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KREIN SYSTEMS

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In memory of Mark Grigorievich Krein, with appreciation of his many great discoveries, on the occasion of his Centennial.

Abstract. In the present paper we extend results of M.G. Krein associated to the spectral problem for Krein systems to systems with matrix valued accelerants with a possible jump discontinuity at the origin. Explicit formulas for the accelerant are given in terms of the matrizant of the system in question. Recent developments in the theory of continuous analogs of the resultant operator play an essential role.

1. Introduction

The following result is due to M.G. Krein, see [14]:

Theorem 1.1. *Let $\mathbf{T} > 0$, and let k be a scalar continuous and hermitian function on the interval $[-\mathbf{T}, \mathbf{T}]$ such that for each $0 < \tau \leq \mathbf{T}$ the corresponding convolution integral operator T_τ on $\mathbf{L}^1[0, \tau]$,*

$$(T_\tau f)(t) = f(t) - \int_0^\tau k(t-s) f(s)\, ds, \quad 0 \leq t \leq \tau, \tag{1.1}$$

is invertible. Let $\gamma_\tau(t,s)$ denote the resolvent kernel

$$\gamma_\tau(t,s) - \int_0^\tau k(t-v)\gamma_\tau(v,s)dv = k(t-s), \quad 0 \leq t, s \leq \tau. \tag{1.2}$$

Consider the entire function

$$\mathcal{P}(\tau,\lambda) = e^{i\lambda\tau}\left(1 + \int_0^\tau e^{-i\lambda x}\gamma_\tau(x,0)dx\right), \tag{1.3}$$

$$\mathcal{P}_*(\tau,\lambda) = 1 + \int_0^\tau e^{i\lambda x}\gamma_\tau(\tau - x, \tau)dx. \tag{1.4}$$

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Then with $a(\tau)=\gamma_\tau(\tau,0)$ *and for* $\lambda\in\mathbb{C}$ *it holds that*

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial\tau}\mathcal{P}(\tau,\lambda)=i\lambda\mathcal{P}(\tau,\lambda)+\mathcal{P}_*(\tau,\lambda)a(\tau),\quad 0\le\tau\le\mathbf{T},\\ \dfrac{\partial}{\partial\tau}\mathcal{P}_*(\tau,\lambda)=\mathcal{P}(\tau,\lambda)a(\tau)^*.\end{array}\right. \tag{1.5}$$

Putting $Y(\tau,\lambda)=\begin{bmatrix}\mathcal{P}(\tau,\lambda) & \mathcal{P}_*(\tau,\lambda)\end{bmatrix}$, the system (1.5) can be rewritten as

$$\frac{\partial}{\partial\tau}Y(\tau,\lambda)=Y(\tau,\lambda)\left(i\lambda\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & a(\tau)\\ a(\tau)^* & 0\end{bmatrix}\right). \tag{1.6}$$

Here $\tau\in[0,\mathbf{T}]$. We call (1.6) a *Krein system* when, as in (1.5), the function a is given by $a(\tau)=\gamma_\tau(\tau,0)$, where $\gamma_\tau(t,s)$ is the resolvent kernel corresponding to some k on $[-\mathbf{T},\mathbf{T}]$ with the properties described in the Theorem 1.1. In that case, following Krein, the function k is called an *accelerant* for (1.6), and we shall refer to a as the *potential associated with the accelerant* k. The functions $\mathcal{P}(\tau,\cdot)$, $\mathcal{P}_*(\tau,\cdot)$ are called *Krein orthogonal functions* at τ associated to the *weight* $\delta-k$, where δ is the delta function.

In this paper we prove the analogue of Theorem 1.1 for systems with accelerants that are allowed to have a jump discontinuity at the origin. We also present explicit formulas for determining the unique accelerant k from the given potential a. As for continuous accelerants in [14], the results are proved not only for scalar functions but also for the matrix-valued case, when in (1.5) the functions $\mathcal{P}$, $\mathcal{P}_*$ and a are $\mathbb{C}^{r\times r}$–valued.

The result expressing the accelerant in terms of the potential referred to in the previous paragraph is based on a recent theorem involving a certain analog $\mathbf{R}(B,D)$ of the resultant operator for a class of entire matrix functions B and D. The resultant $\mathbf{R}(B,D)$ is defined as follows (see Section 3 for more details). Let B and D be of the form

$$B(\lambda)=I_r+\int_{-\tau}^{0}e^{i\lambda u}b(u)du\quad\text{and}\quad D(\lambda)=I_r+\int_{0}^{\tau}e^{i\lambda u}d(u)du,$$

where the functions b and d belong respectively to $\mathbf{L}_1^{r\times r}[-\tau,0]$ and $\mathbf{L}_1^{r\times r}[0,\tau]$. The *resultant* of B and D is the operator defined on the space $\mathbf{L}_1^{r\times r}[-\tau,\tau]$ by:

$$(\mathbf{R}(B,D)q)(u)=\begin{cases}q(u)+\displaystyle\int_{-\tau}^{\tau}d(u-s)q(u)du, & 0\le u\le\tau,\\ q(u)+\displaystyle\int_{-\tau}^{\tau}b(u-s)q(u)du, & -\tau\le u<0.\end{cases}$$

Let us now state our main results.

Theorem 1.2. *Let k be a $r \times r$-matrix valued accelerant on $[-\mathbf{T}, \mathbf{T}]$, with possibly a jump discontinuity at the origin, and let $\gamma_\tau(t, s)$ be the corresponding resolvent kernel as in (1.2). Put*

$$\mathcal{P}(\tau, \lambda) = e^{i\lambda\tau}\left(I_r + \int_0^\tau e^{-i\lambda x}\gamma_\tau(x, 0)dx\right) \tag{1.7}$$

$$\mathcal{P}_*(\tau, \lambda) = I_r + \int_0^\tau e^{i\lambda x}\gamma_\tau(\tau - x, \tau)dx. \tag{1.8}$$

Then $a(\tau) = \gamma_\tau(0, \tau)$, with $0 < \tau \leq \mathbf{T}$, extends to a continuous function on $[0, \mathbf{T}]$ and

$$Y(\tau, \lambda) = \begin{bmatrix} \mathcal{P}(\tau, \lambda) & \mathcal{P}_*(\tau, \lambda) \end{bmatrix},$$

satisfies the Krein system (1.6) with potential a.

For our second main result we need the *matrizant* of (1.6). By definition, this is the unique $\mathbb{C}^{2r \times 2r}$–valued solution $U(\tau, \lambda)$ of (1.6) satisfying the initial condition $U(0, \lambda) \equiv I_{2r}$.

Theorem 1.3. *Let k be a $r \times r$-matrix valued accelerant on $[-\mathbf{T}, \mathbf{T}]$, with possibly a jump discontinuity at the origin, and let a be the corresponding potential. Then k is uniquely determined by a, and k can be obtained from a in the following way. Let $U(\tau, \lambda)$ be the matrizant of (1.6), and put*

$$F(\lambda) = e^{i\lambda\mathbf{T}}\begin{bmatrix} I_r & I_r \end{bmatrix} U(\mathbf{T}, -\lambda)\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \quad G(\lambda) = \begin{bmatrix} I_r & I_r \end{bmatrix} U(\mathbf{T}, -\lambda)\begin{bmatrix} 0 \\ I_r \end{bmatrix}.$$

Then F and G are entire $r \times r$ matrix functions of the form

$$F(\lambda) = I_r + \int_0^{\mathbf{T}} f(x)e^{i\lambda x}\,dx, \quad G(\lambda) = I_r + \int_{-\mathbf{T}}^0 g(x)e^{i\lambda x}\,dx,$$

where f and g are continuous $\mathbb{C}^{r \times r}$-valued functions on $[0, \mathbf{T}]$ and $[-\mathbf{T}, 0]$, respectively. Moreover, the resultant operator $\mathbf{R}(F^\sharp, G^\sharp)$ is invertible, and the function k is given by the formula

$$k = [\mathbf{R}(F^\sharp, G^\sharp)]^{-1}q. \tag{1.9}$$

Here $F^\sharp(\lambda) = F(\bar{\lambda})^*$ and $G^\sharp(\lambda) = G(\bar{\lambda})^*$, where the superscript * means taking adjoints. Finally, q is the function on the interval $[-\mathbf{T}, \mathbf{T}]$ given by

$$q(x) = \begin{cases} f(-x)^*, & -\mathbf{T} \leq x < 0, \\ g(-x)^*, & 0 \leq x \leq \mathbf{T}. \end{cases}$$

To prove Theorem 1.2 we use in an essential way the results of [12]. The proof of Theorem 1.3 is based on recent results of [7] on the continuous analog of the resultant.

In each of the two theorems above our starting point is a given accelerant. In a next paper we plan to study the inverse situation, which includes, in particular, the question whether or not a continuous potential is always generated by an accelerant.

Let us illustrate Theorem 1.2 with an example. Take k to be

$$k(t) = \begin{cases} i, & \text{if} \quad t \in [0, \mathbf{T}], \\ -i, & \text{if} \quad t \in [-\mathbf{T}, 0]. \end{cases} \tag{1.10}$$

Clearly, k is continuous with a jump discontinuity at zero, and k is hermitian. Note that this function is of the form

$$k(t) = \begin{cases} iCe^{-itA}(I-P)B, & t \in [0, \mathbf{T}], \\ -iCe^{-itA}PB, & t \in [-\mathbf{T}, 0], \end{cases} \tag{1.11}$$

with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = B^* = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The formulas from [4] allow us to show that for this k the integral operator T_τ in (1.1) is invertible for $\tau < \frac{\pi}{2}$. Hence k is an accelerant on $[-\mathbf{T}, \mathbf{T}]$ whenever $\mathbf{T} < \frac{\pi}{2}$. Furthermore, again using the formulas from [4], one computes that for each $\tau < \frac{\pi}{2}$ the resolvent kernel associated to k, that is, the solution $\gamma_\tau(t,s)$ of (1.2), is given by

$$\gamma_\tau(t,s) = \begin{cases} \dfrac{ie^{2i(t-s)}}{1+e^{2i\tau}}, & 0 \le s < t \le \tau, \\ \\ \dfrac{-ie^{2i(t-s)}}{1+e^{-2i\tau}}, & 0 \le t < s \le \tau. \end{cases}$$

Direct computations show then that the functions $\mathcal{P}$ and $\mathcal{P}_*$ defined by the formulas (1.5) are equal to

$$\mathcal{P}(\tau, \lambda) = e^{i\lambda\tau} + \frac{2}{1+e^{2i\tau}} \frac{e^{2i\tau} - e^{i\lambda\tau}}{2-\lambda},$$
$$\mathcal{P}_*(\tau, \lambda) = 1 + \frac{2}{1+e^{2i\tau}} \frac{e^{2i\tau} - e^{i\lambda\tau}}{2-\lambda},$$

and that these functions satisfy the system (1.5) with

$$a(\tau) = \frac{2i}{1+e^{-2i\tau}}, \quad \tau \in [0, \mathbf{T}]. \tag{1.12}$$

Other examples will be given in the final two sections of the paper.

We now give the outline of the paper. The rest of the paper consists of five sections. In Section 2 we show that a Krein system can be associated to accelerants with jump discontinuities and prove Theorem 1.2. In Section 3 we review the notion of continuous analogue of the resultant and state the results from [7] used in this paper. The proof of Theorem 1.3 is given in Section 4. The last two sections present examples. In Section 5 we consider the case of accelerants k of the form (1.11), where A, B and C are matrices of appropriate sizes and P is a projection commuting with A. This includes in particular the case when the Fourier transform of k (considered as a function on $\mathbb{R}$) is a rational matrix–valued function vanishing at infinity.

Such functions k have in general a jump discontinuity at the origin. In Section 6 a class of continuous accelerants is elaborated.

2. Krein system for accelerants with jump discontinuity and proof of Theorem 1.2

In the proof of Theorem 1.1 an important role is played by the equations

$$\frac{\partial}{\partial\tau}\gamma_\tau(t,s) = \gamma_\tau(t,\tau)\gamma_\tau(\tau,s), \quad 0 \le s,t \le \tau, \tag{2.1}$$

$$\frac{\partial}{\partial\tau}\gamma_\tau(\tau-t,\tau-s) = \gamma_\tau(\tau-t,0)\gamma_\tau(0,\tau-s), \quad 0 \le t,s \le \tau. \tag{2.2}$$

Equation (2.1) is called the *Krein-Sobolev identity*. The second equation is obtained by replacing in equation (1.2) the function $k(t)$ by $k(-t)$. The corresponding resolvent kernel is equal to $\gamma_\tau(\tau-t,\tau-s)$, as can be seen by a change of variables; see the discussion [5, p. 450] and in particular equation (3.5) there. The above equations have been used by M.G. Krein in [15] to deduce his system (1.5) in the case of a continuous accelerant.

It is known [13, Section 7.3, p. 187] that continuity of the accelerant is not necessary to insure that the Krein-Sobolev identity holds. In fact, when k has a jump discontinuity at the origin appropriate generalizations of (2.1)-(2.2) have been established in [12].

Before presenting the proof of Theorem 1.2, we first review the necessary results from [12]. In what follows k is a $r \times r$ accelerant on $[-\mathbf{T},\mathbf{T}]$ with a possible jump discontinuity at the origin and $\gamma_\tau(t,s)$ is the corresponding resolvent kernel as in (1.2). From [12] we know that the function $(t,s,\tau) \mapsto \gamma_\tau(t,s)$ is continuous on the domain $0 \le s < t \le \mathbf{T}$, $0 < \tau \le \mathbf{T}$ and on the domain $0 \le t < s \le \mathbf{T}$, $0 < \tau \le \mathbf{T}$. Moreover, $(t,s,\tau) \mapsto \gamma_\tau(t,s)$ admits continuous extensions on the closures of these domains. In particular, $a(\tau) = \gamma_\tau(\tau,0)$ is continuous on the left open interval $(0,\mathbf{T}]$ and has a continuous extension to the closed interval $[0,\mathbf{T}]$.

Next, we consider the modifications of equations (2.1)–(2.2). Using the fact that k has a jump discontinuity at the origin, we let k_+ be the function equal to k for $t \neq 0$

$$k_+(0) = \lim_{\substack{h\to 0\\ h>0}} k(h).$$

Similarly, let k_- be the function equal to k for $t \neq 0$ and defined at the origin by

$$k_-(0) = \lim_{\substack{h\to 0\\ h<0}} k(h).$$

One defines $\gamma^u_\tau(t,s)$ and $\gamma^l_\tau(t,s)$ to be the resolvent equations corresponding to the function $k_+(t)$ and $k_-(t)$ respectively. Note that, for $t \neq s$,

$$\gamma_\tau(t,s) = \gamma^u_\tau(t,s) = \gamma^l_\tau(t,s). \tag{2.3}$$

It is proved in [12] that

$$\frac{\partial}{\partial \tau^+}\gamma_\tau^u(t,s) = \gamma_\tau^u(t,\tau)\gamma_\tau^l(\tau,s), \quad 0\le s,t\le\tau, \tag{2.4}$$

$$\frac{\partial}{\partial \tau^+}\gamma_\tau^l(t,s) = \gamma_\tau^u(t,\tau)\gamma_\tau^l(\tau,s), \quad 0\le s,t\le\tau, \tag{2.5}$$

and

$$\frac{\partial}{\partial \tau^-}\gamma_\tau^u(t,s) = \gamma_\tau^u(t,\tau)\gamma_\tau^l(\tau,s), \quad 0\le s,t\le\tau, \tag{2.6}$$

$$\frac{\partial}{\partial \tau^-}\gamma_\tau^l s(t,s) = \gamma_\tau^u(t,\tau)\gamma_\tau^l(\tau,s) \quad 0\le s,t\le\tau, \tag{2.7}$$

where $\frac{\partial}{\partial^+}$ and $\frac{\partial}{\partial^-}$ stand for derivatives from the right and from the left, respectively. See [12, (3.6)-(3.7) p. 274, and p. 278]. It follows that (2.2) becomes

$$\frac{\partial}{\partial \tau^+}\gamma_\tau^u(\tau-t,\tau-s) = \gamma_\tau^u(\tau-t,0)\gamma_\tau^l(0,\tau-s), \tag{2.8}$$

$$\frac{\partial}{\partial \tau^+}\gamma_\tau^l(\tau-t,\tau-s) = \gamma_\tau^u(\tau-t,0)\gamma_\tau^l(0,\tau-s), \tag{2.9}$$

where $0\le t,s\le\tau$, and

$$\frac{\partial}{\partial \tau^-}\gamma_\tau^u(\tau-t,\tau-s) = \gamma_\tau^u(\tau-t,0)\gamma_\tau^l(0,\tau-s), \tag{2.10}$$

$$\frac{\partial}{\partial \tau^-}\gamma_\tau^l(\tau-t,\tau-s) = \gamma_\tau^u(\tau-t,0)\gamma_\tau^l(0,\tau-s), \tag{2.11}$$

also for $0\le t,s\le\tau$.

Proof of Theorem 1.2. We have already proved the continuity of the potential a on $[0,\mathbf{T}]$.

Let $\mathcal{P}$ and $\mathcal{P}_*$ be defined by (1.7) and (1.8). Note that, in view of (2.3), one can replace γ_τ by γ_τ^u or γ_τ^l in the expressions for $\mathcal{P}$ and $\mathcal{P}_*$. Then, using the Krein-Sobolev identity (2.4), we have for $\tau>0$:

$$\begin{aligned}
\frac{\partial}{\partial \tau^+}\mathcal{P}(\tau,\lambda) &= i\lambda\mathcal{P}(\tau,\lambda)+e^{i\lambda\tau}\frac{\partial}{\partial\tau^+}\int_0^\tau e^{-i\lambda x}\gamma_\tau(x,0)\,dx\\
&= i\lambda\mathcal{P}(\tau,\lambda)+\gamma_\tau(\tau,0)+\int_0^\tau e^{i\lambda(\tau-x)}\frac{\partial}{\partial\tau^+}\gamma_\tau(x,0)\,dx\\
&= i\lambda\mathcal{P}(\tau,\lambda)+\gamma_\tau(\tau,0)+\int_0^\tau e^{i\lambda(\tau-x)}\gamma_\tau^u(x,\tau)\gamma_\tau^l(\tau,0)\,dx\\
&= i\lambda\mathcal{P}(\tau,\lambda)+\left(I_n+\int_0^\tau e^{i\lambda(\tau-x)}\gamma_\tau(x,\tau)\,dx\right)\gamma_\tau(\tau,0)\\
&= i\lambda\mathcal{P}(\tau,\lambda)+\left(I_n+\int_0^\tau e^{i\lambda x}\gamma_\tau(\tau-x,\tau)\,dx\right)\gamma_\tau(\tau,0)\\
&= i\lambda\mathcal{P}(\tau,\lambda)+\mathcal{P}_*(\tau,\lambda)\gamma_\tau(\tau,0).
\end{aligned}$$

Here we removed the superscripts u and l using (2.3) and using the fact that the value of an integral does not depend on the value of the integrand at one point. Using now (2.10) we obtain in a similar way that

$$\frac{\partial}{\partial\tau^-}\mathcal{P}(\tau,\lambda) = i\lambda\mathcal{P}(\tau,\lambda) + \mathcal{P}_*(\tau,\lambda)\gamma_\tau(\tau,0).$$

It follows that $\frac{\partial}{\partial\tau}\mathcal{P}(\tau,\lambda)$ exists and that the first equality in (1.5) holds.

Analogously, using the (2.8) and (2.10), we have

$$\begin{aligned}
\frac{\partial}{\partial\tau^\pm}\mathcal{P}_*(\tau,\lambda) &= \frac{\partial}{\partial\tau^\pm}\int_0^\tau e^{i\lambda x}\gamma_\tau(\tau-x,\tau)\,dx \\
&= e^{i\lambda\tau}\gamma_\tau(0,\tau) + \int_0^\tau e^{i\lambda x}\frac{\partial}{\partial\tau^\pm}\gamma_\tau(\tau-x,\tau)\,dx \\
&= e^{i\lambda\tau}\gamma_\tau(0,\tau) + \int_0^\tau e^{i\lambda x}\gamma_\tau(\tau-x,0)\gamma_\tau(0,\tau)\,dx \\
&= e^{i\lambda\tau}\left(I_r + \int_0^\tau e^{i\lambda(x-\tau)}\gamma_\tau(\tau-x,0)\,dx\right)\gamma_\tau(0,\tau), \\
&= e^{i\lambda\tau}\left(I_r + \int_0^\tau e^{i\lambda x}\gamma_\tau(x,0)\,dx\right)\gamma_\tau(0,\tau) \\
&= \mathcal{P}(\tau,\lambda)\gamma_\tau(0,\tau).
\end{aligned}$$

Since k is hermitian, we have $\gamma_\tau(0,\tau) = \gamma_\tau(\tau,0)^*$. Thus $\mathcal{P}$ and $\mathcal{P}_*$ satisfy (1.5), and hence $Y(\tau,\lambda) = \begin{bmatrix}\mathcal{P}(\tau,\lambda) & \mathcal{P}_*(\tau,\lambda)\end{bmatrix}$ satisfies (1.6). □

3. Intermezzo: The continuous analogue of the resultant

We review here the results of [7] needed in the proof of Theorem 1.3. The definition of the resultant operator $\mathbf{R}(B,D)$ has already been given in the introduction. Consider an entire matrix function of the form

$$L(\lambda) = I_r + \int_0^\tau e^{i\lambda x}\ell(x)dx, \quad \ell \in \mathbf{L}_1^{r\times r}[0,\tau]. \tag{3.1}$$

With a slight abuse of terminology, following [7], we call $L(\lambda)$ a *Krein orthogonal matrix function* if there exists a hermitian $\mathbb{C}^{r\times r}$–valued function $k \in \mathbf{L}_1^{r\times r}[-\tau,\tau]$ such that

$$\ell(t) - \int_0^\tau k(t-u)\ell(u)\,du = k(t), \quad 0 \le t \le \tau.$$

In that case we refer to $\delta - k$ as the associate *weight*. The following result is proved in [7, Theorem 5.6].

Theorem 3.1. *Let L be a $\mathbb{C}^{r\times r}$-valued entire function of the form* (3.1). *Then there exists a hermitian matrix function $k \in \mathbf{L}_1^{r\times r}[-\tau,\tau]$ such that L is the Krein orthogonal matrix function with weight $\delta - k$ if and only if there exists a matrix function M of the form*

$$M(\lambda) = I_r + \int_0^\tau e^{i\lambda u}m(u)du, \quad m \in \mathbf{L}_1^{r\times r}[0,\tau], \tag{3.2}$$

such that the following two conditions are satisfied:

$$L(\lambda)L^{\sharp}(\lambda) = M^{\sharp}(\lambda)M(\lambda), \quad \lambda \in \mathbb{C}, \tag{3.3}$$

$$\operatorname{Ker} L^{\sharp}(\lambda) \cap \operatorname{Ker} M(\lambda) = \{0\}, \quad \lambda \in \mathbb{C}. \tag{3.4}$$

Furthermore, when these conditions hold, the function k is given by the formula

$$k = \left[\mathbf{R}(L^{\sharp}, M)\right]^{-1} q, \qquad q(u) = \begin{cases} \ell(-u)^*, & -\tau \le u \le 0, \\ m(u), & 0 \le u \le \tau. \end{cases} \tag{3.5}$$

In [7] the above theorem is derived as a corollary of the following somewhat more general theorem ([7, Theorem 5.5]).

Theorem 3.2. *Given $\ell, m \in \mathbf{L}_1^{r\times r}[0,\tau]$, put*

$$L(\lambda) = I_r + \int_0^\tau e^{i\lambda u}\ell(u)du, \quad M(\lambda) = I_r + \int_0^\tau e^{i\lambda u} m(u)du.$$

Then there is a hermitian matrix function $k \in \mathbf{L}_1^{r\times r}[-\tau,\tau]$ such that

$$\ell(t) - \int_0^\tau k(t-u)\ell(u)\,du = k(t), \quad 0 \le t \le \tau, \tag{3.6}$$

$$m(t) - \int_0^\tau m(u)k(t-u)\,du = k(t), \quad 0 \le t \le \tau, \tag{3.7}$$

if and only if the two conditions (3.3) *and* (3.4) *are satisfied, and in that case the function k is uniquely determined by* (3.5).

In general, a Krein orthogonal matrix function L may have many different weights. This is reflected by the fact that given L as in Theorem 3.1 there may be many different functions M of the form (3.2) satisfying (3.3) and (3.4). However, as soon as M is fixed, then the weight is uniquely determined by (3.5) (as we see from Theorem 3.2).

Remark. If in (3.5) the functions ℓ and m are continuous on the interval $[0,\tau]$, then the function q in the right hand side of (3.5) is a continuous function on $[-\tau,\tau]$ with a possible jump discontinuity at zero. This implies that the function k defined by (3.5) is also continuous on $[-\tau,\tau]$ with a possible jump discontinuity at zero.

4. Proof of Theorem 1.3

Throughout this section k is a $r\times r$-matrix valued accelerant on $[-\mathbf{T},\mathbf{T}]$, with possibly a jump discontinuity at the origin, and we consider the Krein system (1.6) with the potential a defined by k. Furthermore $U(\tau,\lambda)$ will be the matrizant of (1.6).

Our aim is to prove Theorem 1.3. As in Theorem 1.3, put

$$F(\lambda) = e^{i\lambda\mathbf{T}}\begin{bmatrix} I_r & I_r \end{bmatrix} U(\mathbf{T},-\lambda)\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \; G(\lambda) = \begin{bmatrix} I_r & I_r \end{bmatrix} U(\mathbf{T},-\lambda)\begin{bmatrix} 0 \\ I_r \end{bmatrix}.$$

First let us show that

$$F(\lambda) = e^{i\lambda\mathbf{T}}\mathcal{P}(\mathbf{T}, -\lambda) \quad \text{and} \quad G(\lambda) = \mathcal{P}_*(\mathbf{T}, -\lambda), \tag{4.1}$$

where $\mathcal{P}(\mathbf{T}, \lambda)$ and $\mathcal{P}_*(\mathbf{T}, \lambda)$ are defined by (1.7) and (1.8) with $\tau = \mathbf{T}$. To obtain (4.1) note that for each $\lambda \in \mathbb{C}$ the two $r \times 2r$ matrix functions

$$\begin{bmatrix} I_r & I_r \end{bmatrix} U(\tau, \lambda) \quad \text{and} \quad \begin{bmatrix} \mathcal{P}(\tau, \lambda) & \mathcal{P}_*(\tau, \lambda) \end{bmatrix}$$

satisfy the linear differential equation (1.6), and at $\tau = 0$ both functions are equal to $\begin{bmatrix} I_r & I_r \end{bmatrix}$. Thus both have the same initial condition at $\tau = 0$. It follows that these two functions coincide on $0 \leq \tau \leq \mathbf{T}$. For $\tau = \mathbf{T}$ this yields the identities in (4.1).

Using (4.1), we see from the formulas for $\mathcal{P}$ and $\mathcal{P}_*$ in (1.7) and (1.8) that

$$F(\lambda) = I_r + \int_0^{\mathbf{T}} f(x)e^{i\lambda x}dx, \quad G(\lambda) = I_r + \int_{-\mathbf{T}}^0 g(x)e^{i\lambda x}dx, \tag{4.2}$$

with $f(x) = \gamma_{\mathbf{T}}(x, 0)$ on $0 \leq x \leq \mathbf{T}$ and $g(x) = \gamma_{\mathbf{T}}(\mathbf{T}+x, \mathbf{T})$ on the interval $-\mathbf{T} \leq x \leq 0$. In particular, the functions f and g are continuous on their respective domains as desired.

It remains to prove (1.9). To do this we first derive the following lemma.

Lemma 4.1. *The functions $\mathcal{P}$ and $\mathcal{P}_*$ given by* (1.7) *and* (1.8)*, respectively, satisfy the identity*

$$\mathcal{P}(\tau, \lambda)\mathcal{P}^\sharp(\tau, \lambda) = \mathcal{P}_*(\tau, \lambda)\mathcal{P}_*^\sharp(\tau, \lambda) \quad (0 \leq \tau \leq \mathbf{T},\ \lambda \in \mathbb{C}). \tag{4.3}$$

Furthermore, for each $0 \leq \tau \leq \mathbf{T}$ the left hand side in the above identity is a right canonical factorization (that is, for each $0 \leq \tau \leq \mathbf{T}$ the function $\det \mathcal{P}(\tau, \lambda)$ *has no zero in the closed lower half plane) while the right side is a left canonical factorization (that is, for each $0 \leq \tau \leq \mathbf{T}$ the function* $\det \mathcal{P}_*(\tau, \lambda)$ *has no zero in the closed upper half plane).*

Proof. Fix $0 \leq \tau \leq \mathbf{T}$. Recall that the integral operator T_τ defined by (1.1) is selfadjoint and invertible. Let a_τ, b_τ, b_τ, d_τ be the L^1-functions defined by

$$\begin{aligned}
a_\tau(t) - \int_0^\tau k(t-u)a_\tau(u)\,du &= k(t), \quad 0 \leq t \leq \tau, \\
b_\tau(t) - \int_{-\tau}^0 b_\tau(u)k(t-u)\,du &= k(t), \quad -\tau \leq t \leq 0, \\
c_\tau(t) - \int_{-\tau}^0 k(t-u)c_\tau(u)\,du &= k(t), \quad -\tau \leq t \leq 0, \\
d_\tau(t) - \int_0^\tau d_\tau(u)k(t-u)\,du &= k(t), \quad 0 \leq t \leq \tau,
\end{aligned}$$

and put

$$\mathcal{A}_\tau(\lambda) = I + \int_0^\tau e^{i\lambda s} a_\tau(s)\,ds, \quad \mathcal{B}_\tau(\lambda) = I + \int_{-\tau}^0 e^{i\lambda s} b_\tau(s)\,ds,$$

$$\mathcal{C}_\tau(\lambda) = I + \int_{-\tau}^0 e^{i\lambda s} c_\tau(s)\,ds, \quad \mathcal{D}_\tau(\lambda) = I + \int_0^\tau e^{i\lambda s} d_\tau(s)\,ds.$$

In terms of the resolvent kernel $\gamma_\tau(t,s)$ associated with k we have

$$a_\tau(x) = \gamma_\tau(x,0), \qquad b_\tau(-x) = \gamma_\tau(0,x) \quad (0 \le x \le \tau);$$
$$c_\tau(x) = \gamma_\tau(\tau + x, \tau) \quad d_\tau(-x) = \gamma_\tau(\tau, \tau + x) \quad (-\tau \le x \le 0).$$

Note that in this terminology, the functions $\mathcal{P}$ and $\mathcal{P}_*$ given by (1.7) and (1.8) are equal to

$$\mathcal{P}(\tau,\lambda) = e^{i\lambda\tau}\mathcal{A}_\tau(-\lambda), \quad \mathcal{P}_*(\tau,\lambda) = \mathcal{C}_\tau(-\lambda). \tag{4.4}$$

From Theorem 5.3 in [7] we know that

$$\mathcal{A}_\tau(\lambda)\mathcal{B}_\tau(\lambda) = \mathcal{C}_\tau(\lambda)\mathcal{D}_\tau(\lambda), \quad \operatorname{Ker}\mathcal{B}_\tau(\lambda) \cap \operatorname{Ker}\mathcal{D}_\tau(\lambda) = \{0\}. \tag{4.5}$$

Next recall that k is hermitian. This implies that

$$b_\tau(-x) = a_\tau(x)^*, \quad c_\tau(-x) = d_\tau(x)^* \quad (0 \le x \le \tau),$$

and hence $\mathcal{A}_\tau^\sharp(\lambda) = \mathcal{B}_\tau(\lambda)$ and $\mathcal{D}_\tau^\sharp(\lambda) = \mathcal{C}_\tau(\lambda)$. In particular, (4.5) reduces to

$$\mathcal{A}_\tau(\lambda)\mathcal{A}_\tau^\sharp(\lambda) = \mathcal{D}_\tau^\sharp(\lambda)\mathcal{D}_\tau(\lambda), \quad \operatorname{Ker}\mathcal{A}_\tau^\sharp(\lambda) \cap \operatorname{Ker}\mathcal{D}_\tau(\lambda) = \{0\}. \tag{4.6}$$

Finally, since for each $0 \le \tau \le \mathbf{T}$ the operator T_τ in (1.1) is selfadjoint and invertible, it follows that T_τ is strictly positive for each $0 \le \tau \le \mathbf{T}$. Then we know (using the theory of Krein orthogonal functions; see Theorem 8.1.1 in [6]) that the function $\det \mathcal{A}_\tau(\lambda)$ has no zero in the closed upper half plane, and the function $\det \mathcal{D}_\tau^\sharp(\lambda)$ has no zero in the closed lower half plane. Thus $\mathcal{A}_\tau(\lambda)\mathcal{A}_\tau^\sharp(\lambda)$ is a left canonical factorization and $\mathcal{D}_\tau^\sharp(\lambda)\mathcal{D}_\tau(\lambda)$ is a right canonical factorization. Using (4.4) the above remarks provide the proof of the lemma. □

We are now ready to prove (1.9). From (4.1) and (4.3) it follows that

$$F(\lambda)F^\sharp(\lambda) = G(\lambda)G^\sharp(\lambda). \tag{4.7}$$

Moreover the left hand side of this identity is a left canonical factorization and the right hand side is a right canonical factorization. In particular, $\operatorname{Ker} F^\sharp \cap \operatorname{Ker} G^\sharp = \{0\}$. This allows us to apply Theorem 3.2 with $\tau = \mathbf{T}$, $\ell(u) = f(u)$ and $m(u) = g(-u)^*$, where the functions f and g are as in (4.2). In other words, we apply $L = F$ and $M = G^\sharp$. It follows that there exists a unique hermitian $\tilde{k} \in \mathbf{L}_1^{r\times r}[-\mathbf{T}, \mathbf{T}]$ such that

$$f(t) - \int_0^{\mathbf{T}} \tilde{k}(t-s) f(s)\, ds = \tilde{k}(t), \quad 0 \le t \le \mathbf{T}, \tag{4.8}$$

$$g(t) - \int_{-\mathbf{T}}^{0} \tilde{k}(t-s) g(s)\, ds = \tilde{k}(t), \quad -\mathbf{T} \le t \le 0. \tag{4.9}$$

Moreover, $\tilde{k}$ is given by the formula

$$\tilde{k} = [\mathbf{R}(F^\sharp, G^\sharp)]^{-1} q \quad \text{with} \quad q(x) = \begin{cases} f(-x)^*, & -\mathbf{T} \le x \le 0, \\ g(-x)^*, & 0 \le x \le \mathbf{T}. \end{cases}$$

Since $f(x) = \gamma_{\mathbf{T}}(x, 0)$ on $0 \le x \le \mathbf{T}$ and $g(x) = \gamma_{\mathbf{T}}(\mathbf{T}+x, \mathbf{T})$ on the interval $-\mathbf{T} \le x \le 0$, we know from the proof of Lemma 4.1 that (4.8) and (4.9) also hold with $\tilde{k}$ being replaced by the original accelerant k. But then, by the uniqueness statement in Theorem 3.2, the functions $\tilde{k}$ and k coincide. Thus (1.9) holds, which completes the proof of Theorem 1.3.

Remark. In the proof of Lemma 4.1 we used in an essential way the accelerant and its properties. However, this is not necessary. It is possible to give a proof of Lemma 4.1 without any reference to the accelerant. In fact, such a proof can be obtained by using the properties of a canonical differential systems of Dirac type. To see this note that $e^{-i\tau\lambda} Y(\tau, -2\bar{\lambda})^*$ is a solution of a canonical differential system of Dirac type with potential $v(\tau) = -ia(\tau)$ whenever $Y(\tau, \lambda)$ is a solution of (1.6). We will come back to this in a later paper.

5. An example with jump discontinuity: the rational case

In this section we consider the case where the accelerant is of the form

$$k(t) = \begin{cases} iCe^{-itA}(I-P)B, & t > 0, \\ -iCe^{-itA}PB, & t < 0. \end{cases} \tag{5.1}$$

In this expression, A, B and C are matrices of appropriate sizes and P is a projection commuting with A. Motivation for such a form originates with linear system theory. Indeed, let W be a rational $\mathbb{C}^{p\times q}$-valued function, analytic at infinity. Then, as is well-known, W admits a realization of the form

$$W(\lambda) = D + C(\lambda I_N - A)^{-1} B,$$

where $D = W(\infty)$ and $(A, B, C) \in \mathbb{C}^{N\times N} \times \mathbb{C}^{N\times q} \times \mathbb{C}^{p\times N}$. Assume furthermore that A has no real eigenvalues. Then, the function W belongs to the Wiener algebra, and

$$W(\lambda) = D + \int_{\mathbb{R}} e^{i\lambda t} k(t) dt,$$

where k is of the form (5.1) with P being the Riesz projection corresponding to the eigenvalues of A in the upper–half plane. Note that, in general, functions k of the form (5.1) need not have summable entries.

In this section we first take

$$A = \begin{bmatrix} a^{\times} & -bb^* \\ 0 & a^{\times *} \end{bmatrix}, \quad B = \begin{bmatrix} b \\ c^* \end{bmatrix}, \quad C = \begin{bmatrix} -c & -b^* \end{bmatrix}, \tag{5.2}$$

where $(a, b, c) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times k} \times \mathbb{C}^{k\times n}$, and throughout it is assumed that the spectra of a and $a^{\times} = a - bc$ are both in the open upper half-plane. For P we take the Riesz projection of A corresponding to the eigenvalues in the upper–half plane. In other words P is given by

$$P = \begin{bmatrix} I & i\Omega \\ 0 & 0 \end{bmatrix}, \tag{5.3}$$

where Ω is the unique solution of the Lyapunov equation

$$i(\Omega a^{\times *} - a^{\times}\Omega) = bb^*. \tag{5.4}$$

With A, B, C and P as in (5.2) and (5.3), the function

$$W(\lambda) = I_r + \int_{\mathbb{R}} e^{i\lambda t} k(t)dt \tag{5.5}$$

is positive definite on the real line. Conversely, any rational $r \times r$ matrix function W which is positive definite on the real line and analytic at infinity with $W(\infty) = I_r$ can be represented in this way (see [1]).

Proposition 5.1. *When k is of the form* (5.1) *with A, B, C and P being given by* (5.2) *and* (5.3)*, then k is an accelerant on each interval $[-\mathbf{T}, \mathbf{T}]$. Moreover, in this case the corresponding potential is given by*

$$a(\tau) = i\left((I_n + \Omega(Y - e^{-i\tau a^*} Y a^{i\tau a}))^{-1}(b + i\Omega c^*)\right)^*, \tag{5.6}$$

where Ω is given by (5.4)*, and where Y is the solution of the Lyapunov equation*

$$i(Ya - a^*Y) = -c^*c. \tag{5.7}$$

Proof. The fact that the function W in (5.5) is positive definite on the real line implies that for each τ the integral operator T_τ in (1.1) is strictly positive. Hence k is an accelerant on each interval $[-\mathbf{T}, \mathbf{T}]$.

Using Theorem 4.1 in [4] one computes that in this setting

$$\gamma_\tau(0, \tau) = -iC(Pe^{-i\tau(A-BC)}|_{\mathrm{Im}\ P})^{-1}PB.$$

Since A, B and C are given by (5.2), we have

$$A - BC = \begin{bmatrix} a & 0 \\ c^*c & a^* \end{bmatrix}.$$

It then follows, as computed in [1, p.15], that

$$\gamma_\tau(0, \tau) = -i(I_n + \Omega(Y - e^{-i\tau a^*} Y a^{i\tau a}))^{-1}(b + i\Omega c^*),$$

where Ω and Y are given by (5.4) and (5.7), respectively. Since the potential is given by $a(\tau) = \gamma_\tau(\tau, 0)$ and $\gamma_\tau(\tau, 0) = \gamma_\tau(0, \tau)^*$, we see that a is given by (5.6). □

Next we assume that the matrices A, B, and C in (5.1) are given by

$$A = 2\begin{bmatrix} \beta^* & \gamma_2\gamma_2^* \\ 0 & \beta \end{bmatrix}, \quad B = \sqrt{2}\begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}, \quad C = \sqrt{2}\begin{bmatrix} \gamma_1^* & \gamma_2^* \end{bmatrix}, \tag{5.8}$$

where β is a square matrix of order n, and γ_1 and γ_2 are matrices of sizes $n \times r$. Furthermore, we assume that $\beta^* - \beta = i\gamma_2\gamma_2^*$. A triple of matrices β, γ_1 and γ_2 with these properties will be called *admissible*. For the matrix P in (5.1) we take

$$P = \begin{bmatrix} I_n & -iI_n \\ 0 & 0 \end{bmatrix}. \tag{5.9}$$

The fact that the triple of matrices β, γ_1 and γ_2 is assumed to be admissible implies that with A, B, C and P as in (5.8) and (5.9), the function (5.5) is positive semi-definite on the real line. Conversely, any rational $r \times r$ matrix function W which is positive semi-definite on the real line and analytic at infinity with $W(\infty) = I_r$ can be represented in this way (see [9], also [10]).

For information about the connection between the matrices A, B, C and P in (5.2) and (5.3) and those in (5.8) and (5.9), we refer to the introduction of [3].

Proposition 5.2. *Let β, γ_1 and γ_2 be an admissible triple, and put*

$$k(t) = -2(\gamma_1 + i\gamma_2)^* e^{-2it\beta}\gamma_1, \quad k(-t) = k(t)^* \quad (t > 0). \tag{5.10}$$

Then k is an accelerant on each interval $[-\mathbf{T}, \mathbf{T}]$, and the corresponding potential is given by

$$a(\tau) = -2(\gamma_1 + i\gamma_2)^* e^{-i\tau\alpha^*}\Sigma(\tau)^{-1}e^{-i\tau\alpha}\gamma_1, \quad \alpha = \beta - \gamma_1\gamma_2^*, \tag{5.11}$$

where

$$\Sigma(t) = I_n + \int_0^t \Lambda(s)\Lambda(s)^*\, ds, \quad \Lambda(t) = \begin{bmatrix} e^{-it\alpha}\gamma_1 & -e^{it\alpha}(\gamma_1 + i\gamma_2) \end{bmatrix}. \tag{5.12}$$

Proof. Let A, B, C and P be given by (5.8) and (5.9). Put

$$S = \begin{bmatrix} I_n & iI_n \\ 0 & I_n \end{bmatrix}.$$

Then S is invertible, and one computes that

$$A = S\begin{bmatrix} 2\beta^* & 0 \\ 0 & 2\beta \end{bmatrix}S^{-1}, \quad B = \sqrt{2}S\begin{bmatrix} -i(\gamma_1 + i\gamma_2) \\ \gamma_1 \end{bmatrix},$$

$$C = \sqrt{2}\begin{bmatrix} \gamma_1^* & i(\gamma_1 + i\gamma_2)^* \end{bmatrix}S^{-1}, \quad P = S\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}S^{-1}.$$

It follows that

$$\begin{aligned} iCe^{-itA}(I-P)B &= 2i\begin{bmatrix}\gamma_1^* & i(\gamma_1+i\gamma_2)^*\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & e^{-2it\beta}\end{bmatrix}\begin{bmatrix}-i(\gamma_1+i\gamma_2)\\ \gamma_1\end{bmatrix}\\ &= -2(\gamma_1+i\gamma_2)^*e^{-2it\beta}\gamma_1. \end{aligned}$$

Analogously

$$\begin{aligned} -iCe^{-itA}PB &= -2i\begin{bmatrix}\gamma_1^* & i(\gamma_1+i\gamma_2)^*\end{bmatrix}\begin{bmatrix}e^{-2it\beta^*} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-i(\gamma_1+i\gamma_2)\\ \gamma_1\end{bmatrix}\\ &= -2\gamma_1^*e^{-2it\beta^*}(\gamma_1+i\gamma_2). \end{aligned}$$

It follows that k given by (5.10) can be written in the form (5.1) with A, B, C, and P as in (5.8) and (5.9).

Next, we consider $A^\times = A - BC$. We have

$$A^\times = 2\begin{bmatrix}\alpha^* & 0\\ -\gamma_1\gamma_1^* & \alpha\end{bmatrix}, \quad \text{where } \alpha = \beta - \gamma_1\gamma_2^*.$$

The proof of Proposition 4.1 in [9] shows that

$$Pe^{-itA^\times}|_{\mathrm{Im}\ P} = e^{-it\alpha}\Sigma(t)e^{-it\alpha^*}, \quad t \geq 0.$$

Since $\Sigma(t)$ is positive definite, the matrix $\Sigma(t)$ is invertible. Hence the map $Pe^{-itA^\times}|_{\mathrm{Im}\ P}$, viewed as an operator acting on Im P, is invertible. By Theorem 4.3 in [4] this implies that for our k the integral operator T_τ given by (1.1) is invertible for each τ, and

$$\begin{aligned} \gamma_\tau(0,\tau) &= -iC(Pe^{-i\tau(A-BC)}|_{\mathrm{Im}\ P})^{-1}PB\\ &= -2\gamma_1^*e^{i\tau\alpha^*}\Sigma(\tau)^{-1}e^{i\tau\alpha}(\gamma_1+i\gamma_2). \end{aligned}$$

Here we used that

$$PB = \sqrt{2}\begin{bmatrix}-i(\gamma_1+i\gamma_2)\\ 0\end{bmatrix}, \quad C|_{\mathrm{Im}\ P} = \sqrt{2}\gamma_1^*.$$

Since the potential is given by $a(\tau) = \gamma_\tau(\tau,0)$ and $\gamma_\tau(\tau,0) = \gamma_\tau(0,\tau)^*$, we see that a is given by (5.11). □

Remark. Note that the two propositions in this section do not cover the example presented in the introduction. Indeed, when k is given by (1.10), then k is not an accelerant for $[-\pi/2, \pi/2]$.

6. Another class of potentials

We now consider the case where the $\mathbb{C}^{r\times r}$-valued accelerant k admits a representation of the form

(6.1) $$k(t) = Ce^{tA}B, \quad t \in [-\mathbf{T}, \mathbf{T}],$$

where A, B and C are matrices of appropriate sizes. We assume that there exists a hermitian matrix H such that

(6.2) $$HA + A^*H = 0 \quad \text{and} \quad C = B^*H.$$

The latter implies that $k(t)^* = k(-t)$ on $[-\mathbf{T}, \mathbf{T}]$, and hence k is a hermitian kernel. Under certain minimality conditions the converse statement is also true. More precisely, if k given by (6.1) with the pair (A, B) being controllable and the pair (C, A) being observable, then $k(t)^* = k(-t)$ implies that there exists a unique invertible hermitian matrix H such that (6.2) holds.

Let $\tau \in (0, \mathbf{T}]$. As proved in [8], equation (1.2) has a unique solution if and only if the matrix

$$M_\tau = I - \int_0^\tau e^{-sA} B C e^{sA} ds \tag{6.3}$$

is invertible. When this is the case, we have:

$$\gamma_\tau(t,s) = Ce^{tA} M_\tau^{-1} e^{-sA} B, \quad s,t \in [-\tau, \tau]. \tag{6.4}$$

Proposition 6.1. *Assume k is given by* (6.1)*, and let H be a hermitian matrix H such that* (6.2) *holds. Then k is an accelerant if and only if the matrix M_τ in* (6.3) *is non-singular for $0 \le \tau \le \mathbf{T}$. In that case the corresponding potential is given by*

$$a(t) = Ce^{tA} M_t^{-1} B, \quad 0 < t \le \mathbf{T}, \tag{6.5}$$

and the functions

$$\begin{aligned}
\mathcal{P}(\tau,\lambda) &= e^{i\lambda\tau} I_r + C(A - i\lambda I_r)^{-1} \left\{ e^{\tau A} - e^{i\lambda\tau} I_r \right\} M_\tau^{-1} B, \\
\mathcal{P}_*(\tau,\lambda) &= I_r + C(A - i\lambda I_r)^{-1} \left\{ e^{\tau A} - e^{i\lambda\tau} I_r \right\} M_\tau^{-1} e^{-\tau A} B,
\end{aligned}$$

are the associate Krein orthogonal matrix functions.

Proof. Since k is hermitian, the operator T_τ will be strictly positive if and only T_τ is invertible. The latter happens if and only if M_τ is non singular. Thus k is an accelerant if and only if M_τ is non-singular for $0 \le \tau \le \mathbf{T}$.

Assume k to be an accelerant. Then the potential is given by $a(t) = \gamma_t(t, 0)$ on $(0, \mathbf{T}]$. Using (6.4), this yields (6.5). Furthermore, the associate Krein orthogonal function $\mathcal{P}$ for k can be computed as follows:

$$\begin{aligned}
\mathcal{P}(\tau,\lambda) &= e^{i\lambda\tau} \left(I_r + \int_0^\tau e^{-i\lambda x} \gamma_\tau(x,0) dx \right) \\
&= e^{i\lambda\tau} \left(I_r + \int_0^\tau e^{-i\lambda x} C e^{xA} M_\tau^{-1} B dx \right) \\
&= e^{i\lambda\tau} \left(I_r + C \left(\int_0^\tau e^{-i\lambda x} e^{xA} dx \right) M_\tau^{-1} B \right) \\
&= e^{i\lambda\tau} \left(I_r + C(A - i\lambda I)^{-1} \left\{ e^{\tau(A - i\lambda)} - I_r \right\} M_\tau^{-1} B \right) \\
&= e^{i\lambda\tau} I_r + C(A - \lambda I)^{-1} \left\{ e^{\tau A} - e^{i\lambda\tau} I_r \right\} M_\tau^{-1} B.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\mathcal{P}_*(\tau,\lambda) &= I_r + \int_0^\tau e^{i\lambda x}\gamma_\tau(\tau - x,\tau)dx \\
&= I_r + \int_0^\tau e^{i\lambda x} C e^{(\tau-x)A} M_\tau^{-1} e^{-\tau A} B dx \\
&= I_r + Ce^{\tau A}\left(\int_0^\tau e^{i\lambda x} e^{-xA} dx\right) M_\tau^{-1} e^{-\tau A} B \\
&= I_r + Ce^{\tau A}\left((i\lambda - A)^{-1} e^{(i\lambda - A)\tau} - (i\lambda - A)^{-1}\right) M_\tau^{-1} e^{-\tau A} B \\
&= I_r + C(A - i\lambda I_r)^{-1}\left(e^{\tau A} - e^{i\lambda\tau} I_r\right) M_\tau^{-1} e^{-\tau A} B.
\end{aligned}$$

This completes the proof. □

Corollary 6.2. *Assume k is given by* (6.1)*, and assume that* (6.2) *holds with $H = -I$. Then k is an accelerant. In particular, if $r_j > 0$ and $\beta_j \in \mathbb{R}$ for $j = 1, \ldots, n$, then the function*

$$k(t) = -\sum_{\nu=1}^{n} r_n e^{i\beta_\nu t} \tag{6.6}$$

is an accelerant for each each interval $[-\mathbf{T}, \mathbf{T}]$.

Proof. From $H = -I$, we see that the matrix M_τ in (6.3) can be rewritten as

$$M_\tau = I + \int_0^\tau \left(Ce^{sA}\right)^* \left(Ce^{sA}\right) ds.$$

It follows that M_τ is positive definite and hence non-singular for each $\tau \geq 0$. Thus k is an accelerant by Proposition 6.5 above.

Next, consider the function k in (6.6). Since $r_j > 0$ and $\beta_j \in \mathbb{R}$ for each $j = 1, \ldots, n$, we can represent k as in (6.1) by taking

$$A = \operatorname{diag}(i\beta_1, i\beta_2, \ldots, i\beta_n), \quad C = \begin{bmatrix}\sqrt{r_1} & \sqrt{r_2} & \cdots & \sqrt{r_n}\end{bmatrix}, \quad B = -C^*.$$

But then (6.2) holds with $H = I$. By the result of the first paragraph, this shows that k is an accelerant on $[-\mathbf{T}, \mathbf{T}]$ for each $\mathbf{T} > 0$. □

From (6.3) it follows that

$$M_\tau' = \frac{d}{d\tau} M_\tau = -e^{-\tau A} B C e^{\tau A}. \tag{6.7}$$

This together with the explicit formula (6.4) allows one to give a direct proof of the Krein-Sobolev equation (2.1) and of equation (2.2) for accelerants as in (6.1).

The class of accelerants considered in this section includes the restrictions of polynomials to $[-\mathbf{T}, \mathbf{T}]$. On the other hand, when considered for t on the whole real line, k is never integrable (except for the trivial case

$k = 0$). Thus this class of accelerants has a zero intersection with the accelerants considered in the first part of the previous section. Nevertheless the class of potentials corresponding to the accelerants considered in this section shares a number of common properties with the strictly pseudo-exponential potentials. For instance, using (6.7), we have

$$\begin{aligned} a(0) &= CB \\ a'(0) &= CAB + (CB)^2 \\ &\vdots \end{aligned}$$

and there exist non commutative polynomials $f_0, f_1, \ldots$ such that

$$CA^\ell B = f_\ell(v(0), \ldots, v^{(\ell)}(0)), \quad \ell = 0, 1, \ldots$$

Thus, and as for strictly pseudo-exponential potentials (see [2]), one can in principle recover the potential from the values of its first derivatives at the origin (cf., [11], where such results are proved for pseudo-exponential potentials).

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