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# On the Reproducing Kernel Hilbert Spaces Associated With the Fractional and Bi-Fractional Brownian Motions

**Comments**

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# ON THE REPRODUCING KERNEL HILBERT SPACES ASSOCIATED WITH THE FRACTIONAL AND BI–FRACTIONAL BROWNIAN MOTIONS

DANIEL ALPAY AND DAVID LEVANONY

Abstract. We present decompositions of various positive kernels as integrals or sums of positive kernels. Within this framework we study the reproducing kernel Hilbert spaces associated with the fractional and bi-fractional Brownian motions. As a tool, we define a new function of two complex variables, which is a natural generalization of the classical Gamma function for the setting we consider.

## 1. Introduction

In this paper we present expansions of the form

$$\mathbf{K}(t,s)=\sum_{n=1}^{\infty}K^n(t,s)K_n(t,s) \tag{1.1}$$

where $K(t,s)$ and $K_n(t,s)$, $n=1,2,\ldots$ are positive kernels on the real line, for a family of kernels $\mathbf{K}(t,s)$ of the the form

$$\mathbf{K}(t,s)=\varphi(r(t)+r(s))-\varphi(r(t-s)) \tag{1.2}$$

with appropriate choices of functions $r$ and $\varphi$. The functions $r$ we consider are real–valued and such that $r(0)=0$, but it is better to assume them complex–valued and with a possibly non zero $r(0)$ at this stage of the exposition. The case where $\varphi(x)=x$, corresponds to a class of functions introduced by J. von Neumann and I. Schoenberg in the late 1930s (see [38], [44]), and M.G. Krein in the 1940s (see [20] and the 1954 paper [21]; Krein's papers are reprinted in [22]). This class plays an important role both in analysis and in the theory of stochastic processes. To recall the role of the functions $r$ such that the kernel

$$K_r(t,s)=r(t)+r(s)^*-r(t-s)-r(0), \tag{1.3}$$

is positive on the real line, we first give a definition: A *helical arc* is a continuous map $t\mapsto\xi_t$ from the real line into a metric space (with metric $d$) such that $d(\xi_t,\xi_s)$ is a function of $t-s$:

$$d(\xi_t,\xi_s)=\rho(t-s). \tag{1.4}$$

When the metric space is a *real* Hilbert space, it is a consequence of a result of K. Menger (see [29]) that a function $\rho$ satisfies (1.4) if and only if the kernel

$$K_{\rho^2}(t,s)=\rho^2(t)+\rho^2(s)-\rho^2(t-s)$$

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is positive on the real line; see [44, Lemma 1, p. 229]. Using this fact, von Neumann and Schoenberg give in [44, Theorem 1, p. 229] an integral representation of the functions $\rho$ corresponding to a helical arc with values in a real Hilbert space. These are the functions $\rho$ such that:

$$\rho^2(t) = \int_0^\infty \frac{\sin^2 tu}{u^2} d\sigma(u), \tag{1.5}$$

where the positive real measure $\sigma$ is such that

$$\int_1^\infty \frac{d\sigma(u)}{u^2} < \infty.$$

We note the formula (see [44, (1.6) p. 229])

$$\rho^2(t) + \rho^2(s) - \rho^2(t-s) = \int_0^\infty \frac{(1-\cos 2tu)(1-\cos 2su) + \sin 2tu \sin 2su}{2u^2} d\sigma(u)$$

and hence the kernel $\rho^2(t) + \rho^2(s) - \rho^2(t-s)$, which is of the form (1.3), is positive on the real line. The result of von Neumann and Schoenberg expresses in particular that the converse is true: if $r$ is an even real-valued function such that $r(0) = 0$ and $K_r(t,s)$ is positive on the real line, then $r$ is of the form (1.5).

The motivation of von Neumann and Schoenberg comes from imbedding problems of metric spaces into Hilbert spaces; we will not address this question here, but refer the interested reader to [5, pp. 81–84] for further information.

In [20], Krein considers helical arcs defined on an open interval $(a,b) \subset \mathbb{R}$, with values in a *complex* Hilbert space. Condition (1.4) is replaced by the condition on the inner product, that is, by the condition that the function

$$B(t,s) = \langle \xi_{s+v} - \xi_v, \xi_{t+v} - \xi_v \rangle_{\mathcal{H}}$$

does not depend on $v$. Krein states that a function $B(t,s)$ corresponds to a helical arc defined on the real line if and only if it is of the form

$$B(t,s) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u} \frac{e^{-isu}-1}{u} dm(u) \tag{1.6}$$

for an appropriate increasing function $m$; see [20, Theorem 4] and [22, p. 115]. One can then take

$$\xi_t(u) = e^{itu} \quad \text{and} \quad \mathcal{H} = \mathbf{L}_2\left(\frac{dm(u)}{u^2}\right).$$

In particular, $B(t,s)$ defines a positive kernel on $\mathbb{R}$. Setting

$$g(t) = i\gamma t + \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{dm(u)}{u^2}$$

one has

$$B(t,s) = g(t-s) - g(s) - g(s)^*.$$

It follows (see also the paper [23, equations (11.16) and (11.17), p. 402] of M.G. Krein and H. Langer and the preprint [11]) that a function $r(t)$, satisfying to $r(-t) =$

$r(t)^*$, is such that the kernel $K_r(t,s)$ in (1.3) is positive on the real line if and only if it can be written as

$$r(t) = r_0 + i\gamma t - \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{dm(u)}{u^2} \tag{1.7}$$

where $r_0 = r(0)$ and $\gamma$ are real numbers and where $m$ is a positive measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{dm(u)}{u^2+1} < \infty.$$

Furthermore, equation (1.7) implies the representation of the kernel

$$r(t) + r(s)^* - r(t-s) - r(0) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u} \frac{e^{-isu}-1}{u} dm(u) \tag{1.8}$$

as an inner product. See [23, equation (11.17) p. 402].

Since $m$ is positive and hence real valued, we have:

$$\begin{aligned} \operatorname{Re}\{r(t)\} &= \int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2} dm(u) \\ \operatorname{Im}\{r(t)\} &= \int_{\mathbb{R}} \left\{ -\sin(tu) + \frac{tu}{u^2+1} \right\} dm(u) \end{aligned} \tag{1.9}$$

In particular, when the measure is even and $r(0) = 0$, one has

$$r(t) = \int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2} dm(u).$$

With the equality $1 - \cos 2u = 2\sin^2 u$, and a change of variables, one then recognizes formula (1.5) of von Neumann and Schoenberg. Note also that the kernel $K_r(t,s)$ takes the simpler form

$$K_r(t,s) = r(t) + r(s) - r(t-s). \tag{1.10}$$

The choice where $dm(u) = \frac{1}{\pi}|u|^{1-2H}du$ with $H \in (0,1)$, corresponds to covariance function of the fractional Brownian motion with Hurst parameter $H$. Indeed, we have the following computation:

$$\int_0^\infty \frac{1-\cos(tu)}{u^2}|u|^{1-2H}du = \int_0^\infty \frac{1-\cos y}{y^2 t^{-2}} y^{1-2H} t^{-(1-2H)} \frac{dy}{t} = c(1-2H)t^{2H},$$

where

$$c(1-2H) = \int_0^\infty (1-\cos y) y^{-1-2H} dy = \frac{1}{(1-2H)2H} \int_0^\infty y^{1-2H} \cos y dy,$$

where we have assumed that $H \neq 1/2$, and where we have integrated twice by parts. The last integral is computed by integrating the function

$$f(z) = e^{iz} e^{(1-2H)\ln z},$$

where $\ln z$ is the principal value of the logarithm in $\mathbb{C} \setminus \mathbb{R}_-$, along an appropriate contour and is found equal to

$$\Gamma(2-2H)\cos(H\pi),$$

where $\Gamma$ denotes Euler's Gamma function. Thus, $r(t) = \frac{V_H}{2}|t|^{2H}$, where

$$V_H = \frac{\Gamma(2-2H)\cos(H\pi)}{\pi(1-2H)H}.$$

The function

$$K_{H,\alpha}(t,s) = (|t|^{2H} + |s|^{2H})^{\alpha} - |t-s|^{2H\alpha}, \tag{1.11}$$

with

$$0 < \alpha \leq 1 \quad \text{and} \quad 0 < H < 1,$$

is a special case of (1.2) with $r(t) = |t|^{2H}$ and $\varphi(x) = x^{\alpha}$. It was introduced in [5, Exercise 2.12 (h) p. 79], is positive in $\mathbb{R}$, and, by Kolmogorov's theorem, may assume the role of the covariance function of a zero mean Gaussian stochastic process. This process was introduced and first studied by Houdré and Villa in [18], who coined for it the term *bi–fractional Brownian motion*. The classical Brownian motion corresponds to the choice $\alpha = 1$ and $H = 1/2$ while the choice $\alpha = 1$ and $H < 1$ $(H \neq 1/2)$ corresponds, up to a multiplicative constant factor, to the covariance function of the fractional Brownian motion. For $\alpha < 1$, one can view this kernel as a generalization of the covariance function of the fractional Brownian motion. It has been recently the topic of a number of papers; see [18], [32].

It is well known that the reproducing kernel space, associated with the covariance function of the Brownian motion, is the Sobolev space of functions absolutely continuous on the real line and with square summable derivatives; see [28, p. 25], [19]. The reproducing kernel Hilbert space associated with the covariance function of the fractional Brownian motion has also been characterized in a number of places; see for instance [3, Theorem 4.1 p. 948].

We note that Schoenberg [37, (4), p. 788] also considers the multidimensional kernel case.

We now turn to the outline of the paper. The paper consists of five sections besides the introduction. Sections 2 and 3 are of a review nature. The main results appear in Sections 4,5 and 6. In Section 2 we recall some of M.G. Krein's results on positive kernels of the form (1.3) and of the associated reproducing kernel Hilbert spaces. These results have been rediscovered within the setting of the fractional Brownian motion a number of times without reference to Krein's work. In the third section we recall several facts on integrals and products of positive kernels and of the associated reproducing Hilbert kernel spaces. Section 4 contains two proofs of the positivity of the kernel of the fractional Brownian motion. The first is valid for any $H \in (0,1)$ while the second one is valid only for $H \in (0,1/2)$. In both instances, the positivity of the kernel is proved by exhibiting it as a sum of positive kernels of the form (1.1); see formulas (4.1) and (4.8) below. In Section 5 we consider the case of the bi-fractional Brownian motion and associate to its covariance function an expansion of the form (1.1). In Section 6 we study some nonlinear transforms associated with the expansions (1.1).

A note on terminology: Throughout we use the terms positive kernels, positive functions, covariance kernels or covariance functions interchangeably.

## 2. The reproducing kernel Hilbert space $\mathcal{H}(K_r)$

Recall that a function $K(t,s)$ defined on a set $\Omega$ is said to be positive if it is hermitian:

$$K(t,s) = K(s,t)^*, \quad \forall t,s \in \Omega,$$

and if, for every integer $n$ and every choice of points $t_1, \ldots, t_n \in \Omega$, the $n \times n$ matrix with $\ell j$ entry $K(t_\ell, t_j)$ is nonnegative. In preparation for the forthcoming sections (see in particular formula (4.6) below), one also may recall that sums and products of positive functions are still positive functions. The Aronszajn–Moore theorem states that a function $K(t,s)$ is positive on a set $\Omega$ if and only if it can be written as

$$K(t,s) = \langle f_s, f_t \rangle_{\mathcal{C}}, \tag{2.1}$$

where $\mathcal{C}$ is a auxiliary Hilbert space and where $s \mapsto f_s$ is a function from $\Omega$ into $\mathcal{C}$. The space $\mathcal{C}$ can be chosen to be the reproducing kernel Hilbert space associated with $K(t,s)$. This is a Hilbert space of functions on $\Omega$, denoted by $\mathcal{H}(K)$, uniquely determined by the following two conditions: $(a)$ For every $s \in \Omega$ the function

$$K_s \ : \ u \mapsto K(u,s)$$

belongs to $\mathcal{H}(K)$, and: $(b)$ for every $s \in \Omega$ and $F \in \mathcal{H}(K)$

$$\langle F, K_s \rangle_{\mathcal{H}(K)} = F(s). \tag{2.2}$$

Setting $f_s = K_s$ in (2.2), leads to the factorization (2.1).

If $K$is positive on $\Omega$ so is the function $mK$, where $m$ is a strictly positive function. We recall that

$$\|F\|^2_{\mathcal{H}(K)} = m\|F\|^2_{\mathcal{H}(mK)}. \tag{2.3}$$

For future reference we now recall the characterization of the reproducing kernel space when a representation (2.1) is available.

**Lemma 2.1.** *Let $K(t,s)$ be of the form* (2.1) *and let $\mathcal{C}(f)$ be the closed linear span in $\mathcal{C}$ of the functions $f_s$ where $s$ runs throughout $\Omega$. Then the reproducing kernel Hilbert space with reproducing kernel $K(t,s)$ is the set of functions of the form*

$$F(t) = \langle x, f_t \rangle_{\mathcal{C}}, \quad x \in \mathcal{C}(f)$$

*with norm*

$$\|F\|_{\mathcal{H}(K)} = \|x\|_{\mathcal{C}}.$$

The next theorem characterizes the reproducing kernel Hilbert space associated with $K_r(t,s)$. It is a direct consequence of formula (1.8) and Lemma 2.1.

**Theorem 2.2.** *The reproducing kernel Hilbert space $\mathcal{H}(K_r)$ associated with $K_r(t,s)$ consists of functions of the form*

$$\mathcal{H}(K_r) = \left\{ F \ : \ F(t) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u} f(u) dm(u), \right\} \tag{2.4}$$

*where $f$ is in the closed linear span in $\mathbf{L}_2(dm)$ of the functions*

$$\chi_s(u) = \frac{e^{isu}-1}{u}, \quad s \in \mathbb{R}, \tag{2.5}$$

*with the norm*

$$\|F\|_{\mathcal{H}(K_r)} = \|f\|_{\mathbf{L}_2(dm)}. \tag{2.6}$$

Theorem 2.2 is proved through different methods in [3, Theorem 4.1 p. 948] for $dm(u) = \frac{1}{\pi}|u|^{1-2H}du$ with $H \in (1/2, 1)$. See also [36], [17]. We note that in the case $H \in (1/2, 1)$ the covariance kernel can also be expressed in terms of double integrals as follows: one has

$$|t|^{2H} + |s|^{2H} - |t-s|^{2H} = 2H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2}dudv$$

and the associated reproducing kernel Hilbert space can be expressed in terms of the preHilbert space of functions $f$ such that

$$\iint_{\mathbb{R}_+^2} f(u)f(v)|u-v|^{2H-2}dudv < \infty,$$

See [13], [14]. This last space is not complete; see [31, p. 270].

**Remarks:**
$(a)$ In (2.4), one can take $f \in \mathbf{L}_2(m)$, rather than in the stated closed linear span; one then looses in general the uniqueness of $f$ in the representation of $F$ and one has to change (2.6) to $\inf \|f\|_{\mathbf{L}_2(m)}$, where the inf is over all $f$ corresponding to the given function $F$.
$(b)$ An important case is when $dm$ is absolutely continuous with respect to Lebesgue measure and when moreover its derivative satisfies

$$\int_{\mathbb{R}} \frac{\ln m'(u)}{1+u^2}du > -\infty$$

Then $m'$ admits an outer factorization $m' = |h|^2$; see [16].
$(c)$ For any $m$ such that the space $\mathbf{L}_2(dm)$ is infinite dimensional and any $T > 0$, the closed linear span of the functions $\chi_t$ (defined by (2.5)) for $|t| \leq T$ has a special structure. It is a reproducing kernel space of *entire functions* of the kind introduced by L. de Branges; see [7], [8], [15], [16]. Its reproducing kernel is of the form

$$\frac{a(z)a(w)^* - b(z)b(w)^*}{-i(z-w^*)}$$

where the functions $a$ and $b$ are entire functions of finite exponential type. In the case of the fractional Brownian motion, these are the homogeneous de Branges spaces of entire functions (see [8, pp. 184-189]). The functions $a$ and $b$ are then expressed in terms of Bessel functions, a fact already observed (in a slightly different language) by Krein in [21]. See also [17]. We refer to [1] for more information on the reproducing kernel spaces of the kind introduced by de Branges and Rovnyak.
$(d)$ To prove that the kernels $K_{H,1}$ are positive, one can also use the theory of negative kernels (see [5]). Another proof can be found in [27, p. 209]. The associated reproducing kernel Hilbert space is given in for instance in [30] (with different representations of the kernels corresponding to $H < 1/2$ and $H > 1/2$; see [30, p. 136]), in [6, p. 319], and can also be deduced from the model for the fractional Brownian motion given in e.g. [24, p. 50], [36, §7.2 p. 318], [4]. A characterization as operator range can be deduced from [10, Lemma 3.1 p. 182]). When $H = 1/2$, it is well known that the associated reproducing kernel Hilbert space is the Sobolev

space of functions absolutely continuous on the real line and with square summable derivatives; see [28, p. 25], [19].

We now study a relation between the functions $f$ and $F$ of (2.4). We restrict ourselves to the case where $m$ is absolutely continuous with respect to Lebesgue measure, and resort to the theory of distributions, as we now explain. Denote by $\mathcal{S}$ the space of rapidly decreasing functions. These are the infinitely differentiable functions $\sigma(u)$ such that for every choice of $(k,r)\in\mathbb{N}^2$,

$$\sup_{u\in\mathbb{R}}|(1+u^2)^r\sigma^{(k)}(u)|<\infty.$$

We refer the reader to [41] and [42] for more information on this space and in particular on its topology. We denote by $\mathcal{S}'$ the set of continuous linear functionals on $\mathcal{S}$. The elements of $\mathcal{S}'$ are called *tempered distributions*.

**Theorem 2.3.** *A function $F$ of the form* (2.4)*, defines an element in $\mathcal{S}'$ whose derivative is the Fourier transform of the distribution*

$$T_{m'f}\sigma=\int_{\mathbb{R}}\sigma(u)m'(u)f(u)du. \tag{2.7}$$

**Proof:** We first note that the function $m'f$ indeed defines an element in $\mathcal{S}'$ via the formula (2.7) since, for every $\sigma\in\mathcal{S}$, the integral

$$\int_{\mathbb{R}}m'(u)f(u)\sigma(u)du=\int_{\mathbb{R}}\frac{m'(u)}{(1+u^2)^\ell}f(u)(1+u^2)^\ell\sigma(u)du,$$

exists, where $\ell\in\mathbb{N}$ is such that $(1+u^2)^\ell\sigma(u)$ is bounded on the real line. Using Fubini's theorem and integration by parts, we have for $\sigma\in\mathcal{S}$

$$\begin{aligned}
D_F\sigma&=\int_{\mathbb{R}}\sigma'(t)F(t)dt\\
&=\int_{\mathbb{R}}\sigma'(t)\left(\int_{\mathbb{R}}\frac{e^{itu}-1}{u}m'(u)f(u)du\right)dt\\
&=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\sigma'(t)\frac{e^{itu}-1}{u}dt\right)m'(u)f(u)du\\
&=-\int_{\mathbb{R}}\widetilde{\sigma}(u)m'(u)f(u)du,
\end{aligned}$$

where we denote by $\widetilde{\sigma}$ the Fourier transform

$$\widetilde{\sigma}(w)=\int_{\mathbb{R}}\sigma(t)e^{itw}dt.$$

This last equality expresses the fact that $\widetilde{D_F}=T_{m'f}$, and that the distributional derivative of $F$ is the distributional Fourier transform of $m'f$. □

Let us now restrict the above results to $dm(u)=du$. Then, we note that

$$\int_0^\infty\frac{1-\cos(tu)}{u^2}\,du=\frac{\pi|t|}{2}$$

(as is seen by integrating the function $f(z)=\frac{1-e^{itz}}{z^2}$ around an appropriate contour), and so

$$\begin{aligned}\frac{1}{2}\left\{|t|+|s|-|t-s|\right\} &= \frac{1}{\pi}\int_0^\infty \frac{\cos((t-s)u)-\cos(tu)-\cos(su)+1}{u^2}\,du\\ &= \mathrm{Re}\,\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{i(t-s)u}-e^{itu}-e^{-isu}+1}{u^2}du\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{i(t-s)u}-e^{itu}-e^{-isu}+1}{u^2}du\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{itu}-1}{u}\frac{e^{-isu}-1}{u}\,du\\ &= \frac{1}{2\pi}\langle\chi_t,\chi_s\rangle_{\mathbf{L}_2(du)}.\end{aligned}$$

The closed linear span in $\mathbf{L}_2(du)$ of the functions $\chi_t$ ($t\in\mathbb{R}$) is equal to $\mathbf{L}_2(du)$, and the reproducing kernel Hilbert space associated with the kernel $\frac{1}{2}\left\{|t|+|s|-|t-s|\right\}$ is equal to the set of functions $F$ of the form

$$F(t)=\int_{\mathbb{R}}\frac{e^{itu}-1}{u}f(u)du,$$

where $f\in\mathbf{L}_2(du)$, with the norm $\|F\|=\frac{1}{\sqrt{2\pi}}\|f\|_{\mathbf{L}_2(du)}$. This space is known to be the Sobolev space. We regenerate this result as follows. For $f\in\mathbf{L}_1(du)\cap\mathbf{L}_2(du)$, Fubini's theorem shows that

$$F(t)=\int_{\mathbb{R}}\frac{e^{itu}-1}{u}f(u)du=\int_0^t\left(\int_{\mathbb{R}}e^{ius}f(u)du\right)ds=\int_0^t\widetilde{f}(s)ds, \tag{2.8}$$

and

$$\|F\|=\frac{1}{\sqrt{2\pi}}\|f\|_{\mathbf{L}_2(\mathbb{R})}=\|\widetilde{f}\|_{\mathbf{L}_2(\mathbb{R})}.$$

The result for general $f\in\mathbf{L}_2(du)$ follows by approximation. If we restrict $t,s\in[0,T]$ for some $T>0$, then $f$ belongs to the closed linear span in $\mathbf{L}_2(\mathbb{R})$ of the functions $\chi_s$, $s\in[-t,0]$. Thus $f(t)=\int_0^T e^{-iut}g(u)du$ for some $g\in\mathbf{L}_2([0,T])$, and we have

$$F(t)=\int_0^t\left(\int_{\mathbb{R}}e^{isx}\left(\int_0^T e^{-ixu}g(u)du\right)dx\right)ds=\int_0^t g(u)du.$$

## 3. Integrals and products of positive functions

In this section we briefly review a number of results related to reproducing kernel Hilbert spaces. The results are stated for functions positive on the real line because this is the setting we consider, but are valid for functions positive on any set. The first result is in Schwartz's paper [40, p. 170]. We outline the proof in our special case for completeness.

**Theorem 3.1.** *Let $k_u(t,s)=k(t,s,u)$ be a continuous function on $\mathbb{R}^2\times\mathbb{R}_+$ such that for every $u\in\mathbb{R}_+$ the kernel $k(t,s,u)$ is positive on $\mathbb{R}$, and let $q$ be a function positive and increasing on $\mathbb{R}_+$. Assume that $\int_0^\infty k(t,t,u)dq(u)<\infty$. Then the function*

$$\mathbf{k}(t,s)=\int_0^\infty k(t,s,u)dq(u) \tag{3.1}$$

*is positive on the real line and its reproducing kernel Hilbert space is the Hilbert space* $\int_0^\infty \mathcal{H}(k_u)dq(u)$.

**Proof:** First we note that the function (3.1) is well defined. Indeed, from the positivity of $k_u$ we get that

$$|k(t,s,u)|^2 \le k(t,t,u)k(s,s,u)$$

and hence by Cauchy-Schwarz inequality we have

$$\begin{aligned}|\int_0^\infty k(t,s,u)dq(u)| &\le \int_0^\infty k(t,t,u)^{1/2}k(s,s,u)^{1/2}dq(u)\\ &\le \left(\int_0^\infty k(t,t,u)dq(u)\right)^{1/2}\left(\int_0^\infty k(s,s,u)dq(u)\right)^{1/2} < \infty.\end{aligned}$$

Consider now the vector space $\mathcal{M}(\mathbf{k})$ spanned by the finite linear combinations of the functions $\mathbf{k}(t,s)$, with inner product

$$\langle \mathbf{k}(\cdot,w),\mathbf{k}(\cdot,v)\rangle_{\mathcal{M}(\mathbf{k})} = \mathbf{k}(v,w).$$

An element $F \in \mathcal{M}(\mathbf{k})$ can thus be written as a finite sum

$$F(t) = \sum_{\ell=1}^m \mathbf{k}(t,s_\ell)c_\ell = \int_0^\infty \left(\sum_{\ell=1}^m k(t,s_\ell,u)c_\ell\right)dq(u), \quad c_\ell \in \mathbb{C},$$

with norm

$$\|F\|^2_{\mathcal{M}(\mathbf{k})} = \int_0^\infty \|\sum_{\ell=1}^m k(\cdot,s_\ell,u)c_\ell\|^2_{\mathcal{H}(k_u)}dq(u).$$

The space $\mathcal{H}(\mathbf{k})$ is on the one hand the completion of $\mathcal{M}(\mathbf{k})$ with respect to this inner product. By [12, §5, p. 146], this is also the description of the space $\int_0^\infty \mathcal{H}(k_u)dq(u)$. By uniqueness of the completion we have the result. □

Elements of the space $\int_0^\infty \mathcal{H}(k_u)dq(u)$ can be described as follows (see [40, p. 170], [33, pp. 77 and 96]): A function $F$ belongs to $\mathcal{H}(\mathbf{k})$ if and only if it can be written as

$$F(t) = \int_0^\infty x(t,u)dq(u) \tag{3.2}$$

where for every $u$, the function $x_u : t \mapsto x(t,u) \in \mathcal{H}(k_u)$ and the map $u \mapsto x_u$ is weakly measurable. Moreover,

$$\|F\|^2_{\mathcal{H}(\mathbf{k})} = \inf \int_0^\infty \|x_u\|^2_{\mathcal{H}(k_u)}dq(u) \tag{3.3}$$

where the infimum is over all the representations of a given $F$ in (3.2).

We refer the reader to [25], [26], [43] and [12, pp. 139–161] for more information on integrals of Hilbert spaces.

As a special case we have:

**Theorem 3.2.** *Let* $k_n(t,s)$, $n = 1,2,\ldots$ *be positive functions on* $\mathbb{R}$ *and let*

$$\mathbf{k}(t,s) = \sum_{n=1}^\infty k_n(t,s).$$

*Assume that* $\mathbf{k}(t,t) < \infty$ *for all* $t \in \mathbb{R}$. *Then*

$$\mathcal{H}(\mathbf{k}) = \sum_{n=1}^{\infty} \mathcal{H}(k_n) \tag{3.4}$$

*in the following sense:* $\mathcal{H}(\mathbf{k})$ *consists of the functions which can be written as*

$$f(t) = \sum_{n=1}^{\infty} f_n(t), \tag{3.5}$$

*where* $f_n \in \mathcal{H}(k_n)$ *and*

$$\sum_{n=1}^{\infty} \|f_n\|^2_{\mathcal{H}(k_n)} < \infty. \tag{3.6}$$

*Moreover, we have:*

$$\|f\|^2_{\mathcal{H}(\mathbf{k})} = \inf \sum_{n=1}^{\infty} \|f_n\|^2_{\mathcal{H}(k_n)}, \tag{3.7}$$

*where the* inf *is over all the possible decompositions* (3.5) *satisfying* (3.6).

We note that the infimum is in fact a minimum and is achieved for the choice

$$f_n = i_n i_n^* f$$

where $i_n$ is the inclusion map $f \mapsto f$ from $\mathcal{H}(k_n)$ into $\mathcal{H}(\mathbf{k})$. See [2], [33].

The sum (3.4) is in general not direct, and the spaces are called complementary; see [9] for more information on complementation theory.

The following auxiliary result will be instrumental in the following sections. Note that the result is stated in a form convenient for our present purpose.

**Theorem 3.3.** *Let* $k_n(t,s,u), n = 1,2,\dots$ *be a sequence of continuous functions defined on* $\mathbb{R}^2 \times \mathbb{R}_+$ *and assume that for every* $u \in \mathbb{R}_+$ *and every positive integer* $n$ *the function* $k_n(t,s,u)$ *is positive on* $\mathbb{R}$. *Assume that*

$$\sum_{n=1}^{\infty} k_n(t,s,u)$$

*is continuous on* $\mathbb{R}^2 \times \mathbb{R}_+$ *and that for every* $t \in \mathbb{R}$

$$\int_0^{\infty} \left( \sum_{n=1}^{\infty} k_n(t,t,u) \right) du < \infty$$

*Then,*

$$\int_0^{\infty} \left( \sum_{n=1}^{\infty} k_n(t,s,u) \right) du = \sum_{n=1}^{\infty} \int_0^{\infty} k_n(t,s,u) du.$$

**Proof:** For every $u \in \mathbb{R}_+$ and $n \in \mathbb{N}^*$ we have

$$|k_n(t,s,u)|^2 \leq k_n(t,t,u) k_n(s,s,u),$$

and so,

$$\begin{aligned}|\sum_{n=1}^{\infty} k_n(t,s,u)| &\le \sum_{n=1}^{\infty}\sqrt{k_n(t,t,u)}\sqrt{k_n(s,s,u)}\\ &\le \left(\sum_{n=1}^{\infty} k_n(t,t,u)\right)^{1/2}\left(\sum_{n=1}^{\infty} k_n(s,s,u)\right)^{1/2}\end{aligned}$$

and the function $u\mapsto \sum_{n=1}^{\infty} k_n(t,s,u)$ is absolutely summable for every choice of $t,s\in\mathbb{R}$. The result is then a direct consequence of the dominated convergence theorem. □

The next theorem is classical; see for instance [2],[33].

**Theorem 3.4.** *Let $k_1$ and $k_2$ be two functions positive on the real line. Then the function $k_1k_2$ is also positive on the real line. Assume moreover that the functions $k_1$ and $k_2$ are continuous. Then the Hilbert spaces $\mathcal{H}(k_1)$ and $\mathcal{H}(k_2)$ are separable. The space $\mathcal{H}(k_1k_2)$ consists of the restrictions on the diagonal of the elements of tensor product $\mathcal{H}(k_1)\otimes\mathcal{H}(k_2)$. Let $(f_n)_{n=0,\ldots}$ and $(g_n)_{n=0,\ldots}$ be Hilbert space basis of the spaces $\mathcal{H}(k_1)$ and $\mathcal{H}(k_2)$ respectively. Then $\mathcal{H}(k_1k_2)$ consists of the functions of the form*

$$F(t)=\sum_{n,m=0}^{\infty} c_{n,m}f_n(t)g_m(t)$$

*with*

$$\|F\|^2_{\mathcal{H}(k_1k_2)}=\sum_{n,m=0}^{\infty}|c_{n,m}|^2.$$

From this result we obtain that $f_1f_2\in\mathcal{H}(k_1k_2)$ where $f_\ell\in\mathcal{H}(k_\ell)$, $\ell=1,2$, and

$$\|f_1f_2\|_{\mathcal{H}(k_1k_2)}\le\|f_1\|_{\mathcal{H}(k_1)}\|f_2\|_{\mathcal{H}(k_2)}\quad\text{and that}\quad \|f^n\|^2_{\mathcal{H}(k^n)}\le\|f\|^{2n}_{\mathcal{H}(k)} \tag{3.8}$$

where $k$ is a positive kernel. See [34, p 243] for the second inequality.

## 4. Positivity of the kernels $K_{H,1}(t,s)$

In this section we present two decompositions of the form (1.1) of the kernels $K_{H,1}(t,s)$, defined in (1.11). The first is valid for any $H\in(0,1)$ while the second is valid only for $H\in(0,1/2)$.

**Theorem 4.1.** *Let $0<H<1$.*
*(a) It holds that*

$$\begin{aligned}\frac{\Gamma(1-H)}{2H}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}) =&\\ =\int_0^\infty \frac{(1-e^{-u^2t^2})(1-e^{-u^2s^2})}{u^{1+2H}}du+&\\ +\sum_{n=1}^{\infty}\frac{2^{n-1}\Gamma(n-H)}{n!}\frac{t^ns^n}{(t^2+s^2)^{n-H}}.&\end{aligned} \tag{4.1}$$

*(b)A function $F$ belongs to $\mathcal{H}(\frac{2H}{\Gamma(1-H)}K_{H,1})$ if and only if it can be written as*

$$F(t) = \sum_{n=0}^{\infty} F_n(t)$$

*with*

$$F_0(t) = \int_0^\infty \frac{(1-e^{-u^2t^2})x_0(u)}{u^{1+2H}}du,$$

*where $x_0$ belongs to the closed linear span of the functions $1-e^{-u^2s^2}$ $(s \in \mathbb{R})$ in $\mathbf{L}_2(1/u^{1+2H})$, and where for every $n \geq 0$,*

$$F_n(t) = t^n \int_0^\infty e^{-u^2t^2}x_n(u)u^{2n-1-2H}du,$$

*where $x_n$ belongs to the closed linear span of the functions $e^{-u^2s^2}$ $(s \in \mathbb{R})$ in $\mathbf{L}_2(u^{2n-1-2H})$. For any such decomposition,*

$$\|F\|^2 \leq \sum_{n=0}^{\infty} \frac{n!}{2^n}\|x_n\|^2_{\mathbf{L}^2(u^{2n-1-2H})}, \tag{4.2}$$

*and there is a unique decomposition for which equality hold in* (4.2)*.*

**Proof:** The proof is based on an idea of I. Schoenberg. We use the equality (see [39, equation (8) p. 526])

$$|t|^{2H} = \frac{\int_0^\infty(1-e^{-u^2t^2})u^{-1-2H}du}{\int_0^\infty(1-e^{-u^2})u^{-1-2H}du}. \tag{4.3}$$

The equality itself is proved by using the change of variable $u \mapsto u|t|$ in the integral

$$\int_0^\infty (1-e^{-u^2t^2})u^{-1-2H}du, \quad 0 < H < 1.$$

See [39, p. 526]. Moreover, integration by part leads to

$$\int_0^\infty (1-e^{-u^2})u^{-1-2H}du = \frac{\Gamma(1-H)}{2H}.$$

From (4.3) it follows that

$$\begin{aligned}\left(\frac{\Gamma(1-H)}{2H}\right)(|t|^{2H}+|s|^{2H}-|t-s|^{2H}) &= \\ &= \int_0^\infty \frac{(1-e^{-u^2t^2})(1-e^{-u^2s^2})}{u^{1+2H}}du+ \\ &\quad + \int_0^\infty \frac{e^{-u^2t^2}(e^{2u^2ts}-1)e^{-u^2s^2}}{u^{1+2H}}du.\end{aligned} \tag{4.4}$$

The second term is a decomposition of the form (3.1) with

$$k(t,s,u) = e^{-u^2t^2}(e^{2u^2ts}-1)e^{-u^2s^2} \quad \text{and} \quad q'(u) = \frac{1}{u^{1+2H}}.$$

The positivity of $K_{H,1}(t,s)$ follows by using the fact that sums and products of positive functions are still positive. We note that equation (4.4) proves that $K_{H,1}(t,s)$ is a covariance function, but it does not provide a *model* for it (in the sense given in [24, p. 41]). Still, one can be somewhat more explicit by using formulas (3.2) and (3.3): A function $F$ belongs to the reproducing kernel Hilbert space with reproducing kernel

$$M(t,s)=\int_0^\infty \frac{e^{-u^2t^2}(e^{2u^2ts}-1)e^{-u^2s^2}}{u^{1+2H}}du$$

if and only if it can be written as

$$F(t)=\int_0^\infty \frac{f(t,u)}{u^{1+2H}}du$$

where for every $u$ the function $f(t,u)$ belongs to the reproducing kernel Hilbert space $\mathcal{H}(N_u)$ with reproducing kernel

$$N_u(t,s)=e^{-u^2t^2}(e^{2u^2ts}-1)e^{-u^2s^2}$$

and

$$\int_0^\infty \frac{\|f(t,\cdot)\|^2_{\mathcal{H}(N_u)}}{u^{1+2H}}du<\infty.$$

Next, and using Theorem 3.3, write:

$$\begin{aligned}\int_0^\infty \frac{e^{-u^2t^2}(e^{2u^2ts}-1)e^{-u^2s^2}}{u^{1+2H}}du&=\sum_{n=1}^\infty \frac{2^nt^ns^n}{n!}\int_0^\infty e^{-u^2(t^2+s^2)}u^{2n-1-2H}du\\&=\sum_{n=1}^\infty \frac{2^{n-1}\Gamma(n-H)}{n!}\frac{t^ns^n}{(t^2+s^2)^{n-H}},\end{aligned}$$

which recovers (4.1). $\square$

For instance, for a given real number $s$ and for

$$F(t)=\frac{2H}{\Gamma(1-H)}K_{H,1}(t,s)$$

we have $x_0(u)=(1-e^{-u^2s^2})$ and for $n\geq 1$,

$$x_n(u)=\frac{2^n}{n!}s^ne^{-u^2s^2}.$$

For this choice of $x_0,x_1,\ldots$ equality holds in (4.2).

The second approach is also related to Schoenberg's work. Let $dm(u)$ be a positive measure on $[0,\infty)$ subject to the condition

$$\int_1^\infty dm(u)<\infty.$$

(Note that the Lebesgue measure does not satisfy this last requirement.)

The function

$$r(t)=\int_0^\infty (1-e^{-u|t|})dm(u) \tag{4.5}$$

is then defined for all real $t$'s. Such functions $r(t)$ have been introduced in [38, Definition 2, p. 825]. Functions of the form $\sqrt{r(t^2)}$ are exactly the functions such that a separable real Hilbert space $\mathcal{H}$ (with norm denoted by $\|\cdot\|$), endowed with the metric $F(\|x\|)$, can be isometrically imbedded in $\mathcal{H}$. See [38, Theorem 6, p. 828]. In particular, $r$ is of the form (1.5).

**Theorem 4.2.** *Let $r(t)$ be of the form (4.5). Then the kernel $K_r(t,s)$:*

$$K_r(t,s) = r(t) + r(s) - r(t-s).$$

*is positive on the real line. Moreover, with $K_{1,1}(t,s) = |t| + |s| - |t-s| \stackrel{\text{def.}}{=} K(t,s)$,*

$$\begin{aligned} K_r(t,s) = &\int_0^\infty (1-e^{-u|t|})(1-e^{-u|s|})dm(u)+ \\ &+\sum_{n=1}^\infty \frac{K^n(t,s)}{n!}\left(\int_0^\infty u^n e^{-u(|t|+|s|)}dm(u)\right). \end{aligned} \tag{4.6}$$

**Proof:** Using Theorem 3.3 we have:

$$\begin{aligned} K_r(t,s) &= \int_0^\infty \left\{1-e^{-u|t|}+1-e^{-u|s|}+e^{-u|t-s|}-1\right\} m(u)du \\ &= \int_0^\infty (1-e^{-u|t|})(1-e^{-u|s|})dm(u) + \int_0^\infty \left\{e^{-u|t-s|}-e^{-u(|t|+|s|)}\right\} dm(u) \\ &= \int_0^\infty (1-e^{-u|t|})(1-e^{-u|s|})dm(u)+ \\ &\quad + \int_0^\infty e^{-u|t|}(e^{u(|t|+|s|-|t-s|)}-1)e^{-u|s|}dm(u). \end{aligned}$$

which proves the positivity of the kernel since for every $u \geq 0$, one has

$$e^{u(|t|+|s|-|t-s|)} - 1 = \sum_1^\infty \frac{u^n}{n!}(|t|+|s|-|t-s|)^n$$

$\square$

**Remarks:**
$(a)$ The function

$$\Gamma(z,\mu) = \int_0^\infty u^{z-1}e^{-\mu u}dm(u) \tag{4.7}$$

is the generalization for any $m(u)$ of the classical Gamma function. Furthermore, the decomposition (4.6) is conducive in applications with regard to non–linear transforms. See Section 6.
$(b)$ The formula (4.6) allows to characterize the associated reproducing kernel Hilbert space.
$c$) In view of the result of von Neumann and Schoenberg mentioned in the introduction, and as we already noticed above, any function of the form (4.5) is of the form (1.5).

In the case where

$$dm(u) = \frac{e^{-u}}{u}du$$

we have $r(t) = \ln(1+|t|)$. This follows from the formula in [5, Corollary 2.10]:

$$\ln(1+z) = \int_0^\infty (1-e^{-uz})\frac{e^{-u}du}{u}, \quad \text{Re } z \geq 0.$$

It follows that the function

$$q(t,s) = \ln(1+|t|) + \ln(1+|s|) - \ln(1+|t-s|)$$

is positive on $\mathbb{R}$.

The choice

$$dm(u) = \frac{2H}{\Gamma(1-2H)} \frac{1}{u^{1+2H}} du$$

corresponds to the covariance function of the fractional Brownian motion for $H \in (0, 1/2)$ since (see [5, Corollary 2.10 p. 78]),

$$z^{2H} = \frac{2H}{\Gamma(1-2H)} \int_0^\infty (1 - e^{-uz}) \frac{du}{u^{1+2H}},$$

where $H \in (0, 1/2)$ and Re $z \geq 0$. We check this formula for positive $t$ as follows. The change of variable $v = tu$ leads to

$$\int_0^\infty (1 - e^{-uz}) \frac{du}{u^{1+2H}} = t^{2H} \int_0^\infty (1 - e^{-u}) \frac{du}{u^{1+2H}} = \frac{\Gamma(1-2H)}{2H}$$

where the last equality is obtained by integration by parts.

Specializing (4.6) to this case leads to:

**Theorem 4.3.** *Let $H \in (0, 1/2)$. Then:*

$$\begin{aligned} |t|^{2H} + |s|^{2H} - |t-s|^{2H} &= 2H(|t| + |s| - |t-s|) + \\ &\quad + \frac{2H}{\Gamma(1-2H)} \left\{ \int_0^\infty \frac{(1 - e^{-u|t|})(1 - e^{-u|s|})}{u^{1+2H}} du + \right. \\ &\quad \left. + \sum_{n=2}^\infty \frac{K^n(t,s)\Gamma(n-2H)}{n!(|t|+|s|)^{n-2H}} \right\}. \end{aligned} \tag{4.8}$$

*In particular, the space $\mathcal{H}(K_{H,1})$ contains contractively a copy of $\mathcal{H}(K_{1,1})$.*

**Proof:** Indeed, using Theorem 3.3 we have:

$$\begin{aligned} |t|^{2H} + |s|^{2H} - |t-s|^{2H} &= \frac{2H}{\Gamma(1-2H)} \left\{ \int_0^\infty \frac{(1 - e^{-u|t|})(1 - e^{-u|s|})}{u^{1+2H}} du + \right. \\ &\quad \left. + \sum_{n=1}^\infty \frac{K^n(t,s)\Gamma(n-2H)}{n!(|t|+|s|)^{n-2H}} \right\} \\ &= 2H(|t| + s| - |t-s|) + \\ &\quad + \frac{2H}{\Gamma(1-2H)} \left\{ \int_0^\infty \frac{(1 - e^{-u|t|})(1 - e^{-u|s|})}{u^{1+2H}} du + \right. \\ &\quad \left. + \sum_{n=2}^\infty \frac{K^n(t,s)\Gamma(n-2H)}{n!(|t|+|s|)^{n-2H}} \right\}. \end{aligned}$$

$\square$

**Remark:** Formula (4.1) is valid for any $0 < H < 1$ and we see that terms of the form $\Gamma(n-H)$ appear in it ($n \geq 1$). On the other hand, formula (4.8) is valid only for $0 < H < 1/2$, with terms of the form $\Gamma(n-2H)$.

## 5. A general family of positive functions

**Theorem 5.1.** *Let $r(t)$ be a real valued function such that the kernel $K_r(t,s)$, defined by* (1.10)*, is positive on the real line, and let $\varphi$ is of the form* (4.5)*. Then:*

$$\varphi(r(t)+r(s))-\varphi(r(t-s))=\sum_{n=1}^{\infty}\frac{K_r(t,s)^n}{n!}\Gamma(n,r(t)+r(s)) \tag{5.1}$$

*where $\Gamma(z,\mu)$ has been defined in* (4.7)*. In particular the function*

$$V(t,s)=\varphi(r(t)+r(s))-\varphi(r(t-s)) \tag{5.2}$$

*is positive on the real line.*

**Proof:** With $\varphi$ of the form (4.5), one has

$$\begin{aligned}\varphi(r(t)+r(s))-\varphi(r(t-s))&=\int_0^\infty\left\{e^{-ur(t-s)}-e^{-u(r(t)+r(s))}\right\}dm(u)\\&=\int_0^\infty e^{-ur(t)})\left\{e^{u(r(t)+r(s)-r(t-s))}-1\right\}e^{-ur(s)}dm(u)\\&=\sum_{n=1}^\infty\frac{K_r(t,s)^n}{n!}\Gamma(n,r(t)+r(s)).\end{aligned}$$

This allows to conclude the proof since the functions $K_r(t,s)^n$ and $\Gamma(n,r(t)+r(s))$ are positive on the real line, and so is their product. $\square$

We now consider the case $\varphi(z)=z^\alpha$ and obtain the following analogue of Theorem 4.3:

**Theorem 5.2.** *Let $\alpha\in(0,1)$. Then:*

$$(r(t)+r(s))^\alpha-r(t-s)^\alpha=\alpha K_r(t,s)+\frac{\alpha}{\Gamma(1-\alpha)}\sum_{n=2}^\infty\frac{K_r(t,s)^n\Gamma(n-\alpha)}{n!(r(t)+r(s))^{n-\alpha}}. \tag{5.3}$$

*In particular, the space with reproducing kernel $(r(t)+r(s))^\alpha-r(t-s)^\alpha$ contains contractively a copy of $\mathcal{H}(K_r)$.*

**Proof:** We have already recalled that

$$z^\alpha=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty(1-e^{-uz})\frac{du}{u^{1+\alpha}}.$$

Using the preceding arguments we therefore obtain:

$$\begin{aligned}(r(t)+r(s))^\alpha-r(t-s)^\alpha&=\frac{\alpha}{\Gamma(1-\alpha)}\sum_{n=1}^\infty\frac{K_r(t,s)^n}{n!}\int_0^\infty u^ne^{-u(r(t)+r(s))}\frac{du}{u^{1+\alpha}}\\&=\frac{\alpha}{\Gamma(1-\alpha)}\sum_{n=1}^\infty\frac{K_r(t,s)^n}{n!}\frac{1}{(r(t)+r(s))^{n-\alpha}}\Gamma(n-\alpha)\\&=\alpha K_r(t,s)+\frac{\alpha}{\Gamma(1-\alpha)}\sum_{n=2}^\infty\frac{K_r(t,s)^n\Gamma(n-\alpha)}{n!(r(t)+r(s))^{n-\alpha}}.\end{aligned}$$

The last claim of the theorem follows from (5.3) and Theorem 3.2 $\square$

**Remark:** Formula (5.3) is still valid for $\alpha = 1$ since the function $\Gamma(z)$ has a pole at the origin.

## 6. Associated non-linear transforms

We now associate a non-linear transform (in the sense of Saitoh; see [34, Appendix 2, p. 243]) to the decomposition (5.1). We begin with a general result.

**Theorem 6.1.** *Let $K$ and $K_n$, $n = 1, 2, \ldots$ be positive kernels on the real line and define*

$$\mathbf{K}(t, s) = \sum_{n=1}^{\infty} K^n(t, s) K_n(t, s).$$

*Assume that $\mathbf{K}(t, t)$ converges for all $t \in \mathbb{R}$. For every choice of $t_0 \in \mathbb{R}$, the map which to $f \in \mathcal{H}(K)$ associates the function*

$$\varphi(f)(t) = \sum_{n=1}^{\infty} K_n(t, t_0) f(t)^n$$

*maps $\mathcal{H}(K)$ into $\mathcal{H}(\mathbf{K})$. Furthermore,*

$$\|\varphi(f)\|^2_{\mathcal{H}(\mathbf{K})} \leq \sum_{n=1}^{\infty} K_n(t_0, t_0) \|f\|^{2n}_{\mathcal{H}(K)}. \tag{6.1}$$

**Proof:** Using (3.7) and the two inequalities in (3.8) one after the other, we have

$$\begin{aligned}
\|\varphi(f)\|^2_{\mathcal{H}(\mathbf{K})} &\leq \sum_{n=1}^{\infty} \|K_n(\cdot, t_0) f(\cdot)^n\|^2_{\mathcal{H}(K_n K^n)} \\
&\leq \sum_{n=1}^{\infty} \|K_n(\cdot, t_0)\|^2_{\mathcal{H}(K_n)} \|f^n\|^2_{\mathcal{H}(K^n)} \\
&\leq \sum_{n=1}^{\infty} K_n(t_0, t_0) \|f\|^{2n}_{\mathcal{H}(K)}
\end{aligned}$$

since

$$\|K_n(\cdot, t_0)\|^2_{\mathcal{H}(K_n)} = K_n(t_0, t_0).$$

□

As a consequence we have:

**Theorem 6.2.** *Let $t_0 \in \mathbb{R} \backslash \{0\}$, and let $\psi$ be the map which to $f \in \mathcal{H}(K_r)$ associates the function*

$$\psi(f)(t) = \sum_{n=1}^{\infty} \frac{\Gamma(n, r(t) + r(t_0))}{n!} f(t)^n. \tag{6.2}$$

*Then $\psi(f)$ belongs to the reproducing kernel Hilbert space $\mathcal{H}(K_{\psi,r})$ with reproducing kernel $K_{\psi,r}(t, s) = \psi(r(t) + r(s)) - \psi(r(t - s))$ and we have*

$$\|\psi(f)\|^2_{\mathcal{H}(K_{\psi,r})} \leq \sum_{n=1}^{\infty} \frac{\Gamma(n, 2r(t_0))}{n!} \|f\|^{2n}_{\mathcal{H}(K_r)} \tag{6.3}$$

Indeed, the kernel $\frac{\Gamma(n,r(t)+r(s))}{n!}$ is positive on the real line, and

$$\left\|\frac{\Gamma(n,r(\cdot)+r(t_0))}{n!}\right\|^2_{\mathcal{H}\left(\frac{\Gamma(n,r(t)+r(s))}{n!}\right)}=\frac{\Gamma(n,2r(t_0))}{n!};$$

See formula (2.3) if need be.

Similar theorems can be stated for the decompositions (4.6) and (4.8). The case (4.8) will not be pursued here. To present the results pertaining to the case (4.6), we make a number of additional assumptions. First we restrict ourselves to positive $t$ and $s$. The kernel $(|t|+|s|-|t-s|)^n$ then takes the form

$$K^n_{1,1}(t,s)=(|t|+|s|-|t-s|)^n=2^n\max(t^n,s^n),$$

and the corresponding reproducing kernel Hilbert space is equal to the set of functions of the form

$$F(t)=\int_0^{t^n}f(u)du,\quad f\in\mathbf{L}_2(\mathbb{R}),$$

with norm (recall (2.3))

$$\|F\|^2_{\mathcal{H}(K^n_{1,1})}=\frac{\int_0^\infty|f|^2(u)du}{2^n}=\frac{\int_0^\infty\frac{|F'(u)|^2}{nu^{n-1}}du}{2^n}.$$

See [35], [34]. We will also assume that for some $t_0\geq 0$

$$\sum_{n=1}^\infty\frac{\int_0^\infty u^ne^{-2ut_0}dm(u)}{2^n}<\infty. \tag{6.4}$$

We have:

**Theorem 6.3.** *Assume that* (6.4) *holds. The map which to* $F\in\mathcal{H}(K_{1,1})$ *associates*

$$\psi(F)(t)=\sum_{n=1}^\infty\left(\int_0^{t^n}F'(u)du\right)\left(\int_0^\infty u^ne^{-u(t+t_0)}dm(u)\right)$$

*sends* $\mathcal{H}(K_{1,1})$ *into* $\mathcal{H}(K_r)$ *and we have:*

$$\|\psi(F)\|^2_{\mathcal{H}(K_r)}\leq\|F'\|^2_{\mathbf{L}_2(\mathbb{R}^+)}\left\{\sum_{n=1}^\infty\frac{\int_0^\infty u^ne^{-2ut_0}dm(u)}{2^n}\right\}.$$

In the case of the fractional Brownian motion, the above theorem becomes (recall that $K_{H,1}(t,s)$ is defined by (1.11)):

**Theorem 6.4.** *Let* $0<H<1/2$ *and let* $t_0>1/2$. *The map which to* $F\in\mathcal{H}(K_{1,1})$ *associates the*

$$\psi(F)(t)=\frac{2H}{\Gamma(1-2H)}\sum_{n=2}^\infty\left(\int_0^{t^n}F'(u)du\right)\frac{\Gamma(n-2H)}{n!(t+t_0)^{n-2H}}$$

*sends* $\mathcal{H}(K_{1,1})$ *into* $\mathcal{H}(K_{H,1})$ *and we have:*

$$\|\psi(F)\|^2_{\mathcal{H}(K_r)}\leq\|F'\|^2_{\mathbf{L}_2(\mathbb{R}^+)}\sum_{n=2}^\infty\frac{\Gamma(n-2H)}{2^nn!(2t_0)^{n-2H}}. \tag{6.5}$$

We note that the series (6.5) converges for $t_0>1/2$ since $\Gamma(n)=(n-1)!$.

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