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ON FREE STOCHASTIC PROCESSES AND THEIR DERIVATIVES

DANIEL ALPAY, PALLE JORGENSEN, AND GUY SALOMON

Abstract. We study a family of free stochastic processes whose covariance kernels K may be derived as a transform of a tempered measure σ. These processes arise, for example, in consideration non-commutative analysis involving free probability. Hence our use of semi-circle distributions, as opposed to Gaussians. In this setting we find an orthonormal bases in the corresponding non-commutative L^2 of sample-space. We define a stochastic integral for our family of free processes.

Contents

1. Introduction

A number of recent papers have advanced our understanding of Gaussian processes specified by general classes of covariance kernels of the

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form

$$K(t,s)=\int_{\mathbb{R}}\frac{e^{-iut}-1}{u}\frac{e^{ius}-1}{u}d\sigma(u), \tag{1.1}$$

where σ is a positive measure satisfying:

$$\int_{\mathbb{R}}\frac{d\sigma(u)}{u^2+1}<\infty, \tag{1.2}$$

For this family of Gaussian processes, typically having singular generating measures, we established in [2, 3, 5, 4] a versatile extension of Ito calculus to allow for measures not considered in the traditional family of Gaussian processes. We developed an Ito calculus and detailed factorizations for Gaussian processes whose covariance kernels K may be derived as a transform of a tempered measure σ (see (1.4) and (1.5) below). This class in turn includes fractional Brownian motion; so, in particular, processes whose time-increments are not independent (when the associated Hurst parameter is different from $\frac{1}{2}$). We stress that our family of processes allow a rich class when the generating measure σ for the covariance kernel is singular. In our earlier work on this, we introduce a new harmonic analysis which in turn is the basis for our proof of Ito representations for these processes. In the case when σ is a singular measure, these Ito representations go beyond what is known in earlier studies.

Our analysis of Gaussian processes associated to covariance kernels with singular measure σ is motivated in turn by a renewed interest in a harmonic analysis of Fourier decompositions in $\mathbf{L}_2(\sigma)$ for the case when σ is singular and arises from a scale of selfsimilarities; see for example [21, 10, 14, 23]. In order to obtain a more versatile harmonic analysis in the study factorizations, one is naturally led to consideration of independence, but for a host of problems [13, 22], rather than the traditional notion of independence, one needs a related but different notion, that of free independence. The latter arise, for example, in consideration of free products and free probability. In this context, we must therefore use semi-circle distributions, as opposed to Gaussians. As a result, the possibility for orthonormal bases in non-commutative L_2 of sample space entails entirely different algorithms. We resolve this problem in our Theorem 4.2 below. In the remaining part of our paper (sections 7-10), we extend part of the theory of stationary increment processes in the Gaussian case, to the free case of semi-circle free distributions.

In the present paper we construct free stochastic processes with covariance (1.1) and consider associated stochastic integrals. We also consider generalized stochastic processes indexed by the Schwartz space $\mathscr{S}$ of rapidly decreasing smooth functions and with covariance function

$$K(s_1,s_2)=\int_{\mathbb{R}}\widehat{s_1}(u)\overline{\widehat{s_2}(u)}du,\quad s_1,s_2\in\mathcal{S},$$

where now $d\sigma$ is subject to

$$\int_{\mathbb{R}}\frac{d\sigma(u)}{(u^2+1)^N}<\infty \tag{1.3}$$

for some $N\in\mathbb{N}_0$.

Since Fock spaces play an important role in the arguments we begin by setting some notation. Given a real Hilbert space $\mathcal{H}$, we denote the associated symmetric and full Fock spaces by $\Gamma_{\rm sym}(\mathcal{H})$ and $\Gamma(\mathcal{H})$ respectively. These spaces provide the setting for the white noise space and associated problems in the commutative and non-commutative setting respectively.

Recall that Gaussian stochastic processes indexed by the real numbers and with covariance functions of the form (1.1) play an important role in stochastic analysis. The kernel (1.1) can be rewritten as

$$K(t,s)=r(t)+\overline{r(s)}-r(t-s), \tag{1.4}$$

where

$$r(t)=-\int_{\mathbb{R}}\left(e^{-itu}-1-\frac{itu}{u^2+1}\right)\frac{d\sigma(u)}{u^2}. \tag{1.5}$$

The case $r(t)=|t|$ corresponds to the Brownian motion, and more generally $r(t)=|t|^{2H}$ (where $H\in(0,1)$)leads to the fractional Brownian motion. Such stochastic processes were constructed using Hida's white noise space setting in [2] for a family of absolutely continuous $d\sigma$ and in [5] for singular $d\sigma$'s. For the convenience of the reader and for purpose of comparison we will recall in the sequel the white noise space setting. We mention that the white noise space can be built using Minlos theorem or defined as the symmetric Fock space associated to the Lebesgue space $\mathbf{L}_2(\mathbb{R},dx)$. An important point in the white noise space approach is to view the white noise space $\mathcal{W}$ as part of a Gelfand triple

$$\mathcal{S}_1\subset\Gamma_{\rm sym}(\mathbf{L}_2(\mathbb{R},dx))\subset\mathcal{S}_{-1}, \tag{1.6}$$

where $\mathcal{S}_1$ is the Kondratiev space of stochastic test functions and $\mathcal{S}_{-1}$ is the Kondratiev space of stochastic distributions. The processes have (for appropriate classes of functions r) derivatives which belong to $\mathcal{S}_{-1}$. This fact, together with the algebra structure of $\mathcal{S}_{-1}$, allows to define stochastic integrals. See [3].

Let $\mathcal{H}$ be a separable real Hilbert space. Let $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ be a fixed one-dimensional Hilbert space and

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ tensor factors}}.$$

Then,

$$\Gamma(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} = \mathbb{C}\Omega + \mathcal{H} + \mathcal{H} \otimes \mathcal{H} + \cdots , \tag{1.7}$$

with norm

$$\| \sum_{n=0}^{\infty} f_n \|^2 = \sum_{n=0}^{\infty} \| f_n \|^2, \quad \text{where } \ f_n \in \mathcal{H}^{\otimes n}, \tag{1.8}$$

and

$$\Gamma_{\text{sym}}(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}_{\text{sym}}^{\otimes n},$$

where $\mathcal{H}_{\text{sym}}^{\otimes n}$ is the closed subspace in $\mathcal{H}^{\otimes n}$ consisting of all symmetric n-tensors. By general theory, see [18], there is a Gelfand triple

$$E \subset \Gamma_{\text{sym}}(\mathcal{H}) \subset E',$$

a sigma-algebra of subsets in E' (the sigma-algebra generated by cylinders) and a Gaussian measure P on E' such that for $h \in \mathcal{H}$, $\langle \cdot, h \rangle$ extends to a random variable $\widetilde{h}$ on E', with

$$\mathbb{E}(\widetilde{h}) = 0, \quad \text{and} \quad \mathbb{E}(\widetilde{h_1}\widetilde{h_2}) = \langle h_1, h_2 \rangle$$

for all $h_1, h_2 \in \mathcal{H}$, where $\mathbb{E}$ denotes mathematical expectation:

$$\mathbb{E}(F) = \int_{E'} F dP$$

for random variables F on E'.

It is our aim to develop a free stochastic calculus which parallels the above, but nonetheless has quite different features. As in the papers [11, 12] (where the free Brownian motion is defined) we replace the white noise space by the full Fock space associated to $\mathbf{L}_2(\mathbb{R}, dx)$. The new point in the present paper is to view this space (called the non commutative white noise space) as a part of a Gelfand triple analogous to (1.6), where $\mathcal{S}_1$ and $\mathcal{S}_{-1}$ are replaced by their non commutative versions $\widetilde{\mathcal{S}_1}$ and $\widetilde{\mathcal{S}_{-1}}$ respectively. See [8] and Section 5. This approach allows to define the derivatives of the free stochastic processes. More precisely, we build (for certain classes of $d\sigma$'s) a type II_1 von-Neumann algebra $\mathcal{M}_\sigma \subset \mathbf{L}(\Gamma_{\rm sym}(\mathbf{L}_2(\mathbb{R}, dx)))$ (with trace τ) such that

$$\tau(Z_\sigma^*(s)Z_\sigma(t)) = K(t,s)$$

where $Z_\sigma(t) \in \mathcal{M}_\sigma$. When $d\sigma$ is the Lebesgue measure, Z_σ is the non-commutative Brownian motion introduced in [11, 12].

An important result is that (still for certain classes of $d\sigma$'s) we can differentiate the function $t \mapsto Z_\sigma(t)$, and its values are continuous linear operators from $\widetilde{\mathcal{S}_1}$ to $\widetilde{\mathcal{S}_{-1}}$. In the case of the non-commutative Brownian motion, the derivative is the non-commutative counterpart of the white noise. The special structure of $\widetilde{\mathcal{S}_{-1}}$ (see inequality (5.1)) allows to define stochastic integrals in terms of limit of Riemann sums of $\widetilde{\mathcal{S}_{-1}}$-valued functions.

The paper consists of seven sections besides the introduction. Sections 2,3 and 5 are of a survey type. The new results appear in Sections 4, 6, 7, and 8. The commutative setting is briefly outlined in Section 2. Section 3 considers the non-commutative setting. We introduce there in particular a $\mathbf{L}_2$-space associated to a certain non-hyperfinite von Neumann algebra associated to a real Hilbert space. We give an orthonormal basis of this space in terms of the Tchebycheff polynomials of the second kind in Section 4. Section 5 surveys the recently developped theory of non commutative stochastic distributions. Non-commutative processes with correlation functions of the type (1.1) are constructed in Section 6. Their derivatives are considered in Section 7, as well as stochastic integration, and the case of general tempered spectral measures. The algebra $\widetilde{\mathcal{S}_{-1}}$ is an example in a family of similar algebras, all of them carrying an inequality of the form (5.1). The last section briefly discusses this general case.

2. Commutative white noise space

In the commutative setting, a realization of $\Gamma_{\rm sym}(\mathcal{H})$ can be given using the Bochner-Minlos theorem. Assume that $\mathcal{H}$ is infinite dimensional and separable, let $\xi_1, \xi_2, \ldots$ be an orthogonal basis of $\mathcal{H}$, and consider the Schwartz space $\mathscr{S}_{\mathcal{H}}$ of elements $\sum_{n=1}^\infty x_n\xi_n$ (where the $x_1, x_2, \ldots$ are real numbers) such that

$$\sum_{n=1}^\infty x_n^2 n^{2p} < \infty, \quad p = 0, 1, 2, \ldots$$

The space $\mathcal{S}_{\mathcal{H}}$ is nuclear and the Bochner-Minlos theorem (see for instance [20, Appendix A]) implies the existence of a Borel measure P on its strong dual such that

$$e^{-\frac{\|h\|^2}{2}} = \int_{\mathcal{S}'_{\mathcal{H}}} e^{i\langle h,w\rangle} dP(w), \quad h \in \mathcal{S}_{\mathcal{H}}.$$

It follows from this expression that the map which to $h \in \mathcal{S}_{\mathcal{H}}$ associates the following Gaussian random variable

$$Q_h(w) = w(h)$$

extends to an isometry (still denoted by Q_h) from $h \in \mathcal{H}$ into $Q_h \in \mathbf{L}_2(\mathcal{S}'_{\mathcal{H}}, dP)$, and we have

$$\langle Q_h, Q_k\rangle = \langle h, k\rangle \tag{2.1}$$

Before giving an orthogormal basis of $\mathbf{L}_2(\mathcal{S}'_{\mathcal{H}}, \mathcal{B}, P)$ we recall a definition.

Definition 2.1. *The Hermite polynomials $\{h_k\}_{k\in\mathbb{N}_0}$ are defined by*

$$h_k(u) \overset{\text{def.}}{=} (-1)^k e^{\frac{u^2}{2}} \frac{d^k}{du^k}(e^{-\frac{u^2}{2}}), \ \ k = 0, 1, 2 \ldots.$$

An orthogonal basis of $\mathbf{L}_2(\mathcal{S}'_{\mathcal{H}}, \mathcal{B}, P)$ is given by the functions

$$H_\alpha(w) = \prod_{k=1}^{\infty} h_{\alpha_k}(Q_{\xi_k}(w)). \tag{2.2}$$

In this expression, $\xi_1, \xi_2, \ldots,$ denote some pre-assigned orthonornal basis of $\mathcal{H}$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ belongs to the set ℓ of sequences of elements of $\mathbb{N}_0$ indexed by $\mathbb{N}$ and with all entries α_k are equal to 0 at the exception of at most a finite number of k's. We have

$$\mathcal{W} = \Gamma^\circ(\mathcal{H}) = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 \alpha! < \infty \right\} = \mathbf{L}^2(\ell, \nu).$$

The Wick product is defined by

$$H_\alpha \circ H_\beta = H_{\alpha+\beta}, \quad \alpha, \beta \in \ell,$$

and thus is a Cauchy product as in [15]. In terms of the basis, we obtain that

$$f \circ g = \left(\sum_{\alpha\in\ell} f_\alpha H_\alpha\right) \circ \left(\sum_{\alpha\in\ell} g_\alpha H_\alpha\right) = \sum_{\alpha\in\ell} \left(\sum_{\beta\leq\alpha} f_\beta g_{\alpha-\beta}\right) H_\alpha,$$

whenever it makes sense. The space $\mathcal{W}$ is not closed under the Wick product. This motivates the introduction of two spaces, the Kondratiev space $\mathcal{S}_1$ of stochastic test functions, and the Kondratiev space $\mathcal{S}_{-1}$ of stochastic distributions, which closed under the Wick product. These spaces are defined as

$$\mathcal{S}_1 = \left\{ \sum_{\alpha \in \ell} f_\alpha H_\alpha : \sum_{\alpha \in \ell} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} (\alpha!)^2 < \infty \text{ for all } p \in \mathbb{N} \right\},$$

where $(2\mathbb{N})^\alpha = 2^{\alpha_1} \cdot 4^{\alpha_2} \cdot 6^{\alpha_3} \cdots$, and $\mathcal{S}_{-1}$ is defined as:

$$\begin{aligned} \mathcal{S}_{-1} &= \left\{ \sum_{\alpha \in \ell} f_\alpha H_\alpha : \sum_{\alpha \in \ell} |f_\alpha|^2 (2\mathbb{N})^{-\alpha p} < \infty \text{ for some } p \in \mathbb{N} \right\} \\ &= \bigcup_p \mathbf{L}^2(\ell, \mu_{-p}), \end{aligned}$$

where μ_{-p} is the point measure defined by

$$\mu_{-p}(\alpha) = (2\mathbb{N})^{-\alpha p}.$$

Together with the white noise space these two spaces form the Gelfand triple $(\mathcal{S}_1, \mathcal{W}, \mathcal{S}_{-1})$, which plays a key role in the stochastic analysis in [20], and in the theory of stochastic linear systems and stochastic integration developped in [6, 2, 3, 5, 1]. The reason of the importance of this triple is the following result, see [20], which allows to work locally in a Hilbert space setting.

Theorem 2.2 (Våge, 1996)**.** *In the space* $\mathcal{S}_{-1} = \bigcup_p \mathbf{L}^2(\ell, \mu_{-p})$ *it holds that*

$$\|f \circ g\|_q \le A_{q-p} \|f\|_p \|g\|_q, \tag{2.3}$$

(where $\|\cdot\|_p$ *denotes the norm of* $\mathbf{L}^2(\ell, \mu_{-p})$*) for any* $q \ge p+2$*, and for any* $f \in \mathbf{L}_2(\ell, \mu_{-p}), g \in \mathbf{L}_2(\ell, \mu_{-q})$*, and where the number* A_{q-p} *is independent of* f *and* g *and is equal to*

$$A_{q-p} = \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{-\alpha(q-p)} \right)^{\frac{1}{2}}.$$

We refer to [20, p. 118] for a proof of the fact that $A_{q-p} < \infty$. The result is due to Våge; see [25]. See also [7] for a more general result.

3. The Fock space $\Gamma(\mathcal{H})$

This section is essentially of a review nature, and deals with the Fock space $\Gamma(\mathcal{H})$ associated to a real Hilbert space $\mathcal{H}$. For more information we refer in particular to [27, 26, 17]. The source [24] is also very didactic.

We will use the results for the case where $\mathbf{L}_2(d\sigma))$, where $d\sigma$ is a positive Borel measure σ on $\mathbb{R}$ such that (1.3) holds, and consider the full Fock space $\Gamma_\sigma = \Gamma(\mathbf{L}_2(d\sigma))$.

For $h \in \mathcal{H}$ we define ℓ_h to be the operator

$$\ell_h(f) = h \otimes f, \quad f \in \Gamma(\mathcal{H}),$$

f and $T_h = \ell_h + \ell_h^*$. We denote by $\mathcal{M}_\mathcal{H}$ the von Neumann algebra generated by the operators T_h, when h runs through $\mathcal{H}$. It is a II_1 type von Neumann algebra, and we denote by τ its trace. We have

$$\tau(f) = \langle \Omega, f\Omega\rangle_{\Gamma(\mathcal{H})}, \quad f \in \mathcal{M}_\mathcal{H}, \tag{3.1}$$

where Ω is the vacuum vector in (1.7), and, more generally

$$\tau(g^*f) = \langle \Omega, g^*f\Omega\rangle_{\Gamma_\sigma} = \langle g\Omega, f\Omega\rangle_{\Gamma_\sigma}, \quad f, g \in \mathcal{M}_\mathcal{H}, \tag{3.2}$$

where $''$ means double commutant.

Proposition 3.1. *It holds that*

$$\tau(T_h^*T_k) = \langle T_h\Omega, T_k\Omega\rangle = \langle h, k\rangle. \tag{3.3}$$

We note that (3.3) is the counterpart of (2.1).

Proof of Proposition 3.1. For $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in \mathcal{H}$

$$\begin{aligned}\ell_h^*\ell_k(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \ell_h^*(k \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_n)\\ &= \langle h, k\rangle h_1 \otimes h_2 \otimes \cdots \otimes h_n,\end{aligned}$$

and so

$$\ell_h^*\ell_k = \langle h, k\rangle I. \tag{3.4}$$

Thus,

$$\tau(T_h^*T_k) = \langle \ell_h(\Omega) + \ell_h^*(\Omega), \ell_k(\Omega) + \ell_k^*(\Omega)\rangle_{\Gamma(\mathcal{H})} = \langle \ell_h(\Omega), \ell_k(\Omega)\rangle_{\Gamma(\mathcal{H})}$$

since, by definition of the annihilation operator, we have

$$\ell_h^*(\Omega) = \ell_k^*(\Omega) = 0,$$

and where Ω is the vacuum vector, see (1.7), Hence

$$\tau(T_h^*T_k) = \langle \ell_h(\Omega), \ell_k(\Omega)\rangle_{\Gamma(\mathcal{H})} = \langle h, k\rangle$$

in view of (3.4). ☐

The following two propositions will be needed, and are well known; see [26, Theorem 2.6.2, pp.17-18].

Proposition 3.2. *Let* $h \in \Gamma(\mathcal{H})$ *of norm* 1. *Then* $2T_h$ *has as its distribution a semi-circle law* $C_{0,1}$.

Proposition 3.3. *Let* $\mathcal{H}_1, \mathcal{H}_2, \ldots$ *be pairwise orthogonal Hilbert subspaces of* $\mathcal{H}$. *Then, the family of algebras* $\mathcal{M}_{\mathcal{H}_1}, \mathcal{M}_{\mathcal{H}_2}, \ldots$ *is free.*

Proposition 3.4. *Let* Ω *be the empty state of* $\Gamma(\mathcal{H})$. *The map* $f \mapsto f\Omega$ *is one-to-one from the von Neumann algebra* $\mathcal{M}_{\mathcal{H}}$ *onto* $\Gamma(\mathcal{H})$.

Proof. Let $f \in \mathcal{M}_{\mathcal{H}}$ be such that $f\Omega = 0$. Then, $\langle f\Omega, f\Omega\rangle = 0$. But

$$\langle f\Omega, f\Omega\rangle = \tau(f^*f) = 0,$$

and so $f^*f = 0$ (since τ is faithful), and hence $f = 0$. □

Corollary 3.5. *There is a natural unitary isomorphism*

$$\mathbf{L}_2(\mathcal{M}_{\mathcal{H}}, \tau) \overset{W}{\longrightarrow} \Gamma(\mathcal{H}), \tag{3.5}$$

intertwining the respective actions, where $\mathcal{M}$ *is a copy of a non-hyperfinite* II_1 *factor, and where* τ *denotes the trace on* $\mathcal{M}$.

Proof. Using (1.8) one checks that for real valued continuous functions φ and ψ, and $h, k \in \mathcal{H}_{\mathbb{R}}$,

$$\langle \varphi(T_h)\Omega, \psi(T_k)\Omega\rangle = \langle \psi(T_k)\Omega, \varphi(T_h)\Omega\rangle.$$

As a result the state $\langle \Omega, \cdot\Omega\rangle$ extends to a faithful trace on the von-Neumann algebra $\mathcal{M}_{\mathcal{H}}$ generated by $\{T_h,\ h \in \mathcal{H}_{\mathbb{R}}\}$, i.e., $\mathcal{M}_{\mathcal{H}} = \{T_h,\ h \in \mathcal{H}_{\mathbb{R}}\}''$. Hence $\mathcal{M}_{\mathcal{H}}$ is a II_1-factor. It is known (see [26]) to be non-hyperfinite.

It then follows from the uniqueness in the GNS construction that W, defined by

$$W(X) = X\Omega, \quad X \in \mathcal{M}_{\mathcal{H}},$$

extends to an isometric isomorphism with the properties stated in the corollary. □

4. An orthonormal basis

The Tchebycheff polynomials of the second kind are an orthonormal basis of the space $\mathbf{L}_2([-1,1], \sqrt{1-x^2}dx)$. They are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with} \quad x = \cos\theta. \tag{4.1}$$

We have

$$\frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \delta_{mn}, \tag{4.2}$$

where δ_{mn} is Kronecker's symbol.

We now prove a presumably known result on these polynomials.

Lemma 4.1. *Assume $m \geq n$. Then the following linearization formula holds:*

$$U_m U_n = \sum_{k=0}^{n} U_{m-n+2k} \tag{4.3}$$

Proof. We assume $m \geq n$. We have:

$$\begin{aligned}
&\frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} \cdot \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \\
&= \frac{e^{i(m+n+2)\theta} - e^{i(m-n)\theta} + e^{-i(m+n+2)\theta} - e^{-i(m-n)\theta}}{(e^{i\theta} - e^{-i\theta})^2} \\
&= \frac{1}{(e^{i\theta} - e^{-i\theta})} \times \\
&\quad \times \left(\frac{e^{i(m-n)\theta}}{e^{-i\theta}} \cdot \left(\frac{e^{i(2n+2)\theta} - 1}{e^{2i\theta} - 1} \right) + \frac{e^{-i(m-n)\theta}}{e^{i\theta}} \cdot \left(\frac{e^{-i(2n+2)\theta} - 1}{1 - e^{-2i\theta}} \right) \right) \\
&= \frac{1}{(e^{i\theta} - e^{-i\theta})} \times \\
&\quad \times \left(e^{i(m-n+1)\theta} \left(1 + e^{2i\theta} + \cdots + (e^{2i\theta})^n \right) - \right. \\
&\quad \left. - e^{-i(m-n+1)\theta} \left(1 + e^{-2i\theta} + \cdots + (e^{-2i\theta})^n \right) \right) \\
&= \sum_{k=0}^{n} \frac{\sin(m-n+1+2k)\theta}{\sin\theta},
\end{aligned}$$

and hence the result.

□

We denote by $\mathbf{L}_2(\tau)$ the closure of $\mathcal{M}_{\mathcal{H}}$ with respect to τ. In Theorem 4.2 we present an orthonormal basis of $\mathbf{L}_2(\tau)$. In (4.5) in the statement the indices are as follows: The space $\widetilde{\ell}$ denotes the free monoid generated by $\mathbb{N}_0$. We write an element of $1 \neq \alpha \in \widetilde{\ell}$ as

$$\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_k}^{\alpha_k}, \tag{4.4}$$

where $\alpha_1, i_1, \ldots \in \mathbb{N}$ and $i_1, \ldots i_k \in \mathbb{N}_0$ are such that

$$i_1 \neq i_2 \neq \cdots \neq i_{k-1} \neq i_k.$$

Theorem 4.2. *Let $h_0, h_1, h_2, \ldots$ be any orthonormal basis of $\mathbf{L}_2(d\sigma)$. The functions*

$$U_\alpha = U_{\alpha_1}(T_{h_{i_1}}) \cdots U_{\alpha_k}(T_{h_{i_k}}), \tag{4.5}$$

where $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_k}^{\alpha_k} \in \widetilde{\ell}$ *form an orthonormal basis for* $\mathbf{L}_2(\tau)$.

Proof. Let $h_0, h_1, \ldots$ be an orthonormal basis of a real Hilbert space $\mathcal{H}_{\mathbb{R}}$. From [26, 27] we know that the family of non-commutative random variables $T_{h_0}, T_{h_1}, \ldots$ is free, i.e. for any choice of $i_1, i_2, \ldots$ in $\mathbb{N}_0$ such that $i_1 \neq i_2 \neq i_3 \neq \ldots$ and measurable functions $\psi_1, \psi_2, \ldots$ are fixed such that

$$\tau(\psi_j(T_{h_{i_j}})) = 0, \quad j = 1, \ldots, n,$$

it follows that

$$\tau(\psi_1(T_{h_{i_1}})\psi_2(T_{h_{i_2}}) \cdots \psi_n(T_{h_{i_n}})) = 0.$$

The functions $U_0, U_1, \ldots$ in (4.1) form an orthonormal basis (ONB) in $\mathbf{L}_2(d\mu)$, where $d\mu$ is the semi-circle law

$$d\mu(x) = \frac{2}{\pi} 1_{(-1,1)}(x)\sqrt{1-x^2}. \tag{4.6}$$

Hence we have

$$\tau(U_m(T_i)) = 0 \quad \text{for all} \quad m \in \mathbb{N} \text{ and } i \in \mathbb{N},$$
$$\tau(U_m(T_i)^2) = \int_{\mathbb{R}} U_m^2(x) d\mu(x) = 1, \quad \text{for all} \quad m, n \in \mathbb{N}_0,$$

We will be using these basic rules in our verification below of the orthonormality properties of the system (4.5) of non-commutative random variables. Consider

$$\beta = z_{j_1}^{\beta_1} z_{j_2}^{\beta_2} \cdots z_{j_m}^{\beta_m}.$$

We shall show by induction on $|\beta| = \beta_1 + \cdots + \beta_m$, that for every $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}$,

$$\tau(U_\beta^* U_\alpha) = \tau(U_{\beta_m}(T_{j_m}) \cdots U_{\beta_1}(T_{j_1}) U_{\alpha_1}(T_{i_1}) \cdots U_{\alpha_n}(T_{i_n})) = \delta_{\alpha,\beta}.$$

$|\beta| = 0$ implies $\beta = 1$. So,

$$\tau(U_\beta^* U_\alpha) = \tau(U_{\alpha_1}(T_{i_1}) \cdots U_{\alpha_n}(T_{i_n})),$$

which is zero, by freeness, for every $\alpha \neq 1$, and 1 for $\alpha = 1$.

Now assume that $|\beta| > 0$, and consider the following cases.

Case 1: $i_1 \neq j_1$. Then it follows from [26, Theorem 2.6.2, (iii)] that $\tau(U_\beta^* U_\alpha) = 0$.

Case 2: $i_1 = j_1$ and $\alpha_1 \neq \beta_1$. Without the loss of generality we may assume that $\beta_1 > \alpha_1$. By Lemma 4.1,

$$\begin{aligned}\tau(U_\beta^* U_\alpha) &= \sum_{k=0}^{\alpha_1} \tau(U_{\beta_m}(T_{j_m}) \cdots U_{\beta_2}(T_{j_2}) U_{\beta_1-\alpha_1+2k}(T_{i_1}) U_{\alpha_2}(T_{i_2}) \cdots U_{\alpha_n}(T_{i_n})) \\ &= \sum_{k=0}^{\alpha_1} \tau(U_{\beta'}^* U_{\alpha'_k}),\end{aligned}$$

where $\alpha'_k = z_{i_1}^{\beta_1-\alpha_1+2k} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}$ and $\beta' = z_{j_2}^{\beta_2} \cdots z_{j_m}^{\beta_m}$. Since for every $0 \leq k \leq \alpha_1$ we have $\alpha'_k \neq \beta_k$ (because $i_1 = j_1 \neq j_2$), by the induction assumption we obtain $\tau(U_\beta^* U_\alpha) = 0$.

Case 3: $i_1 = j_1$ and $\alpha_1 = \beta_1$. Then again by Lemma 4.1,

$$\begin{aligned}\tau(U_\beta^* U_\alpha) &= \sum_{k=0}^{\alpha_1} \tau(U_{\beta_m}(T_{j_m}) \cdots U_{\beta_2}(T_{j_2}) U_{2k}(T_{i_1}) U_{\alpha_2}(T_{i_2}) \cdots U_{\alpha_n}(T_{i_n})) \\ &= \sum_{k=0}^{\alpha_1} \tau(U_{\beta'}^* U_{\alpha'_k}),\end{aligned}$$

where $\alpha'_k = z_{i_1}^{2k} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}$ and $\beta' = z_{j_2}^{\beta_2} \cdots z_{j_m}^{\beta_m}$.
Since for every $0 < k \leq \alpha_1$ we have $\alpha'_k \neq \beta_k$ (because $i_1 = j_1 \neq j_2$), by the induction assumption we obtain

$$\tau(U_\beta^* U_\alpha) = \tau(U_{\beta'}^* U_{\alpha'_0}) = \tau(U_{\beta_m}(T_{j_m}) \cdots U_{\beta_2}(T_{j_2}) U_{\alpha_2}(T_{i_2}) \cdots U_{\alpha_n}(T_{i_n})) = \delta_{\alpha'_0,\beta'},$$

which is equal to $\delta_{\alpha,\beta}$, since we assume $i_1 = j_1$ and $\alpha_1 = \beta_1$.

Thus,

$$\tau(U_\beta^* U_\alpha) = \delta_{\alpha,\beta}.$$

The proof that the U_α form a complete set of functions relies on Corollary 3.5 as follows. Let $F \in \mathbf{L}_2(\tau)$ such that

$$\langle U_\alpha, F \rangle_\tau = 0, \quad \forall\ \alpha$$

On account of Corollary 3.5 we may decompose

$$F = \sum_{k=0}^{\infty} F_k, \quad \text{where,}\ \ F_k \in \mathcal{H}^{\otimes k}. \tag{4.7}$$

Hence for every α with $|\alpha| = k$,

$$\langle U_{\alpha_1}(T_{h_{i_1}}) \cdots U_{\alpha_n}(T_{h_{i_n}}) \,,\, F_k \rangle_{\mathcal{H}^{\otimes k}} = 0$$

for all appropriate choices of indices. Using Lemma 4.1, the othogonality property (4.2), and the fact that the $T_{h_{i_j}}$ have a semi-circle

$SC_{0,1}$-distribution, we get that

$$\langle G\,,\,F_k\rangle_{\mathcal{H}^{\otimes k}}=0,\quad \forall\, G\in\mathcal{H}^{\otimes k},$$

and so $F_k=0$, and so $F=0$.

□

5. A non-commutative space of stochastic distributions

We here discuss the non-commutative Kondratiev space of stochastic distributions, which was introduced in [8].
For any $p\in\mathbb{Z}$, we denote

$$\mathcal{H}_p=\left\{\sum_{n=1}^{\infty}f_ne_n\,:\,\sum_{n=1}^{\infty}|f_n|^2(2n)^p<\infty\right\}\cong\ell^2(\mathbb{N},(2n)^p),$$

where the (e_n) are the Hermite functions. We note that

$$\cdots\subseteq\mathcal{H}_2\subseteq\mathcal{H}_1\subseteq\mathcal{H}_0\subseteq\mathcal{H}_{-1}\subseteq\mathcal{H}_{-2}\subseteq\cdots,$$

and that $\bigcap_p\mathcal{H}_p$ is the Schwartz space of rapidly decreasing complex smooth functions and $\bigcup_p\mathcal{H}_p$ is its dual, namely the Schwartz space of complex tempered distributions. Let

$$\widetilde{\mathcal{S}}_1=\bigcap_{p\in\mathbb{N}}\Gamma(\mathcal{H}_p),\quad \widetilde{\mathcal{W}}=\Gamma(\mathcal{H}_0),\quad\text{and}\quad \widetilde{\mathcal{S}}_{-1}=\bigcup_{p\in\mathbb{N}}\Gamma(\mathcal{H}_{-p}).$$

Definition 5.1. *The space $\widetilde{\mathcal{S}}_1$ is called the Kondratiev space of non commutative stochastic test functions, and $\widetilde{\mathcal{S}}_{-1}$ is called the Kondratiev space of non commutative stochastic stochastic distributions.*

The following is [8, Theorem 4.1, p. 2314].

Theorem 5.2. *For any $q\geq p+2$ and for any $f\in\Gamma(\mathcal{H}_{-p})$ and $g\in\Gamma(\mathcal{H}_{-q})$ we have*

$$\|f\otimes g\|_q\leq B_{q-p}\|f\|_p\|g\|_q\quad\text{and}\quad\|g\otimes f\|_q\leq B_{q-p}\|f\|_p\|g\|_q \tag{5.1}$$

where $\|\cdot\|_p$ is the norm associated to $\Gamma(\mathcal{H}_{-p})$ and where (with ζ denoting Riemann's zeta function)

$$B_{q-p}^2=\sum_{\alpha\in\widetilde{\ell}}(2\mathbb{N})^{-\alpha(q-p)}=\frac{1}{1-2^{-(q-p)}\zeta(q-p)},$$

Recall that $\Gamma(\mathcal{H}_0)$ is the non commutative white noise space. We have the Gelfand triple

$$\widetilde{\mathcal{S}}_1\subset\Gamma(\mathcal{H}_0)\subset\widetilde{\mathcal{S}}_{-1}.$$

Proposition 5.3. *The algebraic vector space*

$$\bigoplus_{n=0}^{\infty} (\mathscr{S}')^{\otimes n} = \bigoplus_{n=0}^{\infty} \left(\bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p} \right)^{\otimes n}$$

is included in $\widetilde{\mathcal{S}}_{-1} = \bigcup_{p\in\mathbb{N}} \Gamma(\mathcal{H}_{-p})$.

Proof. Since $\bigcup_{p\in\mathbb{N}} \Gamma(\mathcal{H}_{-p})$ is an algebra, it suffices to show that

$$\bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p} \subseteq \bigcup_{p\in\mathbb{N}} \Gamma(\mathcal{H}_{-p}),$$

which is obvious since for any $p \in \mathbb{N}$

$$\mathcal{H}_{-p} \subseteq \Gamma(\mathcal{H}_{-p}).$$

□

Proposition 5.4. *Let $f \in \mathscr{S}'$. Then for any q, such that $\|f\|_{\mathcal{H}_{-q}} < \infty$, we have that*

$$\|\ell_f\|_{B(\Gamma(\mathcal{H}_{-q}))} = \|f\|_{\mathcal{H}_{-q}} \quad \text{and} \quad \|\ell_f^*\|_{B(\Gamma(\mathcal{H}_{q}))} = \|f\|_{\mathcal{H}_{-q}}.$$

Proof. We note that

$$\|\ell_f u\|^2_{\Gamma(\mathcal{H}_{-q})} = \left(\sum_{n\in\mathbb{N}} |f_n|^2 (2n)^{-q} \right) \cdot \left(\sum_{\alpha\in\tilde{\ell}} |u_\alpha|^2 (2\mathbb{N})^{-\alpha q} \right) = \|f\|^2_{\mathcal{H}_{-q}} \|u\|^2_{\Gamma(\mathcal{H}_{-q})}.$$

□

As a consequence of Proposition 5.4 we have:

Corollary 5.5. *For every $f \in \mathscr{S}'$, with $\|f\|_{\mathcal{H}_{-q}} < \infty$, the operator $X_f = \ell_f + \ell_f^*$ is bounded from $\Gamma(\mathcal{H}_q)$ into $\Gamma(\mathcal{H}_{-q})$, and*

$$\|X_f u\|_{\Gamma(\mathcal{H}_{-q})} \leq (2\|f\|_{\mathcal{H}_{-p}}) \|u\|_{\Gamma(\mathcal{H}_q)}. \tag{5.2}$$

In particular, the operator X_f is continuous from $\widetilde{\mathcal{S}_1}$ into $\widetilde{\mathcal{S}_1}$. If $f \in \mathbf{L}_2(\mathbb{R}, dx) = \mathcal{H}_{-0}$, then X_f is continuous from $\widetilde{\mathcal{W}}$ into $\widetilde{\mathcal{W}}$.

Proof. Indeed, we have

$$\begin{aligned}
\|X_f u\|_{\Gamma(\mathcal{H}_{-q})} &\leq \|\ell_f u\|_{\Gamma(\mathcal{H}_{-q})} + \|\ell_f^* u\|_{\Gamma(\mathcal{H}_{-q})} \\
&\leq \|\ell_f u\|_{\Gamma(\mathcal{H}_{-q})} + \|\ell_f^* u\|_{\Gamma(\mathcal{H}_{q})} \\
&= \|f\|_{\mathcal{H}_{-p}} \|u\|_{\Gamma(\mathcal{H}_{-q})} + \|f\|_{\mathcal{H}_{-p}} \|u\|_{\Gamma(\mathcal{H}_{q})} \\
&= \|f\|_{\mathcal{H}_{-p}} \|u\|_{\Gamma(\mathcal{H}_{q})} + \|f\|_{\mathcal{H}_{-p}} \|u\|_{\Gamma(\mathcal{H}_{q})} \\
&= (2\|f\|_{\mathcal{H}_{-p}}) \|u\|_{\Gamma(\mathcal{H}_{q})}.
\end{aligned}$$

In particular, if $f \in \mathcal{H}_0$, then $X_f : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_0)$ is continuous.
Now, if $f \in \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p}$, then since the embedding $\iota : \bigcap \Gamma(\mathcal{H}_p) \hookrightarrow \Gamma(\mathcal{H}_p)$

is continuous (and hence so its dual $\iota^* : \Gamma(\mathcal{H}_{-p}) \hookrightarrow \bigcup \Gamma(\mathcal{H}_{-p})$), we obtain that as a $\bigcap \Gamma(\mathcal{H}_p) \to \bigcup \Gamma(\mathcal{H}_{-p})$ map, X_f is continuous. □

Remark 5.6. In comparing Gelfand triples and generalized functions based on the Gaussian distributions to the free semi-circle case one should keep in mind that both the standard $N(0,1)$ Gaussian variable X_1 and the standard semi-circle random variable T_1 have zero odd moments, and one has

$$E_{\text{sc}}(T_1^{2n}) = \frac{1}{2^n(n+1)} E_{\text{Gauss}}(X_1^{2n})$$

for the even moments. Indeed, standard computations give

$$E_{\text{sc}}(T_1^{2n}) = \frac{1}{2^n(n+1)} \binom{2n}{n} = \frac{1}{2^n(n+1)}(2n-1)!!,$$

and $E_{\text{Gauss}}(X_1^{2n}) = (2n-1)!!$. The corresponding *generating functions* are

$$g_{\text{sc}}(t) = E_{\text{sc}}(e^{tT_1}) = \frac{I_1(t)}{t},$$

where I_1 is the modified Bessel function, while

$$g_{\text{Gauss}}(t) = E_{\text{Gauss}}(e^{tX_1}) = e^{\frac{t^2}{2}}.$$

6. Non commutative stationary increment stochastic processes

Proposition 6.1. *Let $t \mapsto f_t$ be a $\mathbf{L}_2(\mathbb{R}, dx)$-valued function. It defines a free stochastic process $Y_t = X_{f_t}$ such that*

$$\tau(Y_s^* Y_t) = \int_{\mathbb{R}} f_t(u)\overline{f_s(u)}du.$$

Proof. In the proof we set $\ell_{f_t} = \ell_t$ to ease the notation. We have:

$$\begin{aligned}
\tau(Y_s^* Y_t)) &= \langle \ell_s(\Omega) + \ell_s^*(\Omega), \ell_t(\Omega) + \ell_t^*(\Omega)\rangle_\Gamma \\
&= \langle \ell_s(\Omega), \ell_t(\Omega)\rangle_\Gamma + \langle \ell_s(\Omega), \ell_t^*(\Omega)\rangle_\Gamma + \\
&\quad + \langle \ell_s^*(\Omega), \ell_t(\Omega)\rangle_\Gamma + \langle \ell_s^*(\Omega), \ell_t^*(\Omega)\rangle_\Gamma \\
&= \langle \ell_s(\Omega), \ell_t(\Omega)\rangle_\Gamma,
\end{aligned}$$

since

$$\ell_t^*(\Omega) = \ell_s^*(\Omega) = 0.$$

Hence we have

$$\tau(Y_s^* Y_t) = \langle \ell_s(\Omega), \ell_t(\Omega)\rangle_\Gamma = \int_{\mathbb{R}} f_t(u)\overline{f_s(u)}du.$$

□

Following [2] we choose f_t of a special form as follows. First consider a measurable positive function m subject to

$$\int_{\mathbb{R}} \frac{m(u)du}{1+u^2} < \infty. \tag{6.1}$$

We define the operator T_m, defined via

$$\widehat{T_m f}(u) \overset{\text{def.}}{=} \sqrt{m(u)}\widehat{f}(u), \tag{6.2}$$

where $\widehat{f}$ denotes the Fourier transform of f:

$$\widehat{f}(u) = \int_{\mathbb{R}} e^{-iux} f(x)dx.$$

The domain of T_m is

$$\operatorname{dom} T_m = \left\{ f \in \mathbf{L}_2(\mathbb{R}, dx) \ \int_{\mathbb{R}} m(u)|\widehat{f}(u)|^2 du < \infty \right\},$$

and T_m is Hermitian on its domain; see [2]:

$$\langle Tmf, g\rangle = \langle f, T_m g\rangle, \quad \forall f, g \in \operatorname{dom} T_m. \tag{6.3}$$

In view of (6.1) we have that $1_{[0,t]}$ belongs to the domain of the operator T_m.

Definition 6.2. *We set*

$$X_m(t) = \ell_{T_m \mathbf{1}_{[0,t]}} + \ell^*_{T_m \mathbf{1}_{[0,t]}}, \quad t \in \mathbb{R}. \tag{6.4}$$

The case $m(u) = 1$ in the above definition corresponds to the non-commutative Brownian motion. More generally, the case $m(u) = |u|^{1-2H}$ corresponds (up to some multiplicative constant) to the case of the non-commutative fractional Brownian motion with Hurst parameter H.

Proposition 6.3.

$$\tau(X_m(s)^* X_m(t)) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-itu}-1}{u} \frac{e^{ius}-1}{u} m(u)du, \quad t, s \in \mathbb{R}. \tag{6.5}$$

Proof. We take $f_t = T_m \mathbf{1}_{[0,t]} = \ell_t$ in Proposition 6.1 and obtain

$$\begin{aligned} \tau(X_m(s)^* X_m(t)) &= \langle \ell_t(\Omega), \ell_s(\Omega)\rangle_\Gamma \\ &= \langle T_m \mathbf{1}_{[0,t]}, T_m \mathbf{1}_{[0,s]}\rangle_{\mathbf{L}_2(\mathbb{R})} \end{aligned}$$

and, using Plancherel's equality

$$\begin{aligned} &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{T_m \mathbf{1}_{[0,t]}}(u) (\widehat{T_m \mathbf{1}_{[0,s]}})(u)^* du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-itu}-1}{u} \frac{e^{ius}-1}{u} m(u)du, \end{aligned}$$

since the Fourier transform of $\mathbf{1}_{[0,t]}$ is the function $u \mapsto \frac{e^{-iut}-1}{-iu}$. $\square$

7. The derivative of certain operator-valued processes

We first recall a result from [2]. In the proof we make use of the following bounds on the Hermite functions, whose definition we now recall.

Definition 7.1. *The Hermite functions are defined by*

$$\widetilde{h}_k(u) \stackrel{\text{def.}}{=} \frac{h_{k-1}(\sqrt{2}u)e^{-\frac{u^2}{2}}}{\pi^{\frac{1}{4}}\sqrt{(k-1)!}}, \ k = 1, 2, \ldots,$$

where $h_0, h_1 \ldots$ denote the Hermite polynomials (see Definition 2.1 for the latter).

The Hermite functions $\{\widetilde{h}_k\}_{k\in\mathbb{N}}$ form an orthonormal basis of $\mathbf{L}_2(\mathbb{R}, dx)$. In Proposition 7.2 below we study the action of the operator T_m on Hermite functions.

The following proposition outlines the main properties of the Hermite functions which we will need; see [9, p. 349] and the references therein.

Proposition 7.2. [9, p. 349]
of $\mathbf{L}_2(\mathbb{R})$. *Furthermore,*

$$|\widetilde{h}_k(u)| \leq \begin{cases} C & if \quad |u| \leq 2\sqrt{k}, \\ Ce^{-\gamma u^2} & if \quad |u| > 2\sqrt{k}, \end{cases} \tag{7.1}$$

where C and $\gamma > 0$ are constants independent of k. Finally, the Fourier transform of the Hermite function is given by

$$\widehat{\widetilde{h}}_k(u) = \sqrt{2\pi}(-1)^{k-1}\widetilde{h}_k(u). \tag{7.2}$$

Proposition 7.3. *(see* [2, Proposition 3.7 and Lemma 3.8]*) Assume that the function m satisfies a bound of the type:*

$$m(t) \leq \begin{cases} K\,|t|^{-b} & |t| \leq 1, \\ K|t|^{2N} & |t| > 1, \end{cases} \tag{7.3}$$

where $b < 2$, $N \in \mathbb{N}_0$ and $0 < K < \infty$. Then,

$$|(T_m h_n)(t)| \leq C_1 n^{\frac{N+1}{2}} + C_2, \tag{7.4}$$

and

$$|(T_m h_n)(t) - (T_m h_n)(s)| \leq |t-s| \left(D_1 n^{\frac{N+2}{2}} + D_2 \right), \tag{7.5}$$

where C_1, C_2, D_1, D_2 are non-negative constants independent of n.

Theorem 7.4. *Let m be as in Proposition 7.3. There exists a $\mathcal{L}(\widetilde{\mathcal{S}_1}, \widetilde{\mathcal{S}_{-1}})$-valued function $t \mapsto W_m(t)$ such that*

$$\frac{d}{dt} X_m(t) f = W_m(t) f, \quad \forall f \in \widetilde{\mathcal{S}_1}. \tag{7.6}$$

in the topology of $\widetilde{\mathcal{S}_{-1}}$.

Proof. We divide the proof in a number of steps. In the proof, recall that $\widetilde{\mathcal{S}_{-1}} = \cup_{p\in\mathbb{N}}\Gamma(\mathcal{H}_{-p})$ and that an element $f = \sum_{\alpha\in\widetilde{\ell}} f_\alpha U_\alpha \in \widetilde{\mathcal{S}_1} = \cap_{p\in\mathbb{N}}\Gamma(\mathcal{H}_{-p})$ if and only if

$$\sum_{\alpha\in\widetilde{\ell}} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} < \infty, \quad \forall p \in \mathbb{N}. \tag{7.7}$$

STEP 1: *The function $t \mapsto T_m 1_{[0,t]}$ is differentiable in $\mathcal{H}_{-p}$ for $p \geq N+3$ and then*

$$\frac{\mathrm{d}}{\mathrm{dt}} T_m 1_{[0,t]} = \sum_{n=1}^{\infty} \langle T_m \mathbf{1}_{[0,t]}, h_n \rangle h_n, \tag{7.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{L}_2(\mathbb{R}, dx)$.

Using (6.3) we can write:

$$\alpha_n(t) \overset{\text{def.}}{=} \langle T_m \mathbf{1}_{[0,t]}, h_n \rangle = \int_0^t (T_m h_n)(u) du.$$

Using the estimate (7.4) we obtain that

$$\sum_{n\in\mathbb{N}} |\alpha_n'(t)|^2 (2n)^{-p} \leq \sum_{n\in\mathbb{N}} (C_1 n^{\frac{N+1}{2}} + C_2)^2 (2n)^{-p} < \infty \tag{7.9}$$

for $p \geq N+3$, and so the right hand side of (7.8) belongs to $\mathcal{H}_{-p}$ for such p's. Furthermore, with $h \neq 0 \in \mathbb{R}$ we have

$$\begin{aligned} \frac{T_m 1_{[0,t+h]} - T_m 1_{[0,t]}}{h} - \sum_{n=1}^{\infty} \langle T_m \mathbf{1}_{[0,t]}, h_n \rangle h_n = \\ = \sum_{n=1}^{\infty} \langle \frac{T_m 1_{[0,t+h]} - T_m 1_{[0,t]}}{h} - T_m \mathbf{1}_{[0,t]}, h_n \rangle h_n \\ = \sum_{n=1}^{\infty} \frac{\int_t^{t+h} (T_m h_n(u) - T_m h_n(t)) du}{h} h_n \end{aligned}$$

Using (7.5) we see that

$$\|\sum_{n=1}^{\infty}\frac{\int_t^{t+h}(T_mh_n(u)-T_mh_n(t))du}{h}h_n\|_{-p}^2\le K|t-s|^2$$

where

$$K=\sum_{n=1}^{\infty}\left(D_1n^{\frac{N+2}{2}}+D_2\right)^2(2n)^{-p}<\infty$$

for $p\ge N+3$ and hence the result.

STEP 2: *Let* $w_m(t)=\frac{\mathrm{d}}{\mathrm{dt}}T_m1_{[0,t]}$ *Then* $X_{w_m(t)}$ *is a continuous operator from* $\widetilde{\mathcal{S}_1}$ *into* $\widetilde{\mathcal{S}_{-1}}$*.*
This is a direct application of Corollary 5.5.

STEP 3: (7.6) *holds.*

We have for $f\in\widetilde{\mathcal{S}_1}$ and $h\neq 0\in\mathbb{R}$,

$$\left(\frac{X_m(t+h)-X_m(t)}{h}-X_{w_m(t)}\right)f=X_{\Delta(t,h)}f$$

where

$$\Delta(t,h)=\sum_{n=1}^{\infty}\frac{\int_t^{t+h}(T_mh_n(u)-T_mh_n(t))du}{h}h_n$$

and so using (7.5) and Corollary 5.5

$$\begin{aligned}\|\left(\frac{X_m(t+h)-X_m(t)}{h}-X_{w_m(t)}\right)f\|_{-p}&=\|X_{\Delta(t,h)}f\|_{-p}\\&\le K|t-s|\end{aligned}$$

for some finite constant K.

STEP 4: *The operator* $W_m(t)=\frac{\mathrm{d}}{\mathrm{dt}}(\ell_t+\ell_t^*)$ *is continuous from* $\widetilde{\mathcal{S}_1}$ *into* $\widetilde{\mathcal{S}_{-1}}$*.*

This follows from Corollary 5.5 □

As a corollary we have the following construction. Let $d\sigma$ be a positive measure on the real line such that (1.3) is in force. Then, see [4], there exists a continuous operator Q from the Schwartz space $\mathscr{S}$ into $\mathbf{L}_2(\mathbb{R},dx)$ such that

$$\int_{\mathbb{R}}\widehat{\varphi}(u)\overline{\widehat{\psi}(u)}d\sigma(u)=\int_{\mathbb{R}}(Q\varphi)(t)\overline{Q\psi(t)}dt$$

Proposition 7.5. *The free process $X_\sigma(\varphi) = T_{Q\varphi}$ satisfies*

$$\tau(X_\sigma(\psi)^* X_\sigma(\varphi)) = \int_{\mathbb{R}} \widehat{\varphi}(u)\overline{\widehat{\psi}(u)}d\sigma(u).$$

The inequality (5.1) allows to compute stochastic integrals as limit of Riemann sums. The following theorem is the non-commutative counterpart of [3, Theorem 5.1, p. 411].

Theorem 7.6. *Let $f \in \widetilde{\mathcal{S}_1}$ and $t \mapsto Y(t)$, $t \in [a,b]$, be a $\widetilde{\mathcal{S}_{-1}}$-valued function continuous in the strong topology of $\widetilde{\mathcal{S}_{-1}}$. closed interval $[a,b]$. Then, there exists $p \in \mathbb{N}$ (which depends on f) such that the function $t \mapsto Y(t) \otimes (W_m(t)f)$ is $\Gamma(\mathcal{H}_{-p})$-valued, and the integral*

$$\int_a^b Y(u) \otimes W_m f(u)$$

computed as a limit of Riemann sums converges in the form of $\Gamma(\mathcal{H}_{-p})$.

The proof is the same as in [3], the key being the existence in the presence setting of inequality (5.1).

8. The use of other Gelfand triples

In Proposition 7.3 the bounds (7.3) played a key role. Without them, it may happen that, in the notation of the proof of the proposition

$$\sum_{n \in \mathbb{N}} |\alpha_n'(t)|^2 (2n)^{-p} = \infty, \quad \forall p \in \mathbb{N}$$

and then the arguments fail there. This suggest that other Gelfand triples could be used. The Gelfand triple $(\widetilde{\mathcal{S}}_1, \Gamma(\mathbf{L}_2(\mathbb{R}, dx), \widetilde{\mathcal{S}}_{-1})$ belongs to a general family of Gelfand triples in which an inequality of the form (5.1) holds. This is explained in the paper [8], on which is based the present section. We take a separable Hilbert $\mathcal{K}_0$, with orthonormal basis $e_1, e_2, \ldots$. Furthermore, let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than or equal to 1. For any $p \in \mathbb{Z}$, we denote

$$\mathcal{K}_p = \left\{ \sum_{n=1}^\infty f_n e_n : \sum_{n=1}^\infty |f_n|^2 a_n^p < \infty \right\} \cong \mathbf{L}^2(\mathbb{N}, a_n^p).$$

The case $a_n = 2n$ corresponds to the non-commutative Kondratiev space. The choice $a_n = 2^n$ is of special importance, as will appear in the sequel of this section.

For $q \geq p$ we denote by $T_{q,p}$ the embedding $\mathcal{K}_q \hookrightarrow \mathcal{K}_p$. It satisfies

$$\|T_{q,p} a_n^{-q/2} e_n\|_p = a_n^{-(q-p)/2} \|a_n^{-p/2} e_n\|_q,$$

and hence

$$\|T_{q,p}\|_{HS} = \sqrt{\sum_{n\in\mathbb{N}} a_n^{-(q-p)}},$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. The space $\bigcup_{p\in\mathbb{N}} \mathcal{K}_{-p}$ is nuclear if and only for any p there is some $q > p$ such that $\|T_{q,p}\|_{HS} < \infty$, that is, if and only if there exists some $d > 0$ such that $\sum_{n\in\mathbb{N}} a_n^{-d}$ converges. We note that in this case, d can be chosen so that

$$\sum_{n\in\mathbb{N}} a_n^{-d} < 1.$$

We call the smallest integer d which satisfy this inequality the index of $\bigcup_{p\in\mathbb{N}} \mathcal{K}_{-p}$. In the statement $\Gamma(T_{q,p})$ denotes the embedding $\Gamma(\mathcal{K}_{-p}) \hookrightarrow \Gamma(\mathcal{K}_{-p})$, and $\|\cdot\|_p$ denotes the norm associated to $\Gamma(\mathcal{K}_{-p})$.

Theorem 8.1. *If $\bigcup_{p\in\mathbb{N}} \mathcal{K}_{-p}$ is nuclear of index d, then $\bigcup_{p\in\mathbb{N}} \Gamma(\mathcal{K}_{-p})$ is nuclear and has the property that*

$$\|f\otimes g\|_q \leq \|\Gamma(T_{q,p})\|_{HS}\|f\|_p\|g\|_q \text{ and } \|g\otimes f\|_q \leq \|\Gamma(T_{q,p})\|_{HS}\|f\|_p\|g\|_q$$

for all $q \geq p+d$, and where

$$\|\Gamma(T_{q,p})\|_{HS} = \sum_{\alpha\in\widetilde{\ell}} a_{\mathbb{N}}^{-\alpha(q-p)} = \frac{1}{\sqrt{1-\sum_{n\in\mathbb{N}} a_n^{-(q-p)}}}.$$

Let us now take $a_n = 2^n$. Then,

$$\mathcal{K}_p = \left\{ \sum_{n=1}^{\infty} f_n e_n : \sum_{n=1}^{\infty} |f_n|^2 2^{np} < \infty \right\} \cong \ell^2(\mathbb{N}, 2^{np}).$$

We proved in [7] that $\bigcap_p \mathcal{K}_p$ is the space $\mathscr{G}$ of entire holomorphic functions satisfing

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \quad \text{for all } p\in\mathbb{N}.$$

In the argument we recall that use is made of the following formula (see [19])

$$\sum_{n=0}^{\infty} h_n(u)h_n(v)s^n = \pi^{-\frac{1}{2}}(1-s^2)^{-\frac{1}{2}} e^{-\frac{(1+s^2)(u^2+v^2)-4svu}{2(1-s^2)}}. \tag{8.1}$$

The space $\mathscr{G}$ contains strictly the Schwartz space $\mathscr{S}$. We will work in the setting of the Gelfand triple defined by

$$\widetilde{\mathscr{G}}_1 = \bigcap_{p\in\mathbb{N}} \Gamma(\mathcal{K}_p), \quad \widetilde{\mathscr{W}} = \Gamma(\mathcal{K}_0), \quad \text{and} \quad \widetilde{\mathscr{G}}_{-1} = \bigcup_{p\in\mathbb{N}} \Gamma(\mathcal{K}_{-p}).$$

In this case, the term

$$\sum_{n\in\mathbb{N}} |\alpha_n'(t)|^2(2n)^{-p} < \infty$$

is replaced with

$$\sum_{n\in\mathbb{N}} |\alpha_n'(t)|^2 2^{-np} < \infty \tag{8.2}$$

in the proof of the analogue of Theorem 7.4, as we now explain.

Theorem 8.2. *Assume that the function m satisfies a bound of the form*

$$m(t) \le \begin{cases} K\,|t|^{-b} & |t| \le 1, \\ C_1 e^{C_2|t|}, & |t| \ge 1, \end{cases} \tag{8.3}$$

where $b < 2$, $N \in \mathbb{N}_0$ and where C_1 and C_2 are strictly positive numbers. Then there exists a $\mathcal{L}(\widetilde{\mathcal{G}}_1, \widetilde{\mathcal{G}_{-1}})$-valued function $t \mapsto W_m(t)$ such that

$$\frac{d}{dt}X_m(t)f = W_m(t)f, \quad \forall f \in \widetilde{\mathcal{G}}_1. \tag{8.4}$$

Proof of Theorem 8.2. The proof parallels the proof of Theorem 7.4. The main idea is that the new bounds on

$$T_m h_n(t) \quad \text{and} \quad |T_m h_n(t) - T_m h_n(s)|$$

are adapted to the new sequence $a_n = 2^n, n = 1, 2, \ldots$.

STEP 1: Assume that the function m satisfies a bound of the form (8.3). Then,

$$|T_m h_n|(t) \le D_1 e^{D_2\sqrt{n}}$$
$$|T_m h_n(t) - T_m h_n(s)| \le |t-s| D_3 e^{D_4\sqrt{n}},$$

where $D_1, \ldots, D_4$ are strictly positive constants.

The proofs are similar to those in [2]. The key is is to estimate integrals of the form

$$\int_{2\sqrt{n}}^{\infty} \sqrt{m(u)}h_n(u)du, \quad \int_{2\sqrt{n}}^{\infty} u\sqrt{m(u)}h_n(u)du,$$

and

$$\int_0^{2\sqrt{n}} \sqrt{m(u)}h_n(u)du, \quad \int_0^{2\sqrt{n}} u\sqrt{m(u)}h_n(u)du,$$

taking into account the bound (7.1).

STEP 2: *The function* $t \mapsto T_m 1_{[0,t]}$ *is differentiable in* $\mathcal{H}_{-p}$ *for* $p \geq$ *and then equation* (7.8) *holds.*

STEP 3: *Let* $w_m(t) = \frac{\mathrm{d}}{\mathrm{dt}} T_m 1_{[0,t]}$ *Then* $X_{w_m(t)}$ *is a continuous operator from* $\widetilde{\mathcal{S}_1}$ *into* $\widetilde{\mathcal{S}_{-1}}$.

This is a direct application of Corollary 5.5.

STEP 4: (8.4) *holds.*

This is as in the proof of Theorem 7.4. $\square$

Remarks 8.3.
(1) The previous result, together with the existence of a Våge type inequality in $\widetilde{\mathcal{G}_{-1}}$, allows to define stochastic integrals as in Theorem 7.6.
(2) Finally we remark that the analysis in the papers [2, 3] can be extended, in the commutative case, to more general Gelfand triples where a Våge type inequality holds.

We conclude the paper with a table comparing the commutative and free cases.

The setting	Commutative	Free setting
The underlying space	Symmetric Fock space	Full Fock space
Concrete realization	via Bochner-Minlos $\mathbf{L}_2(\mathcal{S}', dP)$	$\mathbf{L}_2(\tau)$
Polynomials	Hermite polynomials	Tchebycheff of the second kind
The building blocks	Functions H_α given by (2.2)	See Theorem 4.2
Distribution law	Gaussian	Semi-circle

References

[1] D. Alpay and H. Attia. An interpolation problem for functions with values in a commutative ring. In *A Panorama of Modern Operator Theory and Related*

Topics, volume 218 of *Operator Theory: Advances and Applications*, pages 1–17. Birkhäuser, 2012.

[2] D. Alpay, H. Attia, and D. Levanony. On the characteristics of a class of Gaussian processes within the white noise space setting. *Stochastic processes and applications*, 120:1074–1104, 2010.

[3] D. Alpay, H. Attia, and D. Levanony. White noise based stochastic calculus associated with a class of Gaussian processes. *Opuscula Mathematica*, 32/3:401–422, 2012.

[4] D. Alpay and P. Jorgensen. Stochastic procesees induced by singular operators. *Numerical Functional Analysis and Optimization*, 33:708–735, 2012.

[5] D. Alpay, P. Jorgensen, and D. Levanony. A class of Gaussian processes with fractional spectral measures. *J. Funct. Anal.*, 261(2):507–541, 2011.

[6] D. Alpay and D. Levanony. Linear stochastic systems: a white noise approach. *Acta Applicandae Mathematicae*, 110:545–572, 2010.

[7] D. Alpay and G. Salomon. New topological $\mathbb{C}$-algebras with applications in linear systems theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15(2):1250011, 30, 2012.

[8] D. Alpay and G. Salomon. Non-commutative stochastic distributions and applications to linear systems theory. *Stochastic Process. Appl.*, 123(6):2303–2322, 2013.

[9] F. Biagini, B. Øksendal, A. Sulem, and N. Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion, stochastic analysis with applications to mathematical finance. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2041):347–372, 2004.

[10] Jana Bohnstengel and Palle Jorgensen. Geometry of spectral pairs. *Anal. Math. Phys.*, 1(1):69–99, 2011.

[11] Marek Bożejko and Eugene Lytvynov. Meixner class of non-commutative generalized stochastic processes with freely independent values. I. A characterization. *Comm. Math. Phys.*, 292(1):99–129, 2009.

[12] Marek Bożejko and Eugene Lytvynov. Meixner class of non-commutative generalized stochastic processes with freely independent values II. The generating function. *Comm. Math. Phys.*, 302(2):425–451, 2011.

[13] Ilwoo Cho and Palle E. T. Jorgensen. Free probability on operator algebras induced by currents in electric resistance networks. *Int. J. Funct. Anal. Oper. Theory Appl.*, 4(1):1–50, 2012.

[14] Dorin E. Dutkay and Palle E. T. Jorgensen. Affine fractals as boundaries and their harmonic analysis. *Proc. Amer. Math. Soc.*, 139(9):3291–3305, 2011.

[15] M. Fliess. Matrices de Hankel. *J. Math. Pures Appl. (9)*, 53:197–222, 1974.

[16] I.M. Gelfand and G.E. Shilov. *Generalized functions. Volume 2*. Academic Press, 1968.

[17] Fumio Hiai and Dénes Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[18] Takeyuki Hida. *Brownian motion*, volume 11 of *Applications of Mathematics*. Springer-Verlag, New York, 1980. Translated from the Japanese by the author and T. P. Speed.

[19] Einar Hille. A class of reciprocal functions. *Ann. of Math. (2)*, 27(4):427–464, 1926.

[20] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic partial differential equations*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1996.

[21] P. E. T. Jorgensen and A. M. Paolucci. States on the Cuntz algebras and p-adic random walks. *J. Aust. Math. Soc.*, 90(2):197–211, 2011.

[22] Palle E. T. Jorgensen and Ilwoo Cho. Symmetry in tensor algebras over Hilbert space. *Illinois J. Math.*, 55(3):977–1013 (2013), 2011.

[23] Palle E. T. Jorgensen, Keri A. Kornelson, and Karen L. Shuman. Families of spectral sets for Bernoulli convolutions. *J. Fourier Anal. Appl.*, 17(3):431–456, 2011.

[24] Alexandre Nou. *Algèbre q-Gaussiennes*. Mémoire de DEA, Université de Franche-Comté, Juin 2000.

[25] G. Våge. Hilbert space methods applied to stochastic partial differential equations. In H. Körezlioglu, B. Øksendal, and A.S. Üstünel, editors, *Stochastic analysis and related topics*, pages 281–294. Birkäuser, Boston, 1996.

[26] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[27] Dan Voiculescu. Symmetries of some reduced free product C^*-algebras. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 556–588. Springer, Berlin, 1985.

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