume that $a(x)$ is not identically equal to 0 (otherwise, multiply R on the left or on the right by a permutation matrix; this does not change the property of R or of R^{-1} being rational). We write (see for instance [52, (0.3), p. 3])

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c(x)a(x)^{-1} & I_n \end{pmatrix} \times \\ \times \begin{pmatrix} a(x) & 0 \\ 0 & d(x) - c(x)a(x)^{-1}b(x) \end{pmatrix} \begin{pmatrix} 1 & a(x)^{-1}b(x) \\ 0 & I_n \end{pmatrix},$$

and so $d(x) - c(x)a(x)^{-1}b(x)$ is invertible for all, but at most a finite number of, values $x \in \mathbb{R}$. We have

$$(4.2) \quad R^{-1}(x) = \begin{pmatrix} 1 & -a(x)^{-1}b(x) \\ 0 & I_n \end{pmatrix} \times \\ \times \begin{pmatrix} a(x)^{-1} & 0 \\ 0 & (d(x) - c(x)a(x)^{-1}b(x))^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c(x)a(x)^{-1} & I_n \end{pmatrix}.$$

The induction hypothesis at rank N implies that $(d(x) - c(x)a(x)^{-1}b(x))^{-1}$ is rational, and so is R^{-1} , as seen from (4.2). \square

Lemma 4.3. *Consider a polynomial $M(p) \in \mathbb{H}^{N \times N}[p]$:*

$$M(p) = \sum_{j=0}^J p^j M_j.$$

Then

$$M(p) = D + pC \star (I - pA)^{-\star} B,$$

where $D = M_0$,

$$A = \begin{pmatrix} 0_N & I_N & 0_N & \cdots \\ 0_N & 0_N & I_N & 0_N & \cdots \\ & \vdots & & & \vdots \\ 0_N & \cdots & \cdots & 0_N & I_N \\ 0_N & 0_N & \cdots & 0_N & 0_N \end{pmatrix},$$

$$B = \begin{pmatrix} 0_N \\ 0_N \\ \vdots \\ I_N \end{pmatrix}, \quad C = (M_J \quad M_{J-1} \quad \cdots \quad M_1).$$

Proof. The equality easily follows from the fact that

$$(I - pA)^{-\star} = \begin{pmatrix} I_N & pI_N & p^2I_N & \cdots & p^J I_N \\ 0_N & I_N & pI_N & \cdots & p^{J-1} I_N \\ & \vdots & & & \vdots \\ 0_N & \cdots & \cdots & I_N & pI_N \\ 0_N & 0_N & \cdots & 0_N & I_N \end{pmatrix}.$$

□

Assume now that the $\mathbb{H}^{N \times N}$ -valued function $f(p)$ can be written as

$$(4.3) \quad f(p) = D + pC \star (I - pA)^{-\star} B,$$

where A, B, C and D are matrices with entries in \mathbb{H} and of appropriate sizes, and that D is invertible. Then,

$$(4.4) \quad f(p)^{-1} = D^{-1} - pD^{-1}C \star (I - p(A - BD^{-1}C))^{-\star} B$$

We also recall the following formula. See for instance [24] for the case of rational functions of a complex variable. The proof is as in the classical case, and will be omitted.

Lemma 4.4. *Let*

$$f_j(p) = D_j + pC_j \star (I_{N_j} - pA_j)^{-\star} B_j, \quad j = 1, 2$$

be two functions admitting realizations of the form (4.3), and respectively $\mathbb{H}^{M \times N}$ and $\mathbb{H}^{N \times R}$ -valued. Then the $\mathbb{H}^{M \times R}$ -valued function $f_1 \star f_2$ can be written in the form (4.3), with $D = D_1 D_2$ and

$$A = \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad D_1 C_2).$$

The proof of the following Lemma is immediate.

Lemma 4.5. *Let*

$$f_j(p) = D_j + pC_j \star (I_{N_j} - pA_j)^{-\star} B_j, \quad j = 1, 2$$

be two functions admitting realizations of the form (4.3), and respectively $\mathbb{H}^{M \times N}$ and $\mathbb{H}^{M \times R}$ -valued. Then the $\mathbb{H}^{M \times (N+R)}$ -valued function $(f_1 \quad f_2)$ can be written in the form (4.3), with $D = (D_1 \quad D_2)$ and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2).$$

In the following theorem,

$$f(p) = \sum_{n=0}^{\infty} \text{diag} \underbrace{(p^n, p^n, \dots, p^n)}_{M \text{ times}} f_n,$$

which we will denote for short by

$$f(p) = \sum_{n=0}^{\infty} p^n f_n,$$

is a series of the quaternionic variable p , with coefficients in $\mathbb{H}^{M \times N}$, converging in a neighborhood of the origin. We say that f is rational if any of the five conditions listed in the theorem holds.

Theorem 4.6. *Let f be a $\mathbb{H}^{M \times N}$ -valued function, slice hyperholomorphic in a neighborhood of the origin. Then, the following are equivalent:*

(1) *There exist matrices A, B and C , of appropriate dimensions, and such that*

$$(4.5) \quad f_n = CA^{n-1}B, \quad n = 1, 2, \dots \quad f_0 = D.$$

(2) *f can be written as (4.3)*

$$f(p) = D + pC \star (I - pA)^{-\star} B.$$

(3) *The right linear span $\mathcal{M}(f)$ of the columns of the functions $R_0 f, R_0^2 f, \dots$ is a finite dimensional right quaternionic Hilbert space.*

(4) *The function $f(x)$ is a rational function from \mathbb{R} into $\mathbb{H}^{M \times N}$.*

(5) *The entries of f are of the form $P \star Q^{-\star}$, where P and Q are slice holomorphic polynomials, and $Q(0) \neq 0$.*

Proof. In the proof we adapt well known arguments from the theory of matrix-valued rational functions to the present case. For similar arguments in the classical case, see for instance [24].

Assume that (1) is in force, and set $D = s_0$. Then, for p such that $|p| \cdot \|A\| < 1$, the series $\sum_{n=0}^{\infty} p^n A^n$ converges in $\mathbb{H}^{N \times N}$, and

$$\sum_{n=0}^{\infty} p^n A^n = (I - pA)^{-\star}, \quad \text{where } I = I_N,$$

and we can write

$$\sum_{n=0}^{\infty} p^n f_n = D + pC \star (I - pA)^{-\star} B,$$

so that (2) is in force.

Assume now (2). Then,

$$R_0^n f = C \star (I - pA)^{-1} A^{n-1} B, \quad n = 1, 2, \dots,$$

so that $\mathcal{M}(f)$ is spanned by the columns of D and the columns of the function $C \star (I - pA)^{-\star}$, and is in particular finite dimensional, so that (3) is in force.

Assume now (3). Then there exists an integer $m_0 \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and $v \in \mathbb{H}^q$, there exist vectors u_1, \dots, u_{m_0} such that

$$(4.6) \quad R_0^{m_0} f v = \sum_{m=1}^{m_0} R_0^m f u_m.$$

Of course, the u_j need not be unique. Let E denote the $\mathbb{H}^{p \times m_0 q}$ -valued slice hyperholomorphic function

$$E = (R_0 f \quad R_0^2 f \quad \dots \quad R_0^{m_0} f).$$

Then, in view of (4.6), there exists a matrix $A \in \mathbb{H}^{m_0q \times m_0q}$ such that

$$R_0E = EA,$$

so that

$$E(p) - E(0) = p \star E(p)A = E(p) \star pA,$$

and so

$$E(p) = E(0) \star (I - pA)^{-\star}$$

and

$$(R_0f)(p) = E(p) \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus we have

$$f(p) - f(0) = pE(0) \star (I - pA)^{-\star} \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It follows that f is of the form (4.3), and so (2) holds. Since (2) implies trivially (1), we have shown that the first three claims are equivalent. We now turn to the equivalence with the other two characterizations.

Assume that (3) holds. Then, the restriction of (4.3) to $p = x \in \mathbb{R}$ gives

$$f(x) = D + xC(I - xA)^{-1}B.$$

and thus f is a rational function of x in view of Lemma 4.2, that is (4) is in force. Assume now (4), and let $f(x) = \frac{M(x)}{m(x)}$ as in (4.1). Since f is assumed defined the origin, we may assume that $m(0) \neq 0$, and $f(x)$ is the restriction to the real line of the slice hyperholomorphic function

$$f(p) = M(p) \star m(p)^{-\star} = m(p)^{-\star} \star M(p).$$

Note that since m has real coefficients we also have $m(p)^{-\star} \star M(p) = m(p)^{-1}M(p)$. So (5) holds. Assume now (5) in force. Then, in view of Lemma 4.4, each of the entries of f admits a realization, and the result follows since the property of realization stays under concatenation. Use Lemma 4.5 and its counterpart for columns rather than rows. \square

We note that a short introduction to realization theory in forms of a sequence of exercises can be found in [3, pp. 328-329].

5. POSITIVE DEFINITE FUNCTIONS AND REPRODUCING KERNELS

The \mathbb{H} -valued function $k(u, v)$ defined for u, v in some set Ω (in the present paper, Ω will be most of the time the unit ball of \mathbb{H}) is called positive definite if it is Hermitian:

$$(5.1) \quad k(u, v) = \overline{k(v, u)}, \quad \forall u, v \in \Omega,$$

and if for every $N \in \mathbb{N}$, every $u_1, \dots, u_N \in \Omega$ and $c_1, \dots, c_N \in \mathbb{H}$ it holds that

$$\sum_{\ell, j=1}^N \overline{c_\ell} k(u_\ell, u_j) c_j \geq 0.$$

(Note that the above sum is a real number in view of (5.1)). As in the complex-valued case, associated to k is a uniquely defined reproducing kernel quaternionic (right)-Hilbert space $\mathcal{H}(k)$, with reproducing kernel $k(u, v)$, meaning that:

- (1) The function $u \mapsto k(u, v)c$ belongs to $\mathcal{H}(k)$ for every choice of $v \in \Omega$ and $c \in \mathbb{H}$, and
- (2) it holds that

$$\overline{c}f(v) = \langle f(\cdot), k(\cdot, v)c \rangle_{\mathcal{H}(k)}$$

for every choice of $f \in \mathcal{H}(k)$ and of $u \in \Omega$ and $c \in \mathbb{H}$. See for instance [13, Theorem 8.4, p. 456] for a proof and [13, Proposition 9.4, p. 458] for a proof of a result that, with similar techniques, proves also the following lemma.

Lemma 5.1. *Let $\mathcal{H}(k)$ be a right reproducing kernel quaternionic Hilbert space of \mathbb{H} -valued functions defined on a set Ω , with reproducing kernel $k(u, v)$. Then, the function f belongs to $\mathcal{H}(k)$ if and only if there exists $M > 0$ such that the kernel*

$$k(u, v) - \frac{f(u)\overline{f(v)}}{M^2} \geq 0.$$

The smallest such M is equal to the norm of f in $\mathcal{H}(k)$.

Reproducing kernels are a main tool in operator theory. Reproducing kernels of the form $c(z\bar{w})$, $z, w \in \mathbb{C}$, where $c(t) = \sum_{n=0}^{+\infty} c_n t^n$, $c_n \geq 0$, for all $n \in \mathbb{N}$ is an analytic function in a neighborhood of the origin are an important case. In the case of Hardy space, Bergman space, and Dirichlet space the functions $c(t)$ are given by

$$c(t) = \frac{1}{1-t} \quad c(t) = \frac{1}{(1-t)^2} \quad c(t) = -\ln(1-t).$$

We will consider the generalization of these kernels to the case in which the variables considered are quaternions, i.e. we will consider

$$(5.2) \quad c(p\bar{q}) = \sum_{n=0}^{+\infty} c_n p^n \bar{q}^n, \quad q, p \in \mathbb{H}.$$

Note that due to the noncommutative nature of quaternions, the series in (5.2) does not coincide with $\sum_{n=0}^{+\infty} c_n (p\bar{q})^n$.

Theorem 5.2. *Let $A \subset \mathbb{N}_0$ and let $(c_n)_{n \in A}$ be a sequence of strictly positive numbers, and assume that*

$$R = \frac{1}{\limsup_{n \in A} c_n^{1/n}} > 0.$$

Then, the function

$$k(p, q) = \sum_{n \in A} c_n p^n \bar{q}^n$$

is positive definite in the ball $|p| < R$. The associated reproducing kernel Hilbert space consists of the functions

$$f(p) = \sum_{n \in A} p^n f_n,$$

with coefficients $f_n \in \mathbb{H}$ such that

$$\sum_{n \in A} \frac{|f_n|^2}{c_n} < \infty.$$

In the case of the Hardy space we have

Proposition 5.3. *The sum of the series $\sum_{n=0}^{+\infty} p^n \bar{q}^n$ is the function $k(p, q)$ given by*

$$(5.3) \quad k(p, q) = (1 - 2\operatorname{Re}(q)p + |q|^2 p^2)^{-1} (1 - pq) = (1 - \bar{p}\bar{q})(1 - 2\operatorname{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}.$$

The kernel $k(p, q)$ is defined for $p \notin [q^{-1}]$ or, equivalently, for $q \notin [p^{-1}]$. Moreover:

- a) $k(p, q)$ is left slice regular in p and right slice regular in q ;
- b) $\overline{k(p, q)} = k(q, p)$.

Proof. Consider the function

$$k_q(z) = \frac{1}{1 - z\bar{q}}, \quad z = x + Iy.$$

The proof of the first equality is an application of the representation formula (2.1). Consider now the function

$$k_p(w) = \frac{1}{1 - p\bar{w}};$$

the representation formula (2.2) gives $k(p, q) = (1 - \bar{p}\bar{q})(1 - 2\operatorname{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}$. The function $(1 - \bar{p}\bar{q})(1 - 2\operatorname{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}$ is right slice regular in the variable q by construction and it is slice regular in the variable p in its domain of definition, since it is the product of a slice regular function and a polynomial with real coefficients. By the identity principle it coincides with the first expression which is slice regular in p by construction. Assertion a) follows. Point b) follows from the chain of equalities

$$\begin{aligned} \overline{k(q, p)} &= \overline{[(1 - \bar{p}\bar{q})(1 - 2\operatorname{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}]} \\ &= (1 - 2\operatorname{Re}(p)q + |p|^2 \bar{q}^2)^{-1} (1 - qp) = k(q, p). \end{aligned}$$

□

We note (see for instance [50, 2]):

Theorem 5.4. *The following are equivalent:*

- (1) *The function s is analytic and contractive in the open unit disk.*
- (2) *The function s is defined in \mathbb{D} and the operator of multiplication by s is a contraction from the Hardy space $H_2(\mathbb{D})$ into itself.*
- (3) *The function s is defined in \mathbb{D} and the kernel*

$$k_s(z, w) = \frac{1 - s(z)\overline{s(w)}}{1 - z\bar{w}}$$

is positive in the open unit disk.

We prove in Theorem 6.2 the counterpart of the above theorem in the quaternionic setting.

Note that k_s can be rewritten as

$$k_s(z, w) = \sum_{n=0}^{\infty} z^n (1 - s(z)\overline{s(w)})\overline{w}^n.$$

This is the form which will be used in the quaternionic setting. See (6.3) in Section 6.

The reproducing kernel Hilbert space with reproducing kernel $k_s(z, w)$ was first introduced and studied by de Branges and Rovnyak; see [29, 28]. This space will be denoted by $\mathcal{H}(s)$. It is equal to the operator range $\text{ran } \sqrt{I - M_s M_s^*}$, where M_s denotes the operator of multiplication by s from $H_2(\mathbb{D})$ into itself, that is, M_s is the commutative version of the operator defined by (6.2) in the following section.

The space $\mathcal{H}(s)$ is the state space for a coisometric realization of s . Furthermore, s is a Schur function if and only if the function s^\sharp defined by

$$(5.4) \quad s^\sharp(z) = \overline{s(\overline{z})}$$

is a Schur function. The space $\mathcal{H}(s^\sharp)$ is the state space for an isometric realization of s . A unitary realization for s is given in terms of the reproducing kernel Hilbert space with reproducing kernel

$$(5.5) \quad D_s(z, w) = \begin{pmatrix} k_s(z, w) & \frac{s(z) - s(\overline{w})}{z - \overline{w}} \\ \frac{s^\sharp(z) - s^\sharp(\overline{w})}{z - \overline{w}} & k_{s^\sharp}(z, w) \end{pmatrix}.$$

For more information on the Schur algorithm, see for instance the papers [61] and the books [2, 19, 48].

6. SCHUR MULTIPLIERS AND SCHWARZ' LEMMA

We will define a finite energy quaternionic signal as a sequence $x = (x_0, x_1, \dots)$ of quaternions such that

$$\|x\|^2 \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} |x_n|^2 < \infty,$$

that is, an element of $\ell_2(\mathbb{N}_0, \mathbb{H})$. Note that $\ell_2(\mathbb{N}_0, \mathbb{H})$ is a quaternionic right Hilbert space when endowed with the \mathbb{H} -valued inner product

$$[x, y] = \sum_{n=0}^{\infty} \overline{y_n} x_n, \quad \text{where } y = (y_0, y_1, \dots).$$

To $x \in \ell_2(\mathbb{N}_0, \mathbb{H})$ we associate the function

$$f_x(p) = \sum_{n=0}^{\infty} p^n x_n$$

of the quaternionic variable p . We denote by \mathbf{H}_2 the right quaternionic vector space of such power series, endowed with the inner product

$$[f_x, f_y] = \sum_{n=0}^{\infty} \overline{y_n} x_n.$$

To ease the notation, we will use the notation $x(p)$ rather than $f_x(p)$:

$$(6.1) \quad x(p) = \sum_{n=0}^{\infty} p^n x_n.$$

Definition 6.1. We set $\ell(\mathbb{N}_0, \mathbb{H})$ denote the set of sequences of quaternions, indexed by \mathbb{N}_0 , and $\ell_0(\mathbb{N}_0, \mathbb{H})$ denote the set of sequences of $\ell(\mathbb{N}_0, \mathbb{H})$ for which at most a finite number of elements are non-zeros.

The slice regular product of $x \in \ell_2(\mathbb{N}_0, \mathbb{H})$ and $y \in \ell_0(\mathbb{N}_0, \mathbb{H})$ is defined by

$$(x \star y)_n = \sum_{m=0}^n x_m y_{n-m}, \quad n = 0, 1, \dots$$

Note that

$$x \star 1 = x, \quad \text{where } 1 = (1, 0, 0, \dots).$$

We now state the counterpart of Theorem 5.4 in the quaternionic setting:

Theorem 6.2. Let $s \in \ell(\mathbb{N}_0, \mathbb{H})$ (see Definition 6.1). Then, the following are equivalent:

- (1) The map $x \mapsto s \star x$, first defined for $x \in \ell_0(\mathbb{N}_0, \mathbb{H})$, extends to a contraction from $\ell_2(\mathbb{N}_0, \mathbb{H})$ into itself.
- (2) The series $s(p) = \sum_{n=0}^{\infty} p^n s_n$ converges in \mathbb{B} and the operator

$$(6.2) \quad (M_s x)(p) = \sum_{n=0}^{\infty} p^n s(p) x_n$$

is a contraction from the quaternionic Hardy space $\mathbf{H}_2(\mathbb{B})$ into itself, where $x(p) = \sum_{n=0}^{\infty} p^n x_n \in \mathbf{H}_2(\mathbb{B})$.

- (3) The series $s(p) = \sum_{n=0}^{\infty} p^n s_n$ converges in \mathbb{B} , and the function

$$(6.3) \quad k_s(p, q) = \sum_{n=0}^{\infty} p^n (1 - s(p) \overline{s(q)}) \overline{q}^n = (1 - s(p) \overline{s(q)}) \star (1 - p \overline{q})^{-\star}$$

is positive definite in the unit ball \mathbb{B} of \mathbb{H} .

Proof. We first assume that (1) is in force. Since $s \star 1 = s$ we note that in fact $s \in \ell_2(\mathbb{N}_0, \mathbb{H})$. The power series $s(p)$ makes thus sense for every p in the open unit ball of \mathbb{H} . We denote by M_s the image of the operator $x \mapsto s \star x$ under the map (6.1). Then, for $x \in \ell_0(\mathbb{N}_0, \mathbb{H})$ we have

$$(6.4) \quad \sum_{n=0}^{\infty} p^n s(p) x_n = \sum_{n=0}^{\infty} p^n \left(\sum_{m=0}^{\infty} p^m s_m \right) x_n = \sum_{n=0}^{\infty} p^n (s \star x)_n,$$

where the sum is finite since $x \in \ell_0(\mathbb{N}_0, \mathbb{H})$. Since in a reproducing kernel Hilbert space, convergence in norm implies pointwise convergence, we see that the operator

M_s is given by (6.4) for every $x \in \ell_2(\mathbb{N}_0, \mathbb{H})$, and hence (2) is in force.

Assume now that (2) holds. Consider the kernel $k(p, q)$ in (5.3), which on $|pq| < 1$ admits the power series expansion

$$k(p, q) = \sum_{n=0}^{\infty} p^n \bar{q}^n.$$

For sake of simplicity we will write $k_q(p)$ instead of $k(p, q)$. Let $q_1, q_2 \in \mathbb{B}$ and $c_1, c_2 \in \mathbb{H}$ and compute

$$\begin{aligned} \overline{c_2} (M_s^* k_{q_1} c_1) (q_2) &= [M_s^* k_{q_1} c_1, k_{q_2} c_2]_{\mathbf{H}_2(\mathbb{B})} \\ &= [k_{q_1} c_1, M_s(k_{q_2} c_2)]_{\mathbf{H}_2(\mathbb{B})} \\ &= [k_{q_1} c_1, \sum_{n=0}^{\infty} p^n s(p) \bar{q}_2^n c_2]_{\mathbf{H}_2(\mathbb{B})} \\ &= \overline{[\sum_{n=0}^{\infty} p^n s(p) \bar{q}_2^n c_2, k_{q_1} c_1]_{\mathbf{H}_2(\mathbb{B})}} \\ &= \overline{c_2} \left(\sum_{n=0}^{\infty} q_2^n \overline{p_1}^n c_1 \right). \end{aligned}$$

Thus, the adjoint of M_s is given by the formula:

$$(M_s^*(k(q)))(p) = \sum_{n=0}^{\infty} p^n \overline{s(q)^n} \bar{q}^n,$$

and (3) is just a rewriting that the operator $I - M_s M_s^*$ is non negative.

Assume now that (3) holds. We consider the relation R_s (that is, the linear subspace) in $\mathbf{H}_2(\mathbb{B}) \times \mathbf{H}_2(\mathbb{B})$ spanned by the pairs

$$\left(\sum_{n=0}^{\infty} p^n \bar{q}^n c, \sum_{n=0}^{\infty} p^n s(p) \overline{s(q)^n} \bar{q}^n c \right),$$

where p runs in \mathbb{B} and c in \mathbb{H} . The domain of R_s is dense, and the positivity of the kernel implies that R_s is a contraction, meaning that if $(f, g) \in R_s$, then

$$\|g\|_{\mathbf{H}_2(\mathbb{B})} \leq \|f\|_{\mathbf{H}_2(\mathbb{B})}.$$

It follows that R_s extends to the graph of an everywhere defined contraction, which we will denote by T . We now compute T^* :

$$\begin{aligned} \overline{c_2} (T^* k_{q_1} c_1) (q_1) &= [T^* k_{q_1} c_1, k_{q_2} c_2]_{\mathbf{H}_2(\mathbb{B})} \\ &= [k_{q_1} c_1, \sum_{n=0}^{\infty} p^n \overline{s(q_2)^n} \bar{q}_2^n c_2]_{\mathbf{H}_2(\mathbb{B})} \\ &= \overline{c_2} \left(\sum_{n=0}^{\infty} q_2^n s(q_2) \bar{q}_1^n c_1 \right) \\ &= \overline{c_2} (M_s(k_{q_2} c_1)) (q_1). \end{aligned}$$

Thus $T^* = M_s$. We obtain (1) by looking at the operator M_s on each p^n . \square

We note that the kernel k_s can also be written as

$$\begin{aligned} k_s(p, q) &= (1 - 2\operatorname{Re}(q)p + |q|^2 p^2)^{-1} (1 - pq) \star (1 - s(p)\overline{s(q)}) \\ &= (1 - 2\operatorname{Re}(q)p + |q|^2 p^2)^{-1} (1 - pq - s(p)\overline{s(q)} + ps(p)q\overline{s(q)}). \end{aligned}$$

The operator M_s is the non-commutative version of the operator of multiplication by s . We note the following properties of the operators M_s :

Proposition 6.3. *Let s_1, s_2 and s be Schur multipliers depending on the quaternionic variable p . Then:*

$$(6.5) \quad M_{s_1} M_{s_2} = M_{s_1 \star s_2},$$

$$(6.6) \quad M_s M_p = M_p M_s = M_{ps}.$$

Proof. The first equality follows from the associativity of the \star -product. The second property is a consequence of Remark 2.9 while the last one follows from the previous ones. \square

The proof of the following proposition follows the case of analytic functions in the open unit disk (see [63, Lemma 1, p. 301] for instance for the latter), and is given for completeness. In the case of analytic functions in a half-plane, the corresponding result is more difficult to prove, and is called the Bochner-Chandrasekharan theorem. See [25, 74].

Remark 6.4. We now discuss a feature which arises because we are working in a non-commutative setting. When considering a vector space on \mathbb{H} , the set of right linear operators acting on V has a structure of vector space over \mathbb{H} only if V is a two sided vector space, in fact right linear operators form a left vector space, see Section 2. However, if in V it is defined a multiplication \star among vectors compatible with the right vector space structure, or at least, if for some $w \in V$ the multiplication $w \star v$ is an operation in V for all $v \in V$, then this multiplication allows to define right linear operators which form a right linear vector space. Thus, in this case, it is not necessary to require that V is two sided vector space. Indeed, we can define a right linear operator M_w :

$$M_w(v) := w \star v.$$

The linearity is obvious and the structure of right linear space on the operators is given by

$$(M_w \lambda)(v) := w \lambda \star v, \quad \lambda \in \mathbb{H}.$$

In particular, if $V = \mathbf{H}_2(\mathbb{B})$ and M is the multiplication on the left by $s(p) \in V$ (assuming that the multiplication is defined for every element in V) then we have $M_s(f)(p) = s(p) \star f(p)$ and if $\pi(p) = a_0 + pa_1 + \dots + p^n a_n$ then $M_s(\pi)(p) = s(p) \star \pi(p) = s(p)a_0 + ps(p)a_1 + \dots + p^n s(p)a_n$.

Proposition 6.5. *Let T be a right-linear contraction from $\mathbf{H}_2(\mathbb{B})$ into itself, which commutes with M_p . Then $T = M_s$ for some Schur multiplier s .*

Proof. Denote $s(p) = (T1)(p)$. Then, for $a \in \mathbb{H} \subset \mathbf{H}_2(\mathbb{B})$ (in other words, in (6.7) and (6.8) below we view $a \in \mathbb{H}$ as a function in $\mathbf{H}_2(\mathbb{B})$ and, by the assumed right linearity, we have $Ta = (T1)a$)

$$(6.7) \quad ((TM_p)(a))(p) = (M_p(Ta))(p) = ((M_p(T1))a)(p) = ps(p)a.$$

More generally, an induction shows that

$$(6.8) \quad ((TM_{p^n})(a))(p) = (M_{p^n}(Ta))(p) = p^n s(p)a, \quad n = 2, 3, \dots$$

Thus, for every polynomial

$$\pi(p) = a_0 + pa_1 + \dots + p^n a_n,$$

we have

$$(6.9) \quad (T\pi)(p) = \sum_{j=0}^n p^j s(p)a_j.$$

Let now $f \in \mathbf{H}_2(\mathbb{B})$, with power series expansion

$$f(p) = \sum_{j=0}^{\infty} p^j a_j,$$

and let f_n be the polynomial

$$f_n(p) = \sum_{j=0}^n p^j a_j.$$

We have $\lim_{n \rightarrow \infty} \|f - f_n\|_{\mathbf{H}_2(\mathbb{B})} = 0$, and so, by continuity of T ,

$$\lim_{n \rightarrow \infty} \|Tf - Tf_n\|_{\mathbf{H}_2(\mathbb{B})} = 0$$

In a reproducing kernel Hilbert space convergence in norm implies pointwise convergence. Thus we have that, for every $p \in \mathbb{B}$,

$$(Tf)(p) = \sum_{j=0}^{\infty} p^j s(p)a_j,$$

where we have used (6.9). Thus, $T = M_s$. Finally, s is a Schur multiplier since T is assumed contractive. \square

The following result is a counterpart of Schwarz's lemma for Schur multipliers.

Theorem 6.6. *Let s be a Schur multiplier, and assume that $s(0) = 0$. Set $s(p) = ps^{(1)}(p)$. Then $s^{(1)}$ is also a Schur multiplier.*

Proof. Let $\mathcal{H}(s)$ be the reproducing kernel quaternionic Hilbert space with reproducing kernel $k_s(p, q)$. Since $s(0) = 0$ we have that $1 = k_s(p, 0) \in \mathcal{H}(s)$, and $\|1\|_{\mathcal{H}(s)}^2 = k_s(0, 0) = 1$. Hence the kernel

$$k_s(p, q) - 1$$

is positive definite in \mathbb{B} ; see Lemma 5.1. Thus

$$\sum_{n=0}^{\infty} p^n \left(1 - ps^{(1)}(p)\overline{s^{(1)}(q)q}\right) \bar{q}^n \geq 1,$$

which can be rewritten as

$$p \left(\sum_{n=0}^{\infty} p^n \left(1 - s^{(1)}(p) \overline{s^{(1)}(q)} \right) \bar{q}^n \right) \bar{q} \geq 0.$$

Thus,

$$\sum_{n=0}^{\infty} p^n (1 - s^{(1)}(p) \overline{s^{(1)}(q)}) \bar{q}^n \geq 0,$$

and $s^{(1)}$ is a Schur multiplier. □

We conclude this section with the following characterization of the space $\mathcal{H}(s)$:

Theorem 6.7. *Let s be a Schur multiplier. Then the associated reproducing kernel Hilbert space $\mathcal{H}(s)$ is equal to the operator range $\text{ran} (I - M_s M_s^*)^{1/2}$ endowed with the norm*

$$\|(I - M_s M_s^*)^{1/2} u\|_{\mathcal{H}(s)} = \|(I - \pi)u\|_{\mathbf{H}_2(\mathbb{B})},$$

where π is the orthogonal projection on the kernel of $(I - M_s M_s^*)^{1/2}$.

The proof is as in the classical case and will be omitted. See [8, Theorem 3.2 p. 16] for instance and [4, Theorem 8.3, p. 119] for a similar proof in the setting of upper triangular operators. We refer to [53] for more on operator ranges and for their relations with de Branges Rovnyak spaces.

7. REALIZATIONS OF SCHUR MULTIPLIERS

The purpose of this section is to prove the following result, which is the counterpart in the quaternionic setting of the realization result for Schur functions. We first recall the following definition (see for instance [5, p. 14] in the complex case).

Definition 7.1. *Let \mathcal{H} be a right quaternionic Hilbert space. The pair $(C, A) \in \mathbf{L}(\mathcal{H}, \mathbb{H}^p) \times \mathbf{L}(\mathcal{H})$ is called closely outer connected if*

$$(7.1) \quad \bigvee_{n=0}^{\infty} \text{ran } A^{*n} C^* = \mathcal{H}.$$

The following is the counterpart of [5, Theorem 2.2.1, p. 49] in the setting of slice holomorphic functions.

Theorem 7.2. *Let s be a function from the open unit ball of \mathbb{H} into \mathbb{H} . Then, s is a Schur multiplier if and only if there exist a right quaternionic Hilbert space \mathcal{H} and a coisometric operator*

$$(7.2) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathbb{H} \longrightarrow \mathcal{H} \oplus \mathbb{H}$$

such that s can be written as a power series

$$s(p) = \sum_{n=0}^{\infty} p^n s_n$$

where

$$(7.3) \quad s_n = \begin{cases} D, & n = 0, \\ CA^{n-1}B, & n = 1, 2, \dots \end{cases}$$

Assume the pair (C, A) closely outer connected. Then, it is unique up to an isometry of right quaternionic Hilbert spaces.

We note that the realization (7.3) is called closely outer connected when the pair (C, A) is closely outer connected.

Using formula (2.12) we have for $\|pA\| < 1$

$$\sum_{n=1}^{\infty} p^n C A^{n-1} B = C \star \left(\sum_{n=1}^{\infty} p^n A^{n-1} \right) B = C \star S_R^{-1}(p^{-1}, A) B,$$

and so

$$(7.4) \quad s(p) = D + C \star S_R^{-1}(p^{-1}, A) B = D + pC \star (I - pA)^{-*} B.$$

The strategy of the proof is as follows. Let \mathbf{s} be a Schur multiplier, and let $\mathcal{H}(\mathbf{s})$ be the associated reproducing kernel quaternionic Hilbert space. As in the classical case, we want to show that $\mathcal{H}(\mathbf{s})$ is the state space, in an appropriate sense in the present setting, of a coisometric realization of \mathbf{s} . We use the same method as in [5] (see in particular p. 50 there), suitably adapted to the non commutativity of \mathbb{H} . One considers the reproducing kernel quaternionic right Hilbert space with reproducing kernel $k_s(p, q)$, and define the relation R (that is the right vector subspace of $(\mathcal{H}(\mathbf{s}) \oplus \mathbb{H}) \times (\mathcal{H}(\mathbf{s}) \oplus \mathbb{H})$) defined as the right linear span of elements of the form

$$\left\{ \begin{pmatrix} k_s(p, q) \bar{q} u \\ \bar{q} v \end{pmatrix}, \begin{pmatrix} (k_s(p, q) - k_s(p, 0)) u + k_s(p, 0) \bar{q} v \\ (\overline{\mathbf{s}(q)} - \overline{\mathbf{s}(0)}) u + \overline{\mathbf{s}(0)} \bar{q} v \end{pmatrix} \right\}.$$

We claim that the relation R is densely defined and isometric. It will follow that R can be extended in a unique way to the graph of an isometric operator from $\mathcal{H}(\mathbf{s}) \oplus \mathbb{H}$ into itself. This operator (or more precisely its adjoint) will give the realization. We first prove some preliminary lemmas and then give the proof of the theorem along the above lines.

Lemma 7.3. *The relation R is isometric and densely defined.*

Proof. We want to prove that

$$(7.5) \quad \begin{aligned} & [k_s(p, q_1) \bar{q}_1 u_1, k_s(p, q_2) \bar{q}_2 u_2]_{\mathcal{H}(s)} + [\bar{q}_1 v_1, \bar{q}_2 v_2]_{\mathbb{H}} = \\ & = [(k_s(p, q_1) - k_s(p, 0)) u_1 + k_s(p, 0) \bar{q}_1 v_1, (k_s(p, q_2) - k_s(p, 0)) u_2 + k_s(p, 0) \bar{q}_2 v_2]_{\mathcal{H}(s)} + \\ & + [(\overline{\mathbf{s}(q_1)} - \overline{\mathbf{s}(0)}) u_1 + \overline{\mathbf{s}(0)} \bar{q}_1 v_1, (\overline{\mathbf{s}(q_2)} - \overline{\mathbf{s}(0)}) u_2 + \overline{\mathbf{s}(0)} \bar{q}_2 v_2]_{\mathbb{H}} \end{aligned}$$

for all choices of $u_1, u_2, v_1, v_2 \in \mathcal{H}$ and q_1, q_2 in the open unit ball of \mathbb{H} . We rewrite this equality as

$$\bar{u}_2 M_1 u_1 + \bar{v}_2 M_2 v_1 + \bar{v}_2 M_3 u_1 + \bar{u}_2 M_4 v_1 = \bar{u}_2 N_1 u_1 + \bar{v}_2 N_2 v_1 + \bar{v}_2 N_3 u_1 + \bar{u}_2 N_4 v_1,$$

and we will show that $M_i = N_i$ for $i = 1, 2, 3, 4$. We write out in details the case $i = 1$, and only outline the other cases.

Checking $M_1 = N_1$: Using the reproducing kernel property, we see that

$$M_1 = q_2 k_s(q_2, q_1) \bar{q}_1.$$

Still with this same property, we have

$$\begin{aligned} N_1 &= k_s(q_2, q_1) - k_s(q_2, 0) - k_s(0, q_1) + k_s(0, 0) + \\ &\quad + (s(q_2) - s(0))(\overline{s(q_1)} - \overline{s(0)}) \\ &= k_s(q_2, q_1) - (1 - s(q_2)\overline{s(0)}) - (1 - s(0)\overline{s(q_1)}) + 1 - |s(0)|^2 + \\ &\quad + s(q_2)\overline{s(q_1)} - s(q_2)\overline{s(0)} - s(0)\overline{s(q_1)} + |s(0)|^2 \end{aligned}$$

and thus the to prove the equality $M_1 = N_1$ is equivalent to check that

$$k_s(q_2, q_1) - q_2 k_s(q_2, q_1)\overline{q_1} = 1 - s(q_2)\overline{s(q_1)},$$

but this is a direct consequence of the definition of the kernel k_s .

Checking $M_2 = N_2$: We now have $M_2 = q_2\overline{q_1}$ and

$$N_2 = q_2 k_s(0, 0)\overline{q_1} + q_2 |s(0)|^2 \overline{q_1} = q_2 \overline{q_1} = M_2.$$

We have $M_3 = M_4 = 0$ and so we now need to check that $N_3 = N_4 = 0$.

Checking $N_3 = 0$: This amounts to verify that

$$q_2(k_s(0, q_1) - k_s(0, 0)) + q_2(s(0)(\overline{s(q_1)} - \overline{s(0)})) = 0,$$

but this is clear from the definition of the kernel since

$$k_s(0, q_1) = 1 - s(0)\overline{s(q_1)} \quad \text{and} \quad k_s(0, 0) = 1 - |s(0)|^2.$$

Checking $N_4 = 0$: This amounts to verify that

$$(k_s(0, 0) - k_s(0, q_2))\overline{q_1} + (s(0) - s(q_2))\overline{s(0)}\overline{q_1} = 0,$$

but this is also plain from the definition of the kernel k_s . □

Lemma 7.4. *R is the graph of a densely defined isometry. Let us denote by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* : \mathcal{H}(s) \oplus \mathbb{H} \longrightarrow \mathcal{H}(s) \oplus \mathbb{H}$$

its extension to all of $\mathcal{H}(s) \oplus \mathbb{H}$. Then

$$(7.6) \quad \begin{aligned} (Af)(p) &= \begin{cases} p^{-1}(f(p) - f(0)), & p \neq 0 \\ f_1, & p = 0, \end{cases} \\ (Bv)(p) &= \begin{cases} p^{-1}(s(p) - s(0))v, & p \neq 0 \\ s_1, & p = 0, \end{cases} \\ Cf &= f(0), \\ Dv &= s(0)v. \end{aligned}$$

Proof. We first check that R is the graph of a densely defined isometry. By definition, the domain of R is the set of $F \in \mathcal{H}(s) \oplus \mathbb{H}$ such that there exists $G \in \mathcal{H}(s) \oplus \mathbb{H}$ such that $(F, G) \in R$. It is therefore dense by construction since the kernels $k_s(\cdot, q)\overline{q}u$ are dense in $\mathcal{H}(s)$. The isometry property implies that

$$(0, G) \in R \implies G = 0.$$

We can thus introduce a densely defined operator T such that

$$G = TF \iff (F, G) \in R.$$

T is a densely defined isometry since R is isometric. As in the case of complex Hilbert spaces, it extends to an everywhere defined isometry. We now compute the operator A . Let $q \in \mathbb{B}$ and $u \in \mathbb{H}$. We have

$$A^*(k_s(\cdot, q)\bar{q})u = (k_s(\cdot, q) - k_s(\cdot, 0))u.$$

Hence, for $f \in \mathcal{H}(s)$ it holds that:

$$\begin{aligned} \bar{u}q(Af)(q) &= [Af, k_s(\cdot, q)\bar{q}u]_{\mathcal{H}(s)} \\ &= [f, k_s(\cdot, q) - k_s(\cdot, 0)]_{\mathcal{H}(s)} \\ &= \bar{u}(f(q) - f(0)) \end{aligned}$$

and hence

$$q(Af)(q) = f(q) - f(0).$$

Similarly we have

$$B^*(k_s(\cdot, q)\bar{q})u = (\overline{s(q)} - \overline{s(0)})u,$$

so that we can write for $v \in \mathbb{H}$:

$$\begin{aligned} \bar{u}q(Bv)(q) &= [Bv, k_s(\cdot, q)u]_{\mathcal{H}(s)} \\ &= [v, (\overline{s(q)} - \overline{s(0)})u]_{\mathbb{H}} \\ &= \bar{u}(s(q) - s(0))v \end{aligned}$$

and hence the formula for B . To compute C we note that $C^*(\bar{q}v) = k_s(\cdot, 0)\bar{q}v$ for every $q, v \in \mathbb{H}$. So, for $f \in \mathcal{H}(s)$ we have:

$$\bar{v}qCf = [Cf, \bar{q}v]_{\mathbb{H}} = [f, k_s(\cdot, 0)\bar{q}v]_{\mathcal{H}(s)} = \bar{v}qf(0),$$

and so $Cf = f(0)$. Finally, it is clear that $D = s(0)$. \square

With these results at hand we turn to the proof of the realization theorem.

Proof of Theorem 7.2: We note that the pair (C, A) in (7.6) is closely outer connected. Let $f \in \mathcal{H}(s)$, with power series

$$f(p) = \sum_{n=0}^{\infty} p^n f_n.$$

We have the formulas

$$f_n = CA^n f, \quad n = 0, 1, 2, \dots$$

and

$$f(p) = C \star (I - pA)^{-\star} f, \quad f \in \mathcal{H}(s).$$

Applying these formulas to the function $B1$ we obtain

$$s(p) - s(0) = pC \star (I - pA)^{-\star} B1.$$

We now turn to the converse and assume that s is given by a power series which converges in \mathbb{B} and for which the coefficients are of the form (7.3). To prove that s is a Schur multiplier, we will check the formula

$$(7.7) \quad 1 - s(p)\overline{s(q)} = U(p)(U(q))^* - pU(p)(U(q))^*\bar{q}, \quad p, q \in \mathbb{B},$$

where

$$(7.8) \quad U(p) = \sum_{n=0}^{\infty} p^n C A^n.$$

We have

$$\begin{aligned} 1 - s(p)\overline{s(q)} &= 1 - (D + \sum_{n=1}^{\infty} p^n C A^{n-1} B1)(D + \sum_{m=1}^{\infty} q^m C A^{m-1} B1)^* \\ &= 1 - DD^* - \sum_{m=1}^{\infty} DB^*(A^{m-1})^* C^* \bar{q}^m - \sum_{n=1}^{\infty} p^n C A^{n-1} B D^* + \\ &\quad - \sum_{n,m=1}^{\infty} p^n C A^{n-1} B B^*(A^{m-1})^* C^* \bar{q}^m \\ &= CC^* + \sum_{m=1}^{\infty} C(A^m)^* C^* \bar{q}^m + \sum_{n=1}^{\infty} p^n C A^n C^* + \\ &\quad - \sum_{n,m=1}^{\infty} p^n C A^{n-1} (I - AA^*)(A^{m-1})^* C^* \bar{q}^m \\ &= U(p)(U(q))^* - pU(p)(U(q))^* \bar{q}, \end{aligned}$$

where U is as in (7.8) and we have used the fact that the operator matrix (7.2) is coisometric.

It follows from (7.7) that

$$(7.9) \quad \sum_{n=0}^{\infty} p^n (1 - s(p)\overline{s(q)}) \bar{q}^n = U(p)U(q)^*,$$

is positive definite in \mathbb{B} , and so s is a Schur multiplier.

Finally we turn to the uniqueness claim. Let

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} : \mathcal{H}_1 \oplus \mathbb{H} \longrightarrow \mathcal{H}_1 \oplus \mathbb{H}$$

and

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} : \mathcal{H}_2 \oplus \mathbb{H} \longrightarrow \mathcal{H}_2 \oplus \mathbb{H}$$

be two closely outer-connected coisometric realizations of s , with state spaces right quaternionic Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. From (7.9) we have

$$U_1(p)(U_1(q))^* = U_2(p)(U_2(p))^*,$$

where U_1 and U_2 are built as in (7.8) from the present realizations. It follows that

$$C_1 A_1^n (A_1^m)^* C_1^* = C_2 A_2^n (A_2^m)^* C_2^*, \quad \forall n, m \in \mathbb{N}_0.$$

In view of the presumed outer-connectedness, the relation in $\mathcal{H}_1 \times \mathcal{H}_2$ defined by

$$((A_1^*)^m C_1^* u, (A_2^*)^m C_2^* u), \quad u \in \mathbb{H}, \quad m \in \mathbb{N}_0,$$

is a densely defined isometric relation with dense range. It is thus the graph of a unitary map U such that:

$$U((A_1^*)^m C_1^* u) = (A_2^*)^m C_2^* u, \quad m \in \mathbb{N}_0, \quad \text{and} \quad u \in \mathbb{H}.$$

Setting $m = 0$ leads to $UC_1^* = C_2^*$, that is

$$(7.10) \quad C_1 = C_2 U.$$

With this equality, writing

$$(UA_1^*)((A_1^*)^m C_1) = A_2^*(A_2^*)^m C_2^* = A_2^* U U^* (A_2^*)^m C_2^* = (A_2^* U)((A_1^*)^m C_1),$$

and taking into account that both pairs (C_1, A_1) and (C_2, A_2) are closely outer-connected, we obtain $A_1 U^* = U^* A_2$, that is

$$(7.11) \quad U A_1 = A_2 U.$$

Since clearly $D_1 = D_2 = s(0)$, it remains only to prove that $UB_1 = B_2$. This follows from the equalities (where we use (7.10) and (7.11))

$$s_n = C_1 A_1^{n-1} B_1 = C_2 A_2^{n-1} B_2 = C_1 A_1^{n-1} U^* B_2, \quad n = 1, 2, \dots$$

and from the fact that (C_1, A_1) is closely outer connected. \square

We note that we have followed the arguments in [5] suitably adapted to the present case. In particular the proof of the uniqueness is adapted from that of [5, Theorem 2.1.3, p. 46].

The preceding theorem gives a characterization of *all* Schur multipliers. A simple example is given by the choice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \bar{a} & \sqrt{1 - |a|^2} \\ \sqrt{1 - |a|^2} & -a \end{pmatrix},$$

where $a \in \mathbb{B}$. The corresponding Schur multiplier $s_a(p)$ is

$$s_a(p) = -a + (1 - |a|^2)(1 - p\bar{a})^{-\star} = (p - a) \star (1 - \bar{a})^{-\star},$$

and is the counterpart of an elementary Blaschke factor. The corresponding space $\mathcal{H}(s_a)$ is finite dimensional and spanned by the function $(1 - p\bar{a})^{-\star}$. We postpone to a future publication the study of finite dimensional (possibly indefinite) de Branges-Rovnyak spaces in the present setting.

We also note that (7.7) suggests an equivalent definition of Schur multiplier: The function s is a Schur multiplier if there is a function $k(p, q)$ positive definite in \mathbb{B} , and such that

$$(7.12) \quad 1 - s(p)\overline{s(q)} = k(p, q) - pk(p, q)\bar{q}, \quad p, q \in \mathbb{B},$$

See also (9.1), (9.2). These equations, as well as (7.12), can be rewritten as sums of positive definite functions, which induce an isometry. For instance, we can rewrite (7.12) as

$$1 + pk(p, q)\bar{q} = k(p, q) + s(p)\overline{s(q)}, \quad p, q \in \mathbb{B},$$

We will not pursue this line of idea here, which is also called the *lurking isometry method*; see [22], and postpone it to a work where we consider functions of several quaternionic variables.

Proposition 7.5. *The reproducing kernel k_s can be written as*

$$(7.13) \quad \begin{aligned} \bar{u}k_s(p, q)v &= [(C \star (I - qA)^{-\star})^*v, (C \star (I - pA)^{-\star})^*u]_{\mathcal{H}(s)} \\ &= \bar{u} \left(\sum_{n=0}^{\infty} p^n C A^n A^{*n} C^* \bar{q}^n \right) v. \end{aligned}$$

Proof. Indeed, both $k_s(p, q)$ and $U(p)(U(q))^* = (\sum_{n=0}^{\infty} p^n C A^n A^{*n} C^* \bar{q}^n)$ satisfy the equation

$$X - pX\bar{p} = 1 - s(p)\overline{s(q)}.$$

□

8. THE SCHUR ALGORITHM IN THE QUATERNIONIC SETTING

Recall that we have denoted by \mathcal{S} the set of functions analytic and contractive in the open unit disk \mathbb{D} . We rewrite the recursion (1.2) in a way which is used in signal processing, and is conducive to generalization to more general cases, and in particular to this case. We write

$$(8.1) \quad \begin{aligned} (1 \quad -s(z)) \begin{pmatrix} 1 & s_0 \\ \bar{s}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} &= (z(1 - s(z)\bar{s}_0) \quad -(s(z) - s_0)) \\ &= (1 - s(z)\bar{s}_0)^{-1} (z \quad -zs^{(1)}(z)) \\ &= (1 - s(z)\bar{s}_0)^{-1} z (1 \quad -s^{(1)}(z)). \end{aligned}$$

Let $s(p) = \sum_{p=0}^{\infty} p^n s_n$ be a Schur multiplier, and assume that $|s_0| < 1$. We now want the counterpart of (8.1) in the quaternionic setting. We set

$$s(p) - s(0) = p\sigma^{(1)}(p).$$

Then, the counterpart of (8.1) in the quaternionic setting is:

$$\begin{aligned} (I \quad -M_s) \begin{pmatrix} I & M_{s_0} \\ M_{\bar{s}_0} & I \end{pmatrix} \begin{pmatrix} M_p & 0 \\ 0 & I \end{pmatrix} &= ((I - M_s M_{\bar{s}_0})M_p \quad -(M_s - M_{s_0})) \\ &= (I - M_s M_{\bar{s}_0}) (M_p \quad -(I - M_s M_{\bar{s}_0})^{-1} M_p M_{\sigma_1}) \\ &= (I - M_s M_{\bar{s}_0}) M_p \star (I \quad -(I - M_s M_{\bar{s}_0})^{-1} M_{\sigma_1}). \end{aligned}$$

The function $s^{(1)}$ defined by

$$M_{s^{(1)}} = (I - M_s M_{\bar{s}_0})^{-1} M_{\sigma_1}$$

will be called the Schur transform of s . In view of Theorem 6.6 it is a Schur multiplier. When $|s^{(1)}(0)| < 1$, the preceding procedure can be iterated. This is the quaternionic counterpart of the Schur algorithm.

9. CONCLUDING REMARKS

Schur analysis has been extended to wide range of other settings. We mention in particular the definitions of the Schur-Agler classes for functions analytic in the polydisk or the unit ball. See [1, 21, 20, 22, 62]. Other settings include the non-commutative case, [9, 23], Riemann compact surfaces [17] functions defined on an homogeneous tree, [18], and others. See for instance [4] for the time-varying case, [10] for the stochastic case and [12] for the case of multi-dimensional systems.

We mentioned in the paper a number of problems which will be considered in future publications. We here list some more:

(1) We have presented a coisometric realization of a Schur multiplier. There are also isometric and unitary realizations, associated associated to the reproducing kernel Hilbert spaces $\mathcal{H}(s^\sharp)$ and $\mathcal{H}(D_s)$, where s^\sharp and D_s are defined in (5.4) and (5.5) respectively. Their counterpart in the setting of slice holomorphic functions will be presented elsewhere.

(2) Let $J \in \mathbb{H}^{n \times n}$ such that $J = J^* = J^{-1}$. A $\mathbb{H}^{n \times n}$ -valued function Θ defined in an open subset Ω of \mathbb{B} will be called J -contractive if the kernel

$$K_\Theta(p, q) = \sum_{n=0}^{\infty} p^n (J - \Theta(p)J(\Theta(q))^*) \bar{q}^n = (1 - \Theta(p)J(\Theta(q))^*) \star (1 - p\bar{q})^{-\star}$$

is positive definite. The study of these functions, and their relations to interpolation problems for Schur multipliers, are an important topic in Schur analysis. In the classical case, these functions Θ are called J -contractive, and are characteristic operator functions; see [65, 64, 66, 30, 31]. Their structure has been described in [69].

(3) We have mentioned Schur-Agler classes associated to the unit ball and the poly-disk. Their counterpart in the setting of Fueter series were presented in [16, 14]. The study of Schur multipliers slice hyperholomorphic in several quaternionic variables should be a topic of interest. Let $p = (p_1, \dots, p_N)$ and $q = (q_1, \dots, q_N)$ vary in \mathbb{B}^N . A Schur-Agler function is defined as a function $s(p)$ such that the

$$(9.1) \quad 1 - s(p)\overline{s(q)} = \sum_{j=1}^N (k_j(p, q) - p_j k_j(p, q)\bar{q}_j), \quad (\text{polydisk case}),$$

or

$$(9.2) \quad 1 - s(p)\overline{s(q)} = k(p, q) - \sum_{j=1}^N p_j k(p, q)\bar{q}_j, \quad (\text{unit ball case}),$$

where the k_j and k are positive definite kernels.

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