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POINT EVALUATION AND HARDY SPACE ON A HOMOGENEOUS TREE

DANIEL ALPAY AND DAN VOLOK

ABSTRACT. We consider stationary multiscale systems as defined by Bas-
seville, Benveniste, Nikoukhah and Willsky. We show that there are deep
analogies with the discrete time non stationary setting as developed by the
first author, Dewilde and Dym. Following these analogies we define a point
evaluation with values in a C^* -algebra and the corresponding “Hardy space”
in which Cauchy’s formula holds. This point evaluation is used to define in
this context the counterpart of classical notions such as Blaschke factors.

1. INTRODUCTION

The purpose of this paper is to explain relationships between the theory of non-
stationary linear systems indexed by the integers and the theory of stationary linear
systems indexed by a homogeneous tree. We restrict ourselves to the case of input/output
systems and postpone the treatment of state space realizations to a
future publication. Our motivation for this study originates with the works of Bas-
seville, Benveniste, Nikoukhah and Willsky (see [12], [11], [10], [9]) where a general
theory of multiscale system is developed. In particular these authors distinguished
special families of stochastic processes (stationary and isotropic) and developed
Schur-Levinson recursions for isotropic processes. They also distinguish a special
family of operators which they call *stationary transfer functions*. We associate to
such functions point evaluations whose domain and range are in a C^* -algebra asso-
ciated to the tree. We explain the relationships with the point evaluation defined
for upper-triangular operators in [2]. The analogies between the two cases allow
interfeeding. In particular, one can pose and solve interpolation problems which
should have applications to the theory of multiscale systems. We plan to consider
this in a future publication. Some of the results presented here were announced in
[4].

The outline of the paper is as follows. In Section 2 we briefly review the setting
developed in [2]. In the third Section we present part of the multiscale system the-
ory relevant to the present study. Although in part of a review nature, the point of
view contains some novelties, in particular by considering the ℓ_2 space associated to
the tree. An important role is played by the so-called Cuntz relations (see [16]). In
the fourth Section we introduce the C^* -algebra of constants and the Hardy space
associated to the tree. In Section 5 we study the properties of the point evaluation

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and the notion of Schur multipliers is studied in Section 6.

We would like to mention that similar techniques were used by Constantinescu and Johnson (see [14], [15]) in a different setting (tensor algebras rather than Cuntz algebras). Elements of system theory for Cuntz algebras are being considered by Ball and Vinnikov in [8] and [7]. Also, the connections between Cuntz algebras and wavelets are studied in the book [22].

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2. THE DISCRETE TIME NON STATIONARY SETTING

We briefly review the nonstationary setting developed in [1] and [2]. We fix a separable Hilbert space \mathcal{N} , the coefficient space, and consider the Hilbert space $\ell_{\mathcal{N}}^2$ of all two sided square summable sequences $f = (f)_{i=-\infty}^{\infty} = (\dots, f_{-1}, \boxed{f_0}, f_1, \dots)$ with components $f_i \in \mathcal{N}$ provided with the standard inner product. The set of bounded linear operators from $\ell_{\mathcal{N}}^2$ into itself is denoted by \mathcal{X} . Let Z denote the bilateral backward shift operator

$$(Zf)_i = f_{i+1}, \quad i = \dots, -1, 0, 1, \dots$$

It is unitary on $\ell_{\mathcal{N}}^2$. An element $A \in \mathcal{X}$ can be represented as an operator matrix (A_{ij}) with $A_{ij} = \pi^* Z^i A Z^{*j} \pi$ where π denotes the injection map: $u \in \mathcal{N} \mapsto (\dots, 0, \boxed{u}, 0, \dots) \in \ell_{\mathcal{N}}^2$. We denote by \mathcal{U} and \mathcal{D} the spaces of upper triangular and diagonal operators :

$$\mathcal{U} = \{A \in \mathcal{X} : A_{ij} = 0, i > j\}, \quad \mathcal{D} = \{A \in \mathcal{X} : A_{ij} = 0, i \neq j\}.$$

For $W \in \mathcal{D}$ we set $W^{(j)} = Z^{*j} W Z^j$ and

$$W^{[0]} = I, \quad W^{[n]} = W W^{(1)} W^{(2)} \dots W^{(n-1)} = (W Z^*)^n Z^n, \quad n \geq 1.$$

Theorem 2.1. *Let $F \in \mathcal{U}$ and $D \in \mathcal{D}$. There exists a unique sequence of operators $F_{[j]} \in \mathcal{D}$, $j = 0, 1, \dots$, namely $(F_{[j]})_{ii} = F_{i-j, i}$, such that*

$$(2.1) \quad F = \sum_{n=0}^{\infty} Z^n F_{[n]}$$

in the sense that $F - \sum_{j=0}^{n-1} Z^j F_{[j]} \in Z^n \mathcal{U}$. The operator $(Z - W)^{-1} (F - D)$ belongs to \mathcal{U} for $W \in \mathcal{D}$ such that $r_{sp}(Z^ W) < 1$ if and only if*

$$(2.2) \quad D = \sum_{n=0}^{\infty} W^{[n]} F_{[n]} \stackrel{\text{def.}}{=} F^\wedge(W).$$

An operator $F = (F_{ij}) \in \mathcal{X}$ is a Hilbert-Schmidt operator if all its entries F_{ij} are Hilbert-Schmidt operators on \mathcal{N} and $\sum_{ij} \text{Tr } F_{ij}^* F_{ij} < \infty$, where Tr stands for trace. The set of these operators will be denoted by \mathcal{X}_2 and it is a Hilbert space with respect to the inner product

$$\langle F, G \rangle_{\mathcal{X}_2} = \sum_{ij} \text{Tr } G_{ij}^* F_{ij} < \infty.$$

The subspaces of upper triangular and diagonal operators which are moreover Hilbert–Schmidt operators on \mathcal{N} will be denoted by \mathcal{U}_2 and \mathcal{D}_2 . The space \mathcal{U}_2 is a reproducing kernel Hilbert space with reproducing kernel

$$\rho_W^{-1} = (I - ZW^*)^{-1} = \sum_0^\infty (ZW^*)^n = \sum_0^\infty Z^n W^{[n]*}$$

in the sense that for all $W \in \Omega$, $E \in \mathcal{D}_2$, and $F \in \mathcal{U}_2$, the operator $\rho_W^{-1}E \in \mathcal{U}_2$ and

$$(2.3) \quad \langle F, \rho_W^{-1}E \rangle_{\mathcal{U}_2} = \text{Tr } E^* F^\wedge(W).$$

This last formula is the non stationary counterpart of Cauchy’s formula for Hardy functions.

The map $W \mapsto F^\wedge(W)$ (which was first introduced in [1]) and its counterpart when one replaces $(Z - W)^{-1}(F - D)$ by $(F - D)(Z - D)^{-1}$ in the above theorem allow to translate most, if not all, of the classical analysis of the Hardy space H_2 to the setting of upper triangular operators. The analogue of the Hardy space H_2 is given by the Hilbert space of upper triangular operators \mathcal{U}_2 . See [3], [6], [17] for sample applications. As already mentioned and as we will see in the sequel, they have analogues in the setting of multiscale system theory.

3. MULTISCALE SYSTEM THEORY

Some of the fundamental notions in the classical theory of discrete time linear systems are that of causality and stationarity. In this section we review the analogues of these notions, introduced by Basseville, Benveniste, Nikoukhah and Willisky in the case of multiscale linear systems.

Let \mathcal{T} be a homogeneous tree of order $q \geq 2$ - an infinite acyclic, undirected, connected graph such that every node has exactly $q + 1$ branches (see [24], [13]). We consider a linear system of the form

$$(3.1) \quad g(t) = (Sf)(t),$$

where the input signal $f = f(t)$ and the output signal $g = g(t)$ belong to the Hilbert space $\ell_2(\mathcal{T})$ of square-summable sequences, indexed by the nodes of \mathcal{T} , and where S is a bounded linear operator on $\ell_2(\mathcal{T})$ (notation: $S \in \mathbf{X}(\mathcal{T})$). Using the notation χ_t for the element of the standard basis of $\ell_2(\mathcal{T})$, supported at the node t , one can write

$$(3.2) \quad (Sf)(t) = \sum_{u \in \mathcal{T}} s_{t,u} f(u), \text{ where } s_{t,u} = [S\chi_u, \chi_t] \in \mathbb{C}$$

and where the sum is absolutely convergent by Cauchy–Schwarz inequality.

According to the standard tree compactification procedure, a boundary point of \mathcal{T} is an equivalence class of infinite paths modulo finite number of edges. Let us distinguish a boundary point of \mathcal{T} and denote it by $\infty_{\mathcal{T}}$. Then for each $t \in \mathcal{T}$ there exists a unique representative \mathcal{T}_t^- of the equivalence class $\infty_{\mathcal{T}}$, starting at t . For a pair of nodes t, s , the paths \mathcal{T}_t^- , \mathcal{T}_s^- have to coincide after a finite number of edges; the first of their common nodes is denoted by $s \wedge t$. The notion of distance $\text{dist}(s, t)$

between the nodes s, t , defined as the number of edges along the path connecting s and t , allows to introduce the partial order

$$s \preceq t \quad \text{if} \quad \text{dist}(s, s \wedge t) \leq \text{dist}(t, s \wedge t)$$

and the equivalence relation

$$s \simeq t \quad \text{if} \quad \text{dist}(s, s \wedge t) = \text{dist}(t, s \wedge t).$$

The equivalence classes, defined with respect to the equivalence relation above, are called horocycles.

Definition 3.1. The multiscale linear system (3.1) is said to be causal if for every node $t \in \mathcal{T}$ the subspace

$$\{f \in \ell_2(\mathcal{T}) : t \preceq \text{support}(f)\}$$

is S -invariant.

In order to analyze Definition 3.1, we consider the primitive shifts on the tree. By convention they act on the right and are defined as follows. The primitive upward shift $\bar{\gamma} : \mathcal{T} \mapsto \mathcal{T}$ is determined by

$$\forall t \in \mathcal{T} : t\bar{\gamma} \preceq t, \quad \text{dist}(t\bar{\gamma}, t) = 1.$$

In the choice of the primitive downward shifts there is some freedom; we assume that some such choice $\alpha_i : \mathcal{T} \mapsto \mathcal{T}$, $1 \leq i \leq q$,

$$\forall t \in \mathcal{T} : \{s \in \mathcal{T} : t \preceq s, \text{dist}(t, s) = 1\} = \{t\alpha_1, \dots, t\alpha_q\},$$

is fixed, as well. Furthermore we consider the primitive shift operators, acting on the left on $\ell_2(\mathcal{T})$ and defined via convolution:

$$\bar{\gamma}f(t) = \frac{1}{\sqrt{q}}f(t\bar{\gamma}), \quad \alpha_i f(t) = f(t\alpha_i).$$

We compute the adjoint operators $\gamma = \bar{\gamma}^*$, $\bar{\alpha}_i = \alpha_i^*$:

$$\gamma f(t) = \frac{1}{\sqrt{q}} \sum_{s\bar{\gamma}=t} f(s), \quad \bar{\alpha}_i f(t) = \begin{cases} f(t\bar{\gamma}), & t = t\bar{\gamma}\alpha_i, \\ 0, & \text{otherwise,} \end{cases}$$

and observe that the following relations hold true:

$$(3.3) \quad \alpha_i \bar{\alpha}_j = \delta_{i,j}, \quad \sum_{i=1}^q \bar{\alpha}_i \alpha_i = 1,$$

$$(3.4) \quad \bar{\gamma} = \frac{1}{\sqrt{q}} \sum_{i=1}^q \bar{\alpha}_i, \quad \gamma \bar{\gamma} = 1.$$

Equations (3.3) are called the Cuntz relations. Equation (3.4) implies that the primitive upward shift operator $\bar{\gamma}$ is an isometry from $\ell_2(\mathcal{T})$ into itself. However, it is not surjective and thus $\bar{\gamma}$ is not unitary (see also (4.2) and (4.3) below). We also observe that for any pair of nodes $t, u \in \mathcal{T}$ there exist a unique choice of indices $i_1, \dots, i_n, j_1, \dots, j_m$, such that

$$t = (t \wedge u)\alpha_{i_n} \cdots \alpha_{i_1}, \quad u = (t \wedge u)\alpha_{j_m} \cdots \alpha_{j_1}$$

(note that, according to the definition of $t \wedge u$, $j_m \neq i_n$). Then for any $f \in \ell_2(\mathcal{T})$ it holds that

$$f(u) = \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} f(t).$$

This observation leads to a multiscale analogue of Theorem 2.1. It can be formulated in terms of the point-wise convergence of a sequence of bounded operators on $\ell_2(\mathcal{T})$: we shall say that a sequence of bounded operators S_n converges point-wise to a bounded operator S if for every $f \in \ell_2(\mathcal{T})$ and $t \in \mathcal{T}$ $\lim_{n \rightarrow \infty} (S_n f)(t) = (Sf)(t)$. We note that on the dense subspace of finitely supported functions the point-wise convergence implies the convergence in strong operator topology.

Theorem 3.2. *Any operator $S \in \mathbf{X}(\mathcal{T})$ can be represented as the point-wise converging series*

$$(3.5) \quad S = \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq j_1, \dots, j_m \leq q \\ j_m \neq i_n}} \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} S_{j_1, \dots, j_m}^{i_1, \dots, i_n},$$

where $S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \in \mathbf{X}(\mathcal{T})$ are diagonal operators, uniquely determined by

$$(3.6) \quad S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \chi_t = 0, \quad t \notin \mathcal{T} \alpha_{j_m} \cdots \alpha_{j_1},$$

$$(3.7) \quad S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \chi_{t \alpha_{j_m} \cdots \alpha_{j_1}} = [S \chi_{t \alpha_{j_m} \cdots \alpha_{j_1}}, \chi_{t \alpha_{i_n} \cdots \alpha_{i_1}}] \chi_{t \alpha_{j_m} \cdots \alpha_{j_1}}, \quad t \in \mathcal{T}.$$

Proof. Let $f \in \ell_2(\mathcal{T})$ and $t \in \mathcal{T}$ be fixed. Using the operators, defined by (3.6), (3.7), one can rewrite (3.2) as follows:

$$\begin{aligned} Sf(t) &= \sum_{u \in \mathcal{T}} [S \chi_u, \chi_t] f(u) \\ &= \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq j_1, \dots, j_m \leq q \\ t \bar{\gamma}^n \alpha_{j_m} \cdots \alpha_{j_1} \neq t \bar{\gamma}^{n-1}}} [S \chi_{t \bar{\gamma}^n \alpha_{j_m} \cdots \alpha_{j_1}}, \chi_t] f(t \bar{\gamma}^n \alpha_{j_m} \cdots \alpha_{j_1}) \\ &= \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq j_1, \dots, j_m \leq q \\ j_m \neq i_n}} \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} S_{j_1, \dots, j_m}^{i_1, \dots, i_n} f(t), \end{aligned}$$

and we obtain (3.5), where the convergence is point-wise.

Furthermore, let $t \in \mathcal{T}$ be fixed and let $S \in \mathbf{X}(\mathcal{T})$ be of the form (3.5), where the coefficients $S_{j_1, \dots, j_m}^{i_1, \dots, i_n}$ are diagonal, then

$$\begin{aligned} S \chi_t &= \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq j_1, \dots, j_m \leq q \\ j_m \neq i_n}} \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \chi_t \\ &= \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq j_1, \dots, j_m \leq q \\ j_m \neq i_n}} [S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \chi_t, \chi_t] \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} \chi_t \\ &= \sum_{n,m=0}^{\infty} \sum_{\substack{1 \leq i_1, \dots, i_n \leq q \\ i_n \neq i_m}} [S_{i_1, \dots, i_m}^{i_1, \dots, i_m} \chi_t, \chi_t] \chi_{t \bar{\gamma}^m \alpha_{i_n} \cdots \alpha_{i_1}}, \end{aligned}$$

where the indices \hat{i}_k are determined by $t\bar{\gamma}^k \alpha_{\hat{i}_k} = t\bar{\gamma}^{k-1}$. Since the sum above is taken over $i_n \neq \hat{i}_m$, all the summands are mutually orthogonal and (3.7) follows. Thus, under the restriction (3.6), the coefficients $S_{j_1, \dots, j_m}^{i_1, \dots, i_n}$ are determined uniquely. \square

Proposition 3.3. *The multiscale linear system (3.1) is causal if, and only if, the coefficients of the representation (3.5) for S satisfy*

$$(3.8) \quad S_{j_1, \dots, j_m}^{i_1, \dots, i_n} = 0, \text{ whenever } n < m.$$

Proof. According to Definition 3.1, the multiscale linear system (3.1) is causal if and only if

$$[S\chi_u, \chi_t] = 0, \text{ whenever } t \not\leq u.$$

Hence (3.8) follows immediately from (3.7). \square

Next we turn to the notion of stationarity. As in the classical case, this should mean translation-invariance. However, here the primitive downward shifts are one-to-one but not onto, while the primitive upward shift is onto, but not one-to-one. In particular, neither is a tree isometry (a tree isometry is a graph automorphism which preserves distances) and hence is not suitable for the role of a translation. Instead, we shall say that a tree isometry $\tau : \mathcal{T} \mapsto \mathcal{T}$ is a primitive translation if for every $t \in \mathcal{T}$

$$(3.9) \quad t\tau\bar{\gamma} \asymp t.$$

Let us analyze the structure of a primitive translation τ . First of all, we note that, since τ is a tree isometry,

$$\text{dist}(t\bar{\gamma}\tau, t\tau) = 1.$$

According to (3.9),

$$t\bar{\gamma}\tau\bar{\gamma} \asymp t\bar{\gamma} \preceq t \asymp t\tau\bar{\gamma}$$

and hence

$$t\bar{\gamma}\tau \preceq t\tau.$$

Therefore, by definition, $t\bar{\gamma}\tau = t\tau\bar{\gamma}$ and we conclude that τ commutes with the primitive upward shift:

$$(3.10) \quad \tau\bar{\gamma} = \bar{\gamma}\tau.$$

Furthermore, we observe that $\tau\bar{\gamma}$ must have a fixed point. Indeed, let $t \in \mathcal{T}$ and let

$$v = t \wedge t\tau = t\bar{\gamma}^n = t\tau\bar{\gamma}^{n+1}, \text{ where } n = \text{dist}(t \wedge t\tau, t).$$

Then, by (3.10),

$$v\tau\bar{\gamma} = t\bar{\gamma}^n \tau\bar{\gamma} = t\tau\bar{\gamma}^{n+1} = v.$$

Thus we obtain a unique sequence of nodes

$$v_j = v\tau^j, \quad j \in \mathbb{Z},$$

satisfying

$$v_j\bar{\gamma} = v_{j-1}, \quad v_j\tau = v_{j+1}.$$

It is called the skeleton of the primitive translation τ . Each node v_j of the skeleton corresponds to the non-homogeneous tree $\mathcal{T}_{v_j}^+$, which is the maximal connected subgraph of \mathcal{T} , satisfying

$$\mathcal{T}_{v_j}^+ \cap \{v_{j-1}, v_j, v_{j+1}\} = v_j.$$

It is mapped isometrically by τ onto $\mathcal{T}_{v_{j+1}}^+$.

The operator of convolution with a primitive translation is unitary on $\ell_2(\mathcal{T})$. By abuse of notation, we denote both the group of the tree isometries, generated by primitive translations, and the group of the corresponding convolution operators by $\mathbf{A}(\mathcal{T})$.

Definition 3.4. The multiscale linear system (3.1) is said to be stationary if S commutes with every $\tau \in \mathbf{A}(\mathcal{T})$.

Lemma 3.5. *The multiscale linear system (3.1) is stationary if and only if the value of the scalar product $[S\chi_u, \chi_t]$ depends only on $\text{dist}(t \wedge u, t)$ and $\text{dist}(t \wedge u, u)$.*

Proof. First of all, we note that, according to Definition 3.4, the multiscale linear system (3.1) is stationary if and only if for every pair of nodes $t, u \in \mathcal{T}$ and every $\tau \in \mathbf{A}(\mathcal{T})$

$$(3.11) \quad [S\chi_u, \chi_t] = [S\chi_{u\tau}, \chi_{t\tau}].$$

So let us assume that $[S\chi_u, \chi_t]$ depends only on $n = \text{dist}(t \wedge u, t)$ and $m = \text{dist}(t \wedge u, u)$ and let $\tau \in \mathbf{A}(\mathcal{T})$. Without loss of generality, we can also assume that τ is a primitive translation. Let us fix now a pair t, u , then it follows from (3.9) that

$$(t \wedge u)\tau = t\bar{\gamma}^n = u\bar{\gamma}^m.$$

Since

$$\text{dist}(t\tau, u\tau) = \text{dist}(t, u) = m + n,$$

we conclude that

$$(t \wedge u)\tau = (t\tau) \wedge (u\tau)$$

and, therefore, (3.11) holds true.

Conversely, let us assume that the multiscale linear system (3.1) is stationary. Then, in view of (3.11), it suffices to prove that for any two pairs of nodes t, u and t', u' , satisfying

$$\text{dist}(t \wedge u, t) = \text{dist}(t' \wedge u', t') = n, \quad \text{dist}(t \wedge u, u) = \text{dist}(t' \wedge u', u') = m,$$

there exists $\tau \in \mathbf{A}(\mathcal{T})$, such that

$$t\tau = t', \quad u\tau = u'.$$

Such an isometry can be constructed as follows. Denote

$$\text{dist}((t \wedge u) \wedge (t' \wedge u'), t \wedge u) = k, \quad \text{dist}((t \wedge u) \wedge (t' \wedge u'), t' \wedge u') = p.$$

In the case $t = u$ we choose arbitrary primitive translations τ_1 and τ_2 , such that

$$t\tau_1\bar{\gamma} = t, \quad t'\tau_2\bar{\gamma} = t',$$

and set

$$\tau = \tau_1^{-k}\tau_2^p.$$

In the case $t \neq u$ we assume, without loss of generality, that $n \neq 0$ and choose τ_1 as above. Then

$$t\tau_1^{-n-k} = t\bar{\gamma}^{-n-k} = (t \wedge u) \wedge (t' \wedge u') = t'\bar{\gamma}^{-n-p}.$$

Now let t' belong to the skeleton of a primitive translation τ_2 . Then

$$\begin{aligned} u\tau_1^{-n-k} &\in \mathcal{T}_{t'\tau_2^{-2n-p}}^+, \quad u' \in \mathcal{T}_{t'\tau_2^{-n}}^+, \\ \text{dist}(u\tau_1^{-n-k}, t'\tau_2^{-2n-p}) &= \text{dist}(u', t'\tau_2^{-n}) = m. \end{aligned}$$

Thus τ_2 can be chosen so that

$$u\tau_1^{-n-k}\tau_2^{n+p} = u',$$

and we set

$$\tau = \tau_1^{-n-k}\tau_2^{n+p}.$$

□

Remark 3.6. In view of the formula (3.7), Lemma 3.5 implies that the multiscale linear system (3.1) is stationary if, and only if, each coefficient $S_{j_1, \dots, j_m}^{i_1, \dots, i_n}$ in the series (3.5) has a constant (except for normalizing zeroes – see (3.6)) diagonal and, moreover, for $t \in \mathcal{T}\alpha_{j_1} \cdots \alpha_{j_m}$ the diagonal entry $[S_{j_1, \dots, j_m}^{i_1, \dots, i_n} \chi_t, \chi_t]$ depends only on n, m . The first condition is the multiscale analogue of the Toeplitz condition. Unlike the discrete time case, here it is weaker than the stationarity condition.

Theorem 3.7. *The multiscale linear system (3.1) is stationary if, and only if,*

$$S \in \overline{\text{span}}_{\mathbb{C}}\{\bar{\gamma}^n \gamma^m : n, m \in \mathbb{Z}_+\},$$

where the closure is taken in the point-wise sense. In this case, the system (3.1) is also causal if, and only if,

$$S \in \overline{\text{span}}_{\mathbb{C}}\{\bar{\gamma}^n \gamma^m : n \geq m\}.$$

Proof. Let us assume first that the multiscale linear system (3.1) is stationary. Then, since

$$\alpha_{j_m} \cdots \alpha_{j_1} = \alpha_{j_m} \cdots \alpha_{j_1} \pi_{j_1, \dots, j_m},$$

where π_{j_1, \dots, j_m} denotes the orthogonal projection onto the subspace of signals supported in $\mathcal{T}\alpha_{j_m} \cdots \alpha_{j_1}$, Remark 3.6 means that the representation (3.5) can be rewritten as

$$S = \sum_{n, m \in \mathbb{Z}^+} \sum_{\substack{1 \leq i_1, \dots, i_n \leq q \\ j_m \neq i_n}} \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} s_{n, m}, \quad s_{n, m} \in \mathbb{C}.$$

But the partial sums of this series belong to $\text{span}_{\mathbb{C}}\{\bar{\gamma}^n \gamma^m : n, m \in \mathbb{Z}_+\}$, since (3.4) leads to

$$\sum_{\substack{1 \leq i_1, \dots, i_n \leq q \\ j_m \neq i_n}} \bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_n} \alpha_{j_m} \cdots \alpha_{j_1} = q^{\frac{m+n}{2}} \bar{\gamma}^n \gamma^m - q^{\frac{m+n-2}{2}} \bar{\gamma}^{n-1} \gamma^{m-1}.$$

If the system (3.1) is also causal then, according to Proposition 3.3, we have $s_{n, m} = 0$ for $n < m$, hence

$$S \in \overline{\text{span}}_{\mathbb{C}}\{\bar{\gamma}^n \gamma^m : n \geq m\}.$$

In order to prove the converse statements, we note first that if S is of the form $S = \bar{\gamma}^n \gamma^m$ then, because of (3.10) and the fact that

$$\tau \in \mathbf{A}(\mathcal{T}) \implies \tau^* = \tau^{-1} \in \mathbf{A}(\mathcal{T}),$$

the system (3.1) is stationary. In the case $n \geq m$ it is also causal, as follows from Proposition 3.3. It only remains to observe that, in view of our Definitions 3.1, 3.4 and the fact that each operator $\tau \in \mathbf{A}(\mathcal{T})$ is a convolution operator, the properties of causality and stationarity are preserved when passing to the point-wise limit. \square

4. STATIONARY MULTISCALE SYSTEMS AND NON-STATIONARY DISCRETE TIME SYSTEMS

The main goal of the present work is to investigate the multiscale systems of the form (3.1) which are both causal and stationary. We denote the Banach algebra of corresponding operators S by $\mathbf{U}(\mathcal{T})$. According to Theorem 3.7,

$$(4.1) \quad \mathbf{U}(\mathcal{T}) = \overline{\text{span}_{\mathbb{C}}\{\bar{\gamma}^n \sigma_m : n, m \in \mathbb{Z}_+\}},$$

where the closure is taken in the point-wise sense and

$$(4.2) \quad \sigma_m = \bar{\gamma}^m \gamma^m, \quad m \in \mathbb{Z}_+.$$

We note that the self-adjoint operator $\sigma_m \in \mathbf{U}(\mathcal{T})$, defined by (4.2), is just the average

$$(4.3) \quad \sigma_m f(t) = \frac{1}{q^m} \sum_{\substack{u \asymp t \\ \text{dist}(u,t) \leq 2m}} f(u)$$

and hence acts on the subspace of functions, supported in a horocycle. Furthermore, since $\bar{\gamma}$ is an isometry,

$$(4.4) \quad \sigma_m \sigma_n = \sigma_{m \vee n},$$

where $m \vee n$ denotes the maximal of two integers m and n . Hence it is more convenient to consider

$$(4.5) \quad \omega_m = \sigma_m - \sigma_{m+1}, \quad m \in \mathbb{Z}_+.$$

Theorem 4.1. *The space $\ell_2(\mathcal{T})$ admits the orthogonal decomposition*

$$(4.6) \quad \ell_2(\mathcal{T}) = \bigoplus_{m=0}^{\infty} \mathcal{W}_m,$$

where the orthogonal projections $\omega_m : \ell_2(\mathcal{T}) \mapsto \mathcal{W}_m$ are given by (4.5). Each subspace \mathcal{W}_m is mapped isometrically onto \mathcal{W}_{m+1} by the upward shift operator $\bar{\gamma}$.

Proof. It follows immediately from (4.2) (4.4) and (4.5) that

$$\omega_m^* = \omega_m, \quad \omega_m \omega_n = \delta_{m,n} \omega_n.$$

In order to show that $\sum_{m=0}^{\infty} \omega_m$ converges in the strong operator topology to the identity operator I , it suffices to note that

$$\sum_{m=0}^n \omega_m = I - \sigma_{n+1}$$

and that, in view of (4.3), the sequence σ_n converges in the strong operator topology to 0.

Finally, we have to show that

$$\bar{\gamma}\mathcal{W}_m = \mathcal{W}_{m+1}, \quad m \in \mathbb{Z}_+.$$

But it follows from (3.4), (4.2) and (4.5) that

$$(4.7) \quad \omega_{m+1} = \bar{\gamma}\omega_m\gamma \text{ and } \bar{\gamma}\omega_m = \omega_{m+1}\bar{\gamma}.$$

□

It follows from Theorem 4.1 that an operator $S \in \mathbf{X}(\mathcal{T})$ can be viewed as an operator matrix with blocks corresponding to the orthogonal decomposition (4.6) and thus the corresponding multiscale linear system (3.1) can be treated as a (non-stationary, in general) discrete time system (see [2] and Section 2). Let us investigate the causal stationary multiscale linear systems from this point of view.

Proposition 4.2. *Let $S \in \mathbf{X}(\mathcal{T})$. Then $S \in \mathbf{U}(\mathcal{T})$ if, and only if,*

$$(4.8) \quad \omega_m S \omega_n = \begin{cases} \bar{\gamma}^{m-n} \omega_n s_{m,n}, & m \geq n, \\ 0, & \text{otherwise,} \end{cases}$$

where $s_{m,n} \in \mathbb{C}$.

Proof. Let us assume that $S \in \mathbf{U}(\mathcal{T})$. Then, by (4.1), there exists a sequence $S_k \in \text{span}_{\mathbb{C}}\{\bar{\gamma}^j \sigma_i : i, j \in \mathbb{Z}_+\}$ which converges to S point-wise. (4.7) implies that

$$\omega_m S_k \omega_n = \begin{cases} \bar{\gamma}^{m-n} \omega_n s_{m,n,k}, & m \geq n, \\ 0, & \text{otherwise,} \end{cases}$$

where $s_{m,n,k} \in \mathbb{C}$. Since for every $m \in \mathbb{Z}_+$ and $t \in \mathcal{T}$ $\omega_m \chi_t$ has a finite support,

$$\lim_{k \rightarrow \infty} \omega_m S_k \omega_n = \omega_m S \omega_n \text{ point-wise.}$$

In particular, there exists $\lim_{k \rightarrow \infty} s_{m,n,k}$, which we can denote by $s_{m,n}$ and thus obtain (4.8).

Conversely, if (4.8) holds, then by Theorem 4.1 we can represent S as the strongly converging series

$$S = \sum_{m=0}^{\infty} \omega_m S = \sum_{m=0}^{\infty} \sum_{n=0}^m \omega_m S \omega_n.$$

Since the convergence in the strong operator topology implies the point-wise convergence, it suffices to apply Theorem 3.7 to complete the proof.

□

From Proposition 4.2 it follows that a multiscale system (3.1) is stationary and causal if, and only if, S is "triangular" with respect to the orthogonal decomposition (4.6) – that is, for every $n \in \mathbb{Z}_+$ the subspace of piece-wise constant functions $\oplus_{m \geq n} \mathcal{W}_m = \text{ran } \sigma_n$ is S -invariant¹ – and, moreover, the blocks of S are complex constants. However, the subspaces \mathcal{W}_m are infinite-dimensional and hence the only Hilbert-Schmidt element of $\mathbf{U}(\mathcal{T})$ is 0. Nevertheless, we can adapt the techniques,

¹In the language of nest algebras (see [5] [19]) S is in the nest algebra associated with the nest $\{\oplus_{m \geq n} \mathcal{W}_m : n \in \mathbb{Z}_+\}$.

developed for the non-stationary discrete time systems, to the present setting.

In order to formulate the appropriate analogue of Theorem 2.1, we consider the space of operators

$$\mathbb{K} = \left\{ \mathbf{c} = \sum_{m=0}^{\infty} c_m \omega_m : c_m \in \mathbb{C}, \sup_{m \in \mathbb{Z}_+} |c_m| < \infty, \right\}$$

where the convergence is in the strong operator topology. According to Theorem 4.1 and Proposition 4.2, \mathbb{K} is a subalgebra of $\mathbf{U}(\mathcal{T})$ and, moreover, a commutative \mathbb{C}^* -algebra², isometric to $\ell_\infty(\mathbb{Z}^+)$. For $\mathbf{c} \in \mathbb{K}$ we shall use the notation

$$\bar{\mathbf{c}} = \mathbf{c}^* = \sum_{m=0}^{\infty} \bar{c}_m \omega_m.$$

We also introduce the notion of the row-wise (with respect to the orthogonal decomposition (4.6)) convergence: we shall say that a sequence of $S_n \in \mathbf{X}(\mathcal{T})$ converges row-wise to $S \in \mathbf{X}(\mathcal{T})$ if for every $m \in \mathbb{Z}_+$ the sequence $\omega_m S_n$ converges to $\omega_m S$ in the operator norm.

Theorem 4.3. *Let $S \in \mathbf{X}(\mathcal{T})$. Then $S \in \mathbf{U}(\mathcal{T})$ if, and only if, it can be represented as a row-wise converging series*

$$(4.9) \quad S = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{s}_k, \quad \mathbf{s}_k \in \mathbb{K}.$$

In this case the operators \mathbf{s}_k are determined uniquely by

$$(4.10) \quad \omega_n \mathbf{s}_k = \gamma^k \omega_{n+k} S \omega_n, \quad n, k \in \mathbb{Z}_+,$$

and it holds that

$$(4.11) \quad \sum_{k=0}^{\infty} \bar{\mathbf{s}}_k \mathbf{s}_k \omega_n = \frac{q^{n+1}}{q-1} \|S \omega_n \chi_t\|^2 \omega_n, \quad t \in \mathcal{T}, n \in \mathbb{Z}^+.$$

Proof. First, let us assume that there exists a sequence $\mathbf{s}_k \in \mathbb{K}$ such that the series $\sum_k \bar{\gamma}^k \mathbf{s}_k$ converges row-wise to an operator $S \in \mathbf{X}(\mathcal{T})$. Then for any $m, n \in \mathbb{Z}_+$

$$\sum_{k=0}^{\infty} \omega_m \bar{\gamma}^k \mathbf{s}_k \omega_n = \omega_m S \omega_n.$$

Then, since

$$\omega_m \bar{\gamma}^k \mathbf{s}_k \omega_n = \delta_{k, m-n} \bar{\gamma}^k \mathbf{s}_k \omega_n,$$

$S \in \mathbf{U}(\mathcal{T})$ by Proposition 4.2. Moreover,

$$\bar{\gamma}^k \mathbf{s}_k \omega_n = \omega_{n+k} S \omega_n, \quad n, k \in \mathbb{Z}_+,$$

hence (4.10) holds true.

Conversely, let us assume that $S \in \mathbf{U}(\mathcal{T})$ and let $m \in \mathbb{Z}_+$. Then, according to Proposition 4.2, we can define $\mathbf{s}_k \in \mathbb{K}$ by (4.10) and observe that

$$\omega_m S = \sum_{n=0}^m \omega_m S \omega_n = \sum_{n=0}^m \omega_m \bar{\gamma}^{m-n} \mathbf{s}_{m-n} = \sum_{n=0}^m \omega_m \bar{\gamma}^n \mathbf{s}_n = \sum_{n=0}^{\infty} \omega_m \bar{\gamma}^n \mathbf{s}_n.$$

²For background on \mathbb{C}^* -algebras we refer the reader to [18].

Finally,

$$\begin{aligned} \|S\omega_n\chi_t\|^2\omega_n &= \sum_{m=n}^{\infty} \|\omega_m S\omega_n\chi_t\|^2\omega_n = \sum_{m=n}^{\infty} \|\bar{\gamma}^{m-n}\mathbf{s}_{m-n}\omega_n\chi_t\|^2\omega_n \\ &= \sum_{m=0}^{\infty} \|\mathbf{s}_m\omega_n\chi_t\|^2\omega_n = \|\omega_n\chi_t\|^2 \sum_{k=0}^{\infty} \bar{\mathbf{s}}_k\mathbf{s}_k\omega_n = \left(\frac{1}{q^n} - \frac{1}{q^{n+1}}\right) \sum_{k=0}^{\infty} \bar{\mathbf{s}}_k\mathbf{s}_k\omega_n, \end{aligned}$$

and we obtain (4.11). \square

Following the analogy with the non-stationary setting, we consider the following ideal of \mathbb{K} :

$$\mathbb{K}_2 = \{\mathbf{c} \in \mathbb{K} : \sum_{m=0}^{\infty} |c_m|^2 < \infty\}.$$

It is a Hilbert space, isometric to $\ell_2(\mathbb{Z}^+)$. We also consider the \mathbb{K} -module³:

$$\mathbf{H}_2(\mathcal{T}) = \{S = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{s}_k : \mathbf{s}_k \in \mathbb{K}_2, \sum_{k=0}^{\infty} \|\mathbf{s}_k\|_2^2 < \infty\}.$$

At this point we consider the power series in the definition above as formal. However, we shall see later on (Proposition 4.5) that such a series converges in the operator norm. This is the analogue of the space of Hilbert-Schmidt operators in the present setting.

Proposition 4.4. *The \mathbb{K} -module $\mathbf{H}_2(\mathcal{T})$, considered as a vector space over \mathbb{C} with the scalar product*

$$(4.12) \quad [F, G] = \sum_{k=0}^{\infty} [\mathbf{f}_k, \mathbf{g}_k],$$

is a Hilbert space.

Proof. The proof is the same as in the case of Hilbert-Schmidt operators, hence we shall give only an outline. Since the Cauchy-Schwarz inequality holds in \mathbb{K}_2 , the inner product (4.12) is well-defined in the whole of $\mathbf{H}_2(\mathcal{T})$, which is, therefore, a pre-Hilbert space. Hence, the Cauchy-Schwarz inequality holds in $\mathbf{H}_2(\mathcal{T})$ as well, and

$$\|F\|_2 = \sqrt{[F, F]}$$

is a well-defined norm. The completeness of $\mathbf{H}_2(\mathcal{T})$ with respect to this norm can now be proved, using the triangle inequality. \square

Proposition 4.5. *The following hold:*

- (1) *The Hilbert space $\mathbf{H}_2(\mathcal{T})$ is contractively contained in $\mathbf{U}(\mathcal{T})$.*
- (2) *Let $S \in \mathbf{U}(\mathcal{T})$ and $t \in \mathcal{T}$. Then $S \in \mathbf{H}_2(\mathcal{T})$ if, and only if,*

$$\sum_{k=0}^{\infty} q^k \|S\omega_k\chi_t\|^2 < \infty.$$

In this case the expression above is equal to $(1 - \frac{1}{q})\|S\|_2^2$.

³For background on modules over a \mathbb{C}^* -algebra see [20], [21], [23].

Proof.

- (1) In view of completeness of $\mathbf{H}_2(\mathcal{T})$, $\mathbf{U}(\mathcal{T})$ and $\ell_2(\mathcal{T})$, it suffices to consider $S \in \mathbf{H}_2(\mathcal{T})$ such that the coefficients $\mathbf{s}_k \in \mathbb{K}_2$ are different from zero only for a finite number of indices k . Then $S \in \mathbf{U}(\mathcal{T})$ and we have

$$\begin{aligned} \|Sf\|^2 &= \sum_{n=0}^{\infty} \|\omega_n Sf\|^2 \leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \|\mathbf{s}_k \omega_{n-k} f\| \right)^2 \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \|\mathbf{s}_k \omega_{n-k}\|^2 \right) \left(\sum_{m=0}^n \|\omega_{n-m} f\|^2 \right) \\ &\leq \sum_{m=0}^{\infty} \|\omega_m f\|^2 \sum_{n,k=0}^{\infty} \|\mathbf{s}_k \omega_n\|^2 = \|f\|^2 \|S\|_2^2. \end{aligned}$$

- (2) It follows from (4.11) that

$$\|S\|_2^2 = \sum_{k=0}^{\infty} \|\mathbf{s}_k\|_2^2 = \frac{q}{q-1} \sum_{k=0}^{\infty} q^k \|S\omega_k \chi_t\|^2,$$

whenever either right-hand side or left-hand side is finite. \square

Remark 4.6. A consequence of Proposition 4.5 is that $\mathbf{H}_2(\mathcal{T})$ is a left ideal in $\mathbf{U}(\mathcal{T})$ and, moreover, for any $S \in \mathbf{U}(\mathcal{T})$ and $F \in \mathbf{H}_2(\mathcal{T})$ the inequality

$$(4.13) \quad \|SF\|_2 \leq \|S\| \|F\|_2$$

holds true.

5. POINT EVALUATION

In this section we exploit the analogy with the non-stationary setting to associate the elements of $\mathbf{U}(\mathcal{T})$ with maps from \mathbb{K} into itself.

Recalling the identity (4.7) and observing that $\omega_0 \bar{\gamma} = 0$, we conclude that for every $\mathbf{c} \in \mathbb{K}$ there exists an element

$$\mathbf{c}^{(1)} = \sum_{n=0}^{\infty} \omega_n c_{n+1} \in \mathbb{K}$$

such that

$$\mathbf{c} \bar{\gamma} = \bar{\gamma} \mathbf{c}^{(1)}.$$

We note that $\|\mathbf{c}^{(1)}\| \leq \|\mathbf{c}\|$ and $(\mathbf{c} \mathbf{d})^{(1)} = \mathbf{c}^{(1)} \mathbf{d}^{(1)}$. Furthermore, we introduce the following notation:

$$\begin{aligned} \mathbf{c}^{(0)} &= \mathbf{c}, \quad \mathbf{c}^{(n+1)} = \left(\mathbf{c}^{(n)} \right)^{(1)}, \\ \mathbf{c}^{[0]} &= 1, \quad \mathbf{c}^{[n+1]} = \mathbf{c}^{[n]} \mathbf{c}^{(n)}, \\ \rho(\mathbf{c}) &= \limsup_{n \rightarrow \infty} \|\mathbf{c}^{[n]}\|^{\frac{1}{n}}, \\ \mathbb{D}(\mathcal{T}) &= \{\mathbf{c} \in \mathbb{K} : \rho(\mathbf{c}) < 1\}. \end{aligned}$$

The set $\mathbb{D}(\mathcal{T})$ is the counterpart of the open unit disk in the present setting.

Definition 5.1. Let $S \in \mathbf{U}(\mathcal{T})$ be given. For $\mathbf{c} \in \mathbb{D}(\mathcal{T})$ we define the point evaluation of S at \mathbf{c} by

$$(5.1) \quad S(\mathbf{c}) = \sum_{n=0}^{\infty} \mathbf{c}^{[n]} \mathbf{s}_n.$$

We note that (5.1) is the analogue of (2.2). We claim that the point evaluation (5.1) is well-defined. Indeed, the convergence of the series (5.1) in \mathbb{K} follows from Theorem 4.3. Also, if $S \in \mathbf{U}(\mathcal{T})$ is such that for every $\mathbf{c} \in \mathbb{D}(\mathcal{T})$ $S(\mathbf{c}) = 0$, then, in particular,

$$S(0) = S(\omega_k) = S(\omega_k + \omega_{k+1}) = \dots = 0.$$

Hence

$$\mathbf{s}_0 = 0, \mathbf{s}_1 \omega_k = 0, \mathbf{s}_2 \omega_k = 0, \dots$$

and $\mathbf{s}_n = 0$ for $n = 0, 1, 2, \dots$. Furthermore, if $F \in \mathbf{H}_2(\mathcal{T})$ then for every $\mathbf{c} \in \mathbb{D}(\mathcal{T})$ $F(\mathbf{c}) \in \mathbb{K}_2$. We list several other properties of the point evaluation in the following

Lemma 5.2. *The following hold:*

- (1) *Let $F, G \in \mathbf{U}(\mathcal{T})$, $\mathbf{p}, \mathbf{q} \in \mathbb{K}$, $\mathbf{c} \in \mathbb{D}(\mathcal{T})$ and assume that $\mathbf{k} \in \mathbb{K}$ is invertible. Then*

$$(5.2) \quad (F\mathbf{p} + G\mathbf{q})(\mathbf{c}) = F(\mathbf{c})\mathbf{p} + G(\mathbf{c})\mathbf{q},$$

$$(5.3) \quad (FG)(\mathbf{c}) = (F(\mathbf{c})G)(\mathbf{c}),$$

$$(5.4) \quad (\bar{\gamma}^n F)(\mathbf{c}) = \mathbf{c}^{[n]} F(\mathbf{c}^{(n)}),$$

$$(5.5) \quad (\mathbf{k}F)(\mathbf{c}) = F(\mathbf{k}^{(1)} \mathbf{k}^{-1} \mathbf{c}) \mathbf{k}.$$

- (2) *Let $F \in \mathbf{H}_2(\mathcal{T})$, $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. Then*

$$(5.6) \quad F - F(\mathbf{c}) = (\bar{\gamma} - \mathbf{c})G,$$

where $G \in \mathbf{H}_2(\mathcal{T})$ is given by

$$(5.7) \quad G = \sum_{n,k=0}^{\infty} \bar{\gamma}^n \left(\mathbf{c}^{(n+1)} \right)^{[k]} \mathbf{f}_{n+k+1}.$$

Proof.

- (1) The relation (5.2) follows immediately from Definition 5.1. Furthermore,

$$\begin{aligned} (\bar{\gamma}^n F)(\mathbf{c}) &= \left(\sum_{k=0}^{\infty} \bar{\gamma}^{n+k} \mathbf{f}_k \right) (\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{c}^{[n+k]} \mathbf{f}_k \\ &= \sum_{k=0}^{\infty} \mathbf{c}^{[k]} \left(\mathbf{c}^{[n]} \right)^{(k)} \mathbf{f}_k = \left(\sum_{k=0}^{\infty} \bar{\gamma}^k \left(\mathbf{c}^{[n]} \right)^{(k)} \mathbf{f}_k \right) (\mathbf{c}) \\ &= \left(\mathbf{c}^{[n]} \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k \right) (\mathbf{c}) = (\bar{\gamma}^n(\mathbf{c})F)(\mathbf{c}), \end{aligned}$$

and, in view of (5.2), we obtain (5.3). Analogously,

$$(\bar{\gamma}^n F)(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{c}^{[n+k]} \mathbf{f}_k = \sum_{k=0}^{\infty} \left(\mathbf{c}^{(n)} \right)^{[k]} \mathbf{f}_k \mathbf{c}^{[n]},$$

and we obtain (5.4). Finally, we note that

$$(\mathbf{k}\bar{\gamma}^n)(\mathbf{c}) = \mathbf{c}^{[n]}\mathbf{k}^{(n)} = \left(\mathbf{c}\mathbf{k}^{(1)}\mathbf{k}^{-1}\right)^{[n]}\mathbf{k},$$

and (5.5) follows from (5.2).

(2) First, we have to check that the series (5.7) belongs to $\mathbf{H}_2(\mathcal{T})$. But

$$\begin{aligned} \sum_{n=0}^{\infty} \left\| \sum_{k=0}^{\infty} \left(\mathbf{c}^{(n+1)}\right)^{[k]} \mathbf{f}_{n+k+1} \right\|_2^2 &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \|\mathbf{c}^{[k]}\| \|\mathbf{f}_{n+k+1}\|_2 \right)^2 \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \|\mathbf{c}^{[k]}\| \right) \left(\sum_{m=0}^{\infty} \|\mathbf{c}^{[m]}\| \|\mathbf{f}_{n+m+1}\|_2^2 \right) \\ &\leq \left(\sum_{k=0}^{\infty} \|\mathbf{c}^{[k]}\| \right)^2 \|F\|_2^2 < \infty, \end{aligned}$$

since $\rho(\mathbf{c}) < 1$. Hence, indeed, $G \in \mathbf{H}_2(\mathcal{T})$.

Now we shall prove (5.6). Without loss of generality, we assume that $F = \bar{\gamma}^m \mathbf{f}$, where $\mathbf{f} \in \mathbb{K}_2$. Then (5.7) means

$$G = \sum_{n=1}^m \bar{\gamma}^{n-1} \left(\mathbf{c}^{(n)}\right)^{[m-n]} \mathbf{f}.$$

In particular, for $m = 1$ we have $G = \mathbf{f}$ and (5.6) holds. For $m \geq 2$ we have

$$(\bar{\gamma}^m - \mathbf{c}^{[m]})\mathbf{f} = (\bar{\gamma} - \mathbf{c})\bar{\gamma}^{m-1}\mathbf{f} + (\bar{\gamma}^{m-1} - \mathbf{c}^{[m-1]})\mathbf{f}\mathbf{c}^{(m-1)}.$$

Applying induction on m , we obtain

$$\begin{aligned} (\bar{\gamma}^m - \mathbf{c}^{[m]})\mathbf{f} &= (\bar{\gamma} - \mathbf{c}) \left(\bar{\gamma}^{m-1} + \sum_{n=1}^{m-1} \bar{\gamma}^{n-1} \left(\mathbf{c}^{(n)}\right)^{[m-n-1]} \mathbf{c}^{(m-1)} \right) \mathbf{f} \\ &= (\bar{\gamma} - \mathbf{c}) \left(\bar{\gamma}^{m-1} + \sum_{n=1}^{m-1} \bar{\gamma}^{n-1} \left(\mathbf{c}^{(n)}\right)^{[m-n]} \right) \mathbf{f} = (\bar{\gamma} - \mathbf{c})G. \end{aligned}$$

□

A consequence of Lemma 5.2 is that $F \in \mathbf{H}_2(\mathcal{T})$ satisfies $F(\mathbf{c}) = 0$ if and only if F is of the form (5.6), where $G \in \mathbf{H}_2(\mathcal{T})$.

Finally, we present an analogue of Cauchy's formula (and of formula (2.3)) for the space $\mathbf{H}_2(\mathcal{T})$.

Theorem 5.3. *Let $F \in \mathbf{H}_2(\mathcal{T})$, $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. Then for every $\mathbf{k} \in \mathbb{K}_2$ it holds that*

$$(5.8) \quad [F(\mathbf{c}), \mathbf{k}] = [F, K_{\mathbf{c}}\mathbf{k}],$$

where $K_{\mathbf{c}} \in \mathbf{U}(\mathcal{T})$ is given by

$$(5.9) \quad K_{\mathbf{c}} = \sum_{n=0}^{\infty} \bar{\gamma}^n \bar{\mathbf{c}}^{[n]} = (1 - \bar{\gamma} \bar{\mathbf{c}})^{-1}.$$

Proof. Since $\mathbf{c} \in \mathbb{D}(\mathcal{T})$, there exists $\epsilon \in (0, 1)$, such that for n sufficiently large $\|\mathbf{c}\| \leq \epsilon^n$. It follows that the series (5.9) converges absolutely in $\mathbf{U}(\mathcal{T})$ and defines an operator $K_{\mathbf{c}}$, which satisfies

$$K_{\mathbf{c}}(1 - \overline{\gamma} \overline{\mathbf{c}}) = (1 - \overline{\gamma} \overline{\mathbf{c}})K_{\mathbf{c}} = 1.$$

The formula (5.8) follows immediately from (5.1). \square

6. SCHUR MULTIPLIERS

Let us recall that, in view of Proposition 4.5 and the subsequent remarks, for any $S \in \mathbf{U}(\mathcal{T})$ the multiplication operator $\mathcal{M}_S F = SF$ is a bounded linear operator from $\mathbf{H}_2(\mathcal{T})$ into itself.

Definition 6.1. $S \in \mathbf{U}(\mathcal{T})$ is called a Schur multiplier if the multiplication operator \mathcal{M}_S is a contraction in $\mathbf{H}_2(\mathcal{T})$.

Theorem 6.2. An element $S \in \mathbf{U}(\mathcal{T})$ is a Schur multiplier if and only if the map $\mathcal{K}_S : \mathbb{D}(\mathcal{T}) \times \mathbb{D}(\mathcal{T}) \mapsto \mathbb{K}$, defined by

$$(6.1) \quad \mathcal{K}_S(\mathbf{c}, \mathbf{d}) = \sum_{n=0}^{\infty} \mathbf{c}^{[n]} \left(1 - S(\mathbf{c})\overline{S(\mathbf{d})}\right)^{(n)} \overline{\mathbf{d}}^{[n]}$$

is positive: for any $m \in \mathbb{Z}^+$, $\mathbf{c}_0, \dots, \mathbf{c}_m \in \mathbb{D}(\mathcal{T})$, $\mathbf{k}_0, \dots, \mathbf{k}_m \in \mathbb{K}_2$ it holds that

$$(6.2) \quad \sum_{i,j=0}^m [\mathcal{K}_S(\mathbf{c}_i, \mathbf{c}_j) \mathbf{k}_j, \mathbf{k}_i] \geq 0.$$

Proof. Let $S \in \mathbf{U}(\mathcal{T})$, $\mathbf{c} \in \mathbb{D}(\mathcal{T})$, $\mathbf{k} \in \mathbb{K}_2$. Then

$$\begin{aligned} \mathcal{M}_S^*(K_{\mathbf{c}}\mathbf{k}) &= \sum_{n,m=0}^{\infty} \overline{\gamma}^n \omega_m [\mathcal{M}_S^*(K_{\mathbf{c}}\mathbf{k}), \overline{\gamma}^n \omega_m] = \sum_{n,m=0}^{\infty} \overline{\gamma}^n [K_{\mathbf{c}}\mathbf{k}, S\overline{\gamma}^n \omega_m] \omega_m \\ &= \sum_{n,m=0}^{\infty} \overline{\gamma}^n [\mathbf{k}, (S\overline{\gamma}^n \omega_m)(\mathbf{c})] \omega_m = \sum_{n,m=0}^{\infty} \overline{\gamma}^n [\overline{S(\mathbf{c})}^{(n)} \overline{\mathbf{c}}^{[n]} \mathbf{k}, \omega_m] \omega_m \\ &= \sum_{n=0}^{\infty} \overline{\gamma}^n \overline{S(\mathbf{c})}^{(n)} \overline{\mathbf{c}}^{[n]} \mathbf{k}. \end{aligned}$$

It follows that

$$\begin{aligned} &[K_{\mathbf{c}_2}\mathbf{k}_2, K_{\mathbf{c}_1}\mathbf{k}_1] - [\mathcal{M}_S^*(K_{\mathbf{c}_2}\mathbf{k}_2), \mathcal{M}_S^*(K_{\mathbf{c}_1}\mathbf{k}_1)] \\ &= [K_{\mathbf{c}_2}(\mathbf{c}_1)\mathbf{k}_2, \mathbf{k}_1] - \sum_{n=0}^{\infty} [\overline{S(\mathbf{c}_2)}^{(n)} \overline{\mathbf{c}_2}^{[n]} \mathbf{k}_2, \overline{S(\mathbf{c}_1)}^{(n)} \overline{\mathbf{c}_1}^{[n]} \mathbf{k}_1] \\ &= \sum_{n=0}^{\infty} [\mathbf{c}_1^{[n]} \left(1 - S(\mathbf{c}_1)^{(n)} \overline{S(\mathbf{c}_2)}^{(n)}\right) \overline{\mathbf{c}_2}^{[n]} \mathbf{k}_2, \mathbf{k}_1] = [\mathcal{K}_S(\mathbf{c}_1, \mathbf{c}_2)\mathbf{k}_2, \mathbf{k}_1], \end{aligned}$$

where \mathcal{K}_S is given by (6.1). Since elements of the form

$$F = \sum_{\ell=0}^m K_{\mathbf{c}_\ell} \mathbf{k}_\ell$$

are dense in $\mathbf{H}_2(\mathcal{T})$, S is a Schur multiplier if and only if for any such element F it holds that

$$[F, F] - [\mathcal{M}_S^* F, \mathcal{M}_S^* F] \geq 0,$$

which, in view of the computations above, is equivalent to the positivity condition (6.2). \square

Below, following the analysis of [2, p. 86–90], we give an example of a Schur multiplier. Let $\mathbf{a} \in \mathbb{D}(\mathcal{T})$. Then $K_{\mathbf{a}}(\mathbf{a}) \geq 0$ (in the sense of \mathbb{K}) and, since

$$K_{\mathbf{a}}(\mathbf{a}) = 1 + \bar{\mathbf{a}}\mathbf{a}K_{\mathbf{a}}(\mathbf{a})^{(1)},$$

it is also invertible. Hence the element

$$L_{\mathbf{a}} = K_{\mathbf{a}}(\mathbf{a})^{(1)}K_{\mathbf{a}}(\mathbf{a})^{-1}$$

is also positive and invertible. Since

$$L_{\mathbf{a}}^{[k]} = K_{\mathbf{a}}(\mathbf{a})^{(k)}K_{\mathbf{a}}(\mathbf{a})^{-1},$$

we have $\rho(L_{\mathbf{a}}) \leq 1$.

Definition 6.3. The operator

$$B_{\mathbf{a}} = (\bar{\gamma} - \mathbf{a})(1 - L_{\mathbf{a}}\bar{\mathbf{a}}\bar{\gamma})^{-1}\sqrt{L_{\mathbf{a}}} \in \mathbf{U}(\mathcal{T})$$

is called the Blaschke factor, corresponding to \mathbf{a} .

Proposition 6.4. *The multiplication operator $\mathcal{M}_{B_{\mathbf{a}}}$ is an isometry in $\mathbf{H}_2(\mathcal{T})$.*

Proof. First of all, we note that, according to (4.12), $\mathcal{M}_{\bar{\gamma}}$ is an isometry in $\mathbf{H}_2(\mathcal{T})$, and that

$$L_{\mathbf{a}}^{(m)} = L_{\mathbf{a}^{(m)}}, \quad B_{\mathbf{a}}\bar{\gamma}^m = \bar{\gamma}^m B_{\mathbf{a}^{(m)}}.$$

Hence it is enough to show that for any $m \in \mathbb{Z}^+$ and $\mathbf{p}, \mathbf{q} \in \mathbb{K}_2$

$$[B_{\mathbf{a}}\bar{\gamma}^m \mathbf{p}, B_{\mathbf{a}} \mathbf{q}] = \delta_{m,0}[\mathbf{p}, \mathbf{q}].$$

To check this, we rewrite $B_{\mathbf{a}}$ in the form

$$B_{\mathbf{a}} = \left(\bar{\gamma} K_{\mathbf{a}^{(1)}} K_{\mathbf{a}}(\mathbf{a})^{(1)-1} - \mathbf{a} \right) \sqrt{L_{\mathbf{a}}}.$$

Then for $m > 0$ we have

$$\begin{aligned} [B_{\mathbf{a}}\bar{\gamma}^m \mathbf{p}, B_{\mathbf{a}} \mathbf{q}] &= [B_{\mathbf{a}^{(1)}}\bar{\gamma}^{m-1} \mathbf{p}, K_{\mathbf{a}^{(1)}} K_{\mathbf{a}}(\mathbf{a})^{(1)-1} \sqrt{L_{\mathbf{a}}} \mathbf{q}] \\ &= [B_{\mathbf{a}^{(1)}}(\mathbf{a}^{(1)})^{(m-1)} \mathbf{a}^{(1)[m-1]} \mathbf{p}, K_{\mathbf{a}}(\mathbf{a})^{(1)-1} \sqrt{L_{\mathbf{a}}} \mathbf{q}] = 0. \end{aligned}$$

Analogously,

$$\begin{aligned} [B_{\mathbf{a}} \mathbf{p}, B_{\mathbf{a}} \mathbf{q}] &= [K_{\mathbf{a}^{(1)}} K_{\mathbf{a}}(\mathbf{a})^{(1)-1} \sqrt{L_{\mathbf{a}}} \mathbf{p}, K_{\mathbf{a}^{(1)}} K_{\mathbf{a}}(\mathbf{a})^{(1)-1} \sqrt{L_{\mathbf{a}}} \mathbf{q}] + [\bar{\mathbf{a}}\mathbf{a}L_{\mathbf{a}} \mathbf{p}, \mathbf{q}] \\ &= [(L_{\mathbf{a}} K_{\mathbf{a}}(\mathbf{a})^{(1)-1} + 1 - K_{\mathbf{a}}(\mathbf{a})^{-1}) \mathbf{p}, \mathbf{q}] = [\mathbf{p}, \mathbf{q}]. \end{aligned}$$

\square

As a corollary of Lemma 5.2 and Proposition 6.4 we obtain that an element $F \in \mathbf{H}_2(\mathcal{T})$ vanishes at the point $\mathbf{c} \in \mathbb{D}(\mathcal{T})$ if and only if it can be written as $F = B_{\mathbf{c}}G$ where $G \in \mathbf{H}_2(\mathcal{T})$ is such that $[G, G] = [F, F]$. More generally, one can consider the following homogeneous interpolation problem:

$$(6.3) \quad \text{Given } \mathbf{c}_1, \dots, \mathbf{c}_N \in \mathbb{D}(\mathcal{T}) \text{ find all } F \in \mathbf{H}_2(\mathcal{T}) \text{ such that} \\ F(\mathbf{c}_j) = 0, \quad j = 1, \dots, N.$$

We assume that it is possible to recursively define invertible \mathbf{k}_j by

$$\mathbf{k}_1 = 1, \quad \mathbf{k}_{j+1} = (B_{\mathbf{k}_1^{(1)}\mathbf{k}_1^{-1}\mathbf{c}_1} B_{\mathbf{k}_2^{(1)}\mathbf{k}_2^{-1}\mathbf{c}_2} \cdots B_{\mathbf{k}_j^{(1)}\mathbf{k}_j^{-1}\mathbf{c}_j}) (\mathbf{c}_{j+1}).$$

Then F is a solution of the interpolation problem (6.3) if and only if

$$(6.4) \quad F = (B_{\mathbf{k}_1^{(1)}\mathbf{k}_1^{-1}\mathbf{c}_1} B_{\mathbf{k}_2^{(1)}\mathbf{k}_2^{-1}\mathbf{c}_2} \cdots B_{\mathbf{k}_N^{(1)}\mathbf{k}_N^{-1}\mathbf{c}_N}) G,$$

where $G \in \mathbf{H}_2(\mathcal{T})$ satisfies $[G, G] = [F, F]$. Indeed, assume that

$$F = G_0 = \dots = B_{\mathbf{k}_1^{(1)}\mathbf{k}_1^{-1}\mathbf{c}_1} B_{\mathbf{k}_2^{(1)}\mathbf{k}_2^{-1}\mathbf{c}_2} \cdots B_{\mathbf{k}_n^{(1)}\mathbf{k}_n^{-1}\mathbf{c}_n} G_n,$$

where $G_n \in \mathbf{H}_2(\mathcal{T})$. Then

$$0 = f(\mathbf{c}_{n+1}) = (\mathbf{k}_{n+1} G_n)(\mathbf{c}_{n+1}) = G_n(\mathbf{k}_{n+1}^{(1)} \mathbf{k}_{n+1}^{-1} \mathbf{c}_{n+1}),$$

and hence

$$G_n = B_{\mathbf{k}_{n+1}^{(1)}\mathbf{k}_{n+1}^{-1}\mathbf{c}_{n+1}} G_{n+1},$$

where $G_{n+1} \in \mathbf{H}_2(\mathcal{T})$, and (6.4) follows by induction.

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