2013

Convex Cones of Generalized Positive Rational Functions and Nevanlinna-Pick Interpolation

Daniel Alpay
Chapman University, alpay@chapman.edu

Izchak Lewkowicz
Ben Gurion University of the Negev

Recommended Citation
D. Alpay and I. Lewkowicz. Convex cones of generalized positive rational functions and Nevanlinna-Pick interpolation. Linear Algebra and its Applications, vol. 438 (2013) 3949-3966.

Convex Cones of Generalized Positive Rational Functions and Nevanlinna-Pick Interpolation

CONVEX CONES OF GENERALIZED POSITIVE RATIONAL FUNCTIONS AND THE NEVANLINNA-PICK INTERPOLATION

DANIEL ALPAY AND IZCHAK LEWKOWICZ

Dedicated to appreciated colleagues Avraham Berman, Moshe Goldberg and Raphael Loewy

Abstract. Scalar rational functions with a non-negative real part on the right half plane, called positive, are classical in the study of electrical networks, dissipative systems, Nevanlinna-Pick interpolation and other areas. We here study generalized positive functions, i.e with a non-negative real part on the imaginary axis. These functions form a Convex Invertible Cone, **cic** in short, and we explore two partitionings of this set: (i) into (infinitely many non-invertible) convex cones of functions with prescribed poles and zeroes in the right half plane and (ii) each generalized positive function can be written as a sum of even and odd parts. The sets of even generalized positive and odd functions form sub**cic**s.

It is well known that the Nevanlinna-Pick interpolation problem is not always solvable by positive functions. Unfortunately, there is no computationally simple procedure to carry out this interpolation in the framework of generalized positive functions. Through examples it is illustrated how the two above partitionings of generalized positive functions can be exploited to introduce simple ways to carry out the Nevanlinna-Pick interpolation.

Finally we show that only some of these properties are carried over to rational generalized bounded functions, mapping the imaginary axis to the unit disk.

1. Introduction

1.1. **Historical perspective.** Functions which are analytic in the open right half-plane $\mathbb{C}_+$ and with a non-negative real part there

$$\text{Re } p(s) \geq 0, \qquad s \in \mathbb{C}_+, \tag{1.1}$$

here denoted by $\mathcal{P}$, play an important in the theory of electrical networks they were first studied around 1930 by W. Cauer, [22], [23] and O. Brune (who first coined the name *positive* for such functions; see [20, Definition 1, p. 25], see also [21]). These functions also serve as the corner stone of the theory of linear dissipative systems (a.k.a absolutely stable), see e.g. [12, Theorem 2.7.1], [17, 3.18], [45] and [47].

1991 *Mathematics Subject Classification.* Primary: 26C15, 30C40, 30E05, 47A68; Secondary: 47B50, 93D10, 94C05.

Key words and phrases. positive real functions, Nevanlinna functions, Carathéodory functions, Cauer classes, convex invertible cones of rational functions, Nevalinna Pick interpolation.

This research is partially supported by BSF grant no 2010117.

D. Alpay thanks the Earl Katz family for endowing the chair which supported his research. This research is part of the European Science Foundation Networking Program HCAA.

One can weaken condition (1.1) and assume that a function p is analytic almost everywhere on the imaginary axis satisfying

$$\text{Re } p(s) \geq 0, \qquad s \in i\mathbb{R}.$$

These functions will be called *generalized positive* and thus denoted by $\mathcal{GP}$. They were first addressed more than forty years ago by B.D.O. Anderson and J.B. Moore in [11].

We shall denote by $\mathbb{C}_-$ the open left half of the complex plane. We also denote the closed right half plane by $\overline{\mathbb{C}_+}$ $(= i\mathbb{R} \cup \mathbb{C}_+)$. We here consider the field of scalar rational functions of a complex variable s with complex coefficients. Throughout this work we denote by $\mathcal{GP}$ the set of scalar *rational* generalized positive functions and by $\mathcal{P}$ its subset of positive functions.

In the sequel, we shall find it convenient to denote for an arbitrary rational function $g(s)$,

$$g^{\#}(s) := (g(-s^*))^*.$$

The following result appeared in [8] (to help the reader we shall use $p \in \mathcal{P}$ and $\psi \in \mathcal{GP}$):

Theorem 1.1. *A rational function $\psi(s)$ is generalized positive if and only if it admits the factorization*

$$\psi(s) = g(s)p(s)g^{\#}(s), \qquad s \in \mathbb{C}, \tag{1.2}$$

where $p \in \mathcal{P}$ and g is rational such that both g and g^{-1} analytic in $\mathbb{C}_-$ and [1] $\deg(g) \in [0,\ \deg(\psi)]$.
Moreover, one can always[2] *find $s_o \in i\mathbb{R}$ so that in* (1.2) *the functions $p(s)$ and $g(s)$ are uniquely determined by $0 \neq \psi(s_o) = p(s_o)$ and $g(s_o) = 1$.*

Factorization results of the form of (1.2) are well known in other frameworks. To name but three:
(i) Schur functions, analytically mapping the open unit disk to its closure, rational generalized Schur functions mapping the unit circle to the closed unit disk, see e.g. [46], [24], [1]. Factorization result of generalized Schur functions appeared in [38, Theorem 3.2].
(ii) Carathéodory functions, mapping the open unit disk to $\overline{\mathbb{C}_+}$ and the rational generalized (=pseudo) Carathéodory functions, mapping the unit circle to $\overline{\mathbb{C}_+}$. Factorization result of generalized Carathéodory functions appeared in [28, Theorem 3.1].
(iii) Nevanlinna functions, analytically mapping the closed upper half plane to its closure and the rational generalized Nevanlinna functions mapping the real axis to the closed upper half plane. Factorization of generalized Nevanlinna functions appeared in parallel in [31] and [33] and further explored and extended to operator valued functions in [42], [43] and [44].

An extended version of $\mathcal{GP}$ functions was introduced by M.G. Kreĭn and H. Langer in a long and celebrated series of papers of which we cite only [38], [39] (note that they studied functions meromorphic in the open upper half plane, or in the open disk).

[1] Recall, the degree of a rational function is taken to be the maximum between the degrees of the numerator and the denominator.

[2] assuming $\psi(s) \not\equiv 0$

Important part of the existing research on $\mathcal{GP}$ functions is neither confined to scalar functions nor to rational functions. Restricting the discussion to scalar rational functions mapping the imaginary axis into the right half-plane, enabled us in [8] to obtain an elementary proof for the factorization (1.2). On the expense of generality, to keep the exposition simple, we here adhere to the case of scalar rational functions.

1.2. **The current work.** Traditionally, $\mathcal{GP}$ functions were studied almost uniquely by mathematicians. They were addressed in the framework of not necessarily rational functions. In contrast, Electrical engineers have long been interested in rational positive functions (impedance of R-L-C electrical circuits).

In this work we take a challenging task try to simultaneously address both audiences. Thus, depending on their background, some readers may find part of the statements nearly obvious and others not clear at all. Moreover, we try to address skeptical questions like:
(An engineer): "Why should I be interested in the extension of positive functions to the $\mathcal{GP}$ framework?"
(A mathematician): "Why should I be interested in scalar rational $\mathcal{GP}$ functions if the operator-valued non-rational case has already been addressed ?"

As already pointed out above, positive functions have been a corner-stone in system theory. We thus believe that a sufficiently good motivation for a researcher in this field to explore $\mathcal{GP}$ functions, is to gain a proper perspective on the subset of $\mathcal{P}$ functions. In particular to understand which of the properties of $\mathcal{P}$ exist in the larger set $\mathcal{GP}$ and which are peculiar to positive functions.

As a prime example we point out that rational $\mathcal{GP}$ functions (bounded at infinity) may be characterized through the Generalized Positive Real Lemma (in the positive case a.k.a. the Kalman-Yakubovich-Popov Lemma) see e.g. [11], [32] for early accounts and most recently [9].

One can examine this gap between $\mathcal{GP}$ functions and its Hurwitz stable subset of $\mathcal{P}$ functions, from the Matrix Lyapunov Equation point of view. It has been of interest to identify which of the properties of the Lyapunov Equation for Hurwitz stable matrices are carried over to the general inertia case, see e.g. [36, Chapter 2 and Section 4.4] or the relation to the dimension of the controllable subspace in [40]. Resorting to the matrix Lyapunov equation was not just a metaphor, it is part of the above mentioned Generalized Positive Real Lemma, see e.g. [9], [11], [32].

Recall that convex sets are an essential ingredient in optimization. As an illustration, a typical control engineering problem would be: "Find among all stabilizing controllers the one which minimizes a certain property". Such problems are easier to address if the set of all controllers or closed loop systems is convex. Convex sets also serve as a model for uncertainty, e.g. the celebrated Kharitonov Theorem for checking the Hurwitz stability of a polytope of polynomials, see [16]. Hence, one is motivated in studying convex sets of rational functions which are Hurwitz stable (and then look for stable minimum phase). In Section 2 we identify maximal convex cones of rational functions with prescribed poles and zeroes in the open right (or left) half plane.

Using the above result, we introduce in Section 3 a partitioning of all $\mathcal{GP}$ functions into "small", yet maximal, convex cones denoted by $\mathcal{GP}_g$, with prescribed poles and zeroes in the right half plane. Each of these sets is a replica of $\mathcal{P}$ functions.

Recall that a convex cone which in addition is closed under inversion is called a Convex Invertible Cone, **cic** in short[3] , see e.g. [25], [26] and [27]. It is easy to see that the set of $\mathcal{GP}$ functions is closed under positive scaling, summation and inversion, i.e. a **cic** and $\mathcal{P}$ is a sub**cic** of it. More precisely, $\mathcal{P}$ is a maximal **cic** of functions which are analytic in $\mathbb{C}_+$, see [27, Proposition 4.1.1] and Proposition 2.2 below.

In Section 4 partition each $\mathcal{GP}$ function to a sum of *even* and *odd* generalized positive functions. It turns out that the sets of *even* and *odd* generalized positive functions are sub**cic**s of $\mathcal{GP}$. First, the set of all $\mathcal{O}dd$ functions (i.e. generalized lossless) which is of particular interest, is then studied. In Section 5 even $\mathcal{GP}$ functions are explored. As a by-product of this partitioning it is shown that it is only within the larger $\mathcal{GP}$ set that a positive function can always be written as a sum of even and odd part.

From an applications point of view, Nevanlinna-Pick type interpolation was originally motivated by the design of the driving point impedance of R-L-C electrical circuits and subsequently by H_∞ control, both restricted to positive functions. There are good reasons to study Nevanlinna-Pick interpolation over $\mathcal{GP}$ functions:

Recall that in the positive case solution exists, if and only if, the Pick matrix associated with the data points is positive semidefinite. In the $\mathcal{GP}$ framework, this restriction is removed, namely for almost any set of data points a solution exists[4]. Note that in some applications, Hurwitz stability is not required e.g. when the data is not associated with an input-output dynamical system or when the system at hand is not passive.

Nevanlinna-Pick interpolation problem of generalized Schur and Nevanlinna functions has been well addressed in the literature: For generalized Nevanlinna functions see e.g. [3], [10], [5], [18], [29, Section 3] and [4]. For generalized Schur functions see e.g. [2], [13], [15], [19], [30] and [41] Nonetheless from computational point of view the known procedure is involved.

In each of the Sections: 3, 4 and 5, we illustrate through examples how can one exploit the new structural results to simplify the Nevanlinna-Pick interpolation problem. A careful examination of this examples suggests directions for future research. Some of them are stated in Section 7.

Recall that a function is called *bounded*, denoted by $f_b \in \mathcal{B}$, (commonly the real case is addressed) if it analytically maps $\mathbb{C}_+$ to the closed unit disk, see e.g. [12, Chapter 7], [17, Section 6.5] and $f_{gb} \in \mathcal{GB}$ is *generalized bounded* if it maps the imaginary axis to the closed unit disk, see e.g. [32]. It is known that through the Cayley transform positive functions may be identified with bounded functions. In Section 6 properties of generalized bounded functions, which do not trivially follow from this Cayley transform are explored.

[3]Strictly speaking, this means that whenever the inverse exists, it also belongs to the set, e.g. the set of positive semidefinite matrices is a **cic**. In contrast, the open upper half of $\mathbb{C}$ is not.

[4]For generalized Schur function this follows from [30] and by appropriate Cayley transforms (of the functions and of the variable) this is true for $\mathcal{GP}$ functions as well.

In particular it is shown that one cannot easily mimic Section 3 to obtain a partitioning of all rational generalized bounded functions to a union of sets with prescribed poles and zeroes outside the unit disk.

2. Maximal convex sets of rational functions with prescribed poles and zeroes in $\mathbb{C}_+$ or in $\mathbb{C}_-$

In this section we consider poles and zeroes of sums of rational functions. Up to possible cancellations, poles of a sum are the union of the poles original functions. However, in general little can be said about the zeroes of a sum. We now characterize maximal convex sets of rational functions with prescribed poles and zeroes in $\mathbb{C}_+$ (or in $\mathbb{C}_-$).
We begin with some preliminaries. We find it convenient to define the following sets

$$\begin{array}{rl} \mathcal{G}_- := & \text{all rational functions with poles and zeroes in } \mathbb{C}_- \ , \\ \mathcal{G}_+ := & \text{all rational functions with poles and zeroes in } \mathbb{C}_+ \ . \end{array} \tag{2.1}$$

Note that $g \in \mathcal{G}_-$ is equivalent to $g^{\#} \in \mathcal{G}_+$.

Example 2.1. All degree one real functions with poles and zeroes in the $\overline{\mathbb{C}_+}$ are given by

$$\mathcal{G}_1 = \left\{ \frac{as - b}{cs - d} \ : \ ab \geq 0, \ cd \geq 0, \ ad \neq bc. \right\}. \tag{2.2}$$

Now $\mathcal{G}_1^{\#} = \left\{ \frac{as+b}{cs+d} \ : \ ab \geq 0, \ cd \geq 0, \ ad \neq bc. \right\}$, and all degree one real functions within $\mathcal{P}$ is a subset of $\mathcal{G}_1^{\#}$ where in addition $a, \ c \geq 0$. Indeed, up to inversion, all degree one real functions in $\mathcal{P}$ are of the form $as + b$ or $\frac{a}{s} + b$ with $a, b \geq 0$ (in electrical circuits terminology, the driving point impedance of R-L or R-C networks, respectively). $\square$

Next, for a *given* g_+ in $\mathcal{G}_+$, see (2.1), let $\tilde{\mathcal{G}}_{g_+}$ be the set of all rational functions with prescribed poles and zeroes in $\mathbb{C}_+$ given by,

$$\tilde{\mathcal{G}}_{g_+} := \left\{ \frac{n_-(s)}{d_-(s)} g_+(s) \ : \ n_-(s), d_-(s) \text{ polynomials with roots in } \overline{\mathbb{C}_-} \right\}. \tag{2.3}$$

Obviously, if in (2.3) $g \in \tilde{\mathcal{G}}_{g_+}$ satisfies $\frac{n_-(s)}{d_-(s)} \equiv \text{const.}$ then in fact $g \in \mathcal{G}_+$ (2.1). The set $\tilde{\mathcal{G}}_{g_+}$ can not be convex as both $-g_+(s)$ and $sg_+(s)$ belong to it, but their sum has an additional zero in $\mathbb{C}_+$. Yet, it is of interest to identify maximal convex subsets of $\tilde{\mathcal{G}}_{g_+}$ in (2.3). To this end, denote by $\mathcal{G}_o$ the set of rational functions, which along with their inverses, are analytic in both open half planes. Namely, whenever $g_o \in \mathcal{G}_o$, it is of the form

$$g_o(s) = c \prod_{j=1}^{m} (s - ir_j)^{\eta_j} \ , \tag{2.4}$$

where $r_j \in \mathbb{R}$ are all distinct, η_j are integers (positive or negative) and $c \in \mathbb{C}$. We shall use the convention that $\prod_1^0 = 1$, so that also $g(s) \equiv \text{const.}$ belongs to this set.

For three given functions $g_+ \in \mathcal{G}_+$, $g_- \in \mathcal{G}_-$ see (2.1), and g_o see (2.4), define,

$$\mathcal{G}_{g_+, \ g_-, \ g_o} := \{ g_+(s) p(s) g_o(s) g_-(s) \ : \ p \in \mathcal{P} \ \}. \tag{2.5}$$

By construction, in $\mathbb{C}_+$ the poles and zeroes of all functions[5] in $\mathcal{G}_{g_+,\, g_-,\, g_o}$ are exactly those of $g_+(s)$. On $i\mathbb{R}$ poles and zeroes are almost fixed in the following sense. Considering (2.4), functions in $\mathcal{G}_{g_+,\, g_-,\, g_o}$ have factors of the form $(s-ir_j)^{\eta_j+1}$, $(s-ir_j)^{\eta_j-1}$ or $(s-ir_j)^{\eta_j}$ depending on $p(s)$ having at $s=ir_j$, a zero, a pole, neither zero nor pole, respectively.

We can now describe convex sets of functions with prescribed poles and zeroes in $\mathbb{C}_+$ and on $i\mathbb{R}$, almost prescribed in the above sense.

Proposition 2.2. *The following is true:*

(i) *The set $\mathcal{P}$ is a maximal convex invertible cone,* **cic***, of rational functions analytic in $\mathbb{C}_+$.*

(ii) $\mathcal{G}_{g_+,g_-,g_o}$ *in* (2.5)*, is a maximal convex set of rational functions with prescribed poles and zeroes in $\mathbb{C}_+$.*
In fact, if $\phi \notin \mathcal{G}_{g_+,g_-,g_o}$, one can always find $\psi \in \mathcal{G}_{g_+,g_-,g_o}$ so that $(\phi+\psi)\notin \tilde{\mathcal{G}}_{g_+}$ (2.3)*.*

The fact that the set $\mathcal{P}$ is a convex invertible cone, **cic**, is classical, see e.g. [17, 5.6]. The result in item (i) is a small variation of [27, Proposition 4.1.1].

Proof : (i) We first show that if $h(s)$ is a non-positive function, one can always find a positive function p so that $h+p$ has a zero in $\mathbb{C}_+$. Indeed, let $h\notin\mathcal{P}$ be given. By definition there are points in $\mathbb{C}_+$ which are mapped by $h(s)$ to $\mathbb{C}_-$, i.e. there exist $\alpha,\gamma>0$, $\beta,\delta\in\mathbb{R}$ so that $h(s)_{|s=\alpha+i\beta}=-\gamma+i\delta$. Take now $p(s)=\frac{\gamma}{\alpha}s-i(\frac{\beta\gamma}{\alpha}+\delta)$. Then clearly $p\in\mathcal{P}$ and $(p+h)(s)_{|s=\alpha+i\beta}=0$, i.e. a zero in $\mathbb{C}_+$. Next, note that $\frac{1}{p+h}$ is not analytic in $\mathbb{C}_+$. Since $\mathcal{P}$ is closed under inversion (i.e. $p\in\mathcal{P}$ is equivalent to $\frac{1}{p}\in\mathcal{P}$), this part is established.
(ii) Let $g_-\in\mathcal{G}_-$, $g_+\in\mathcal{G}_+$ see (2.1) and $g_o\in\mathcal{G}_o$, see (2.4), be given and let $\phi(s)$ be a rational function not in $\mathcal{G}_{g_+,g_o,g_-}$. To avoid triviality assume that $\phi\in\{\tilde{\mathcal{G}}_{g_+}\diagdown\mathcal{G}_{g_+,g_-,g_o}\}$. Next, denote, $h(s)=g_o(s)^{-1}g_+(s)^{-1}\phi(s)g_-(s)^{-1}$. Then by (2.5), $h\notin\mathcal{P}$ (else ϕ would have been in $\mathcal{G}_{g_+,g_-,g_o}$). Take now $\psi(s)=g_+(s)g_o(s)p(s)g_-(s)$ with the above $g_+(s)$, $g_o(s)$, $g_-(s)$ and $p(s)$ as in part (a) of this proof. By construction, $p\in\mathcal{P}$ and thus $\psi\in\mathcal{G}_{g_+,g_-,g_o}$, but $(\phi+\psi)\notin\tilde{\mathcal{G}}_{g_+}$ since this function has an additional zero in $\mathbb{C}_+$. (If this additional zero coincides with an existing pole, both in $\mathbb{C}_+$, still $(\phi+\psi)\notin\tilde{\mathcal{G}}_{g_+}$). Thus, this part of the claim is established and the proof is complete. □

As an illustration we have the following.

Example 2.3. Take $g_+(s)$, $g_o(s)$ and $g_-(s)$ in (2.5) to be fixed polynomials. A maximal (up to scaling) convex set of polynomials whose roots in $\mathbb{C}_+$ are those of g_+, is given by

$$\{g_+(s)(s+a)g_o(s)g_-(s)\ :\ a\in\overline{\mathbb{C}_+}\ \}.$$

Indeed, $p(s)=s+a$, $a\in\overline{\mathbb{C}_+}$ are the only polynomials in $\mathcal{P}$. □

[5] excluding the zero function

We conclude this section by stating the analogous results for the left half plane. First, we denote by $\mathcal{P}^{\#}$ the set of *para-positive* functions,

$$\mathcal{P}^{\#} := \{ \ \psi \ : \ \psi^{\#} \in \mathcal{P} \ \}. \tag{2.6}$$

Thus, functions in $\mathcal{P}^{\#}$ map $\overline{\mathbb{C}_-}$ to $\overline{\mathbb{C}_+}$. In particular, $\mathcal{P}^{\#} \subset \mathcal{GP}$. We can now state results which are dual to Proposition 2.2.

Observation 2.4. *The following is true:*

(i) *The set $\mathcal{P}^{\#}$ is a maximal convex invertible cone,* **cic***, of rational functions analytic in $\mathbb{C}_-$.*

(ii) *Let $g_- \in \mathcal{G}_-$, $g_+ \in \mathcal{G}_+$ and $g_o \in \mathcal{G}_o$ be given. The set*

$$\{g_+(s)p^{\#}(s)g_o(s)g_-(s) \ : \ p \in \mathcal{P} \ \}$$

is a maximal convex set whose poles and zeroes in $\mathbb{C}_-$ are precisely those of $g_-(s)$.

3. Convex partitioning of $\mathcal{GP}$ functions

We now address ourselves to subsets of generalized positive functions within $\mathcal{G}_{g_+,\ g_-,\ g_o}$ in (2.5), namely sets of the form $\mathcal{G}_{g_+,\ g_-,\ g_o} \cap \mathcal{GP}$. To this end, we introduce the following set, using (2.1) and (2.4),

$$\overline{\mathcal{G}_+} := \{g_+(s)g_o(s) \ : \ g_+ \in \mathcal{G}_+, \ g_o \in \mathcal{G}_o \ \}. \tag{3.1}$$

Note that $g \in \overline{\mathcal{G}_+}$ means that both g and g^{-1} are analytic in $\mathbb{C}_-$. For example, all degree one functions in $\overline{\mathcal{G}_+}$ are given by (the real subset $\mathcal{G}_1$ was described in (2.2)),

$$\hat{\mathcal{G}} = \left\{ \frac{as-b}{cs-d} \ : \ \mathrm{Re}(a^*b) \geq 0, \ \mathrm{Re}(c^*d) \geq 0, \ ad \neq bc \right\}. \tag{3.2}$$

Using this notation, we shall hereafter simply write

$$\mathcal{GP}_g := \mathcal{G}_{g_+,\ g_-,\ g_o} \cap \mathcal{GP}.$$

By Theorem 1.1, for given $g \in \overline{\mathcal{G}_+}$ this can be equivalently written as

$$\mathcal{GP}_g = \{ \ gpg^{\#} \ : \ p \in \mathcal{P} \}. \tag{3.3}$$

For a given $g \in \overline{\mathcal{G}_+}$, the set $\mathcal{GP}_g$ is a replica of $\mathcal{P}$. Nevertheless, the picture in $\mathcal{GP}_g$ is richer.

Example 3.1. We here illustrate two properties of the set $\mathcal{GP}_g$ (3.3), where $g \in \overline{\mathcal{G}_+}$ is given:
(a) If $\psi_j = gp_jg^{\#}$ with g fixed, $\deg p_1 > \deg p_2$, does not always imply $\deg \psi_1 > \deg \psi_2$.
(b) In this set, it is only in $\mathbb{C}_+$ that the poles and zeroes are fixed. On $i\mathbb{R}$ they are almost prescribed, and in $\mathbb{C}_-$ they are not fixed.
Take $g(s) = \frac{s-2}{s}$ ($g \in \mathcal{G}_1$, see (2.2)). Thus, $\mathcal{GP}_g = \left\{ \frac{s^2-4}{s^2}p(s) \ : \ p \in \mathcal{P} \right\}$. We here mention, but five interesting samples,

$$
\begin{array}{ccc}
 & p(s) & \psi(s)=g(s)p(s)g^{\#}(s) \\
\text{(i)} & \frac{s}{s+2} & \frac{s-2}{s} \\
\text{(ii)} & \frac{s}{(s+2)^2} & \frac{s-2}{s(s+2)} \\
\text{(iii)} & 1 & \frac{s^2-4}{s^2} \\
\text{(iv)} & \frac{s+2}{s} & \frac{(s+2)^2(s-2)}{s^3} \\
\text{(v)} & \frac{s(s+2i)}{s+i} & \frac{(s^2-4)(s+2i)}{s(s+i)}
\end{array}.
$$

(a) These five functions are ordered so that the degree of $\psi_j(s)$ is non-decreasing. In contrast, the degree of the corresponding $p_j(s)$ "fluctuates".
(b) In $\mathbb{C}_+$, there is always a zero with a unit multiplicity at $s=+2$.
On $i\mathbb{R}$, there is a pole at the origin. Its generic multiplicity is two, but it may also be one or three (i.e. at the origin p has a pole e.g. (iv), no pole nor zero e.g. (iii), or a zero e.g. (i), (ii), (v) respectively). ψ may have additional poles or zeroes, see e.g. (v).
In $\mathbb{C}_-$ poles and zeroes are not fixed, see e.g. the point $s=-2$. $\square$

Using the notation of (3.1) Theorem 1.1 may be formulated as saying that having $\psi\in\mathcal{GP}$ is equivalent to $\psi(s)=g(s)p(s)g^{\#}(s)$ for some $g\in\overline{\mathcal{G}_+}$ and some $p\in\mathcal{P}$. Thus, we can use the last result to introduce a *convex* partitioning of all $\mathcal{GP}$ functions. The proof is immediate and thus omitted.

Observation 3.2. *Let $\mathcal{GP}_g$ be as in* (3.3). *Then,*

(i) *$\mathcal{GP}_g$ is a convex sub-cone of $\mathcal{GP}$.*
(ii) *For $g_1,g_2\in\overline{\mathcal{G}_+}$ $\mathcal{GP}_{g_2}=\left(\frac{g_2}{g_1}\right)\mathcal{GP}_{g_1}\left(\frac{g_2}{g_1}\right)^{\#}$.*
(iii) *Let $g_1,g_2\in\overline{\mathcal{G}_+}$ be so that $g_2\not\equiv cg_1$, for some constant c, then, $\mathcal{GP}_{g_1}\cap\mathcal{GP}_{g_2}=\{0\}$.*
(iv) *$\mathcal{GP}=\bigcup_{g\in\overline{\mathcal{G}_+}}\mathcal{GP}_g$.*
(v) *$(\mathcal{GP}_g)^{-1}=\mathcal{GP}_{(g^{\#})^{-1}}$.*

Proof. Items (i), (ii) and (v) are immediate from (3.3). Item (iv) follows from Theorem 1.1 along with (3.3).
As to item (iii), assume that there exits ψ within $\mathcal{GP}_{g_1}\cap\mathcal{GP}_{g_2}$ for some $g_1,g_2\in\overline{\mathcal{G}_+}$. We shall find it convenient to factorize $g_j=g_{o,j}g_{+,j}$ with $j=1,2$. where $g_{o,1},g_{o,2}\in\mathcal{G}_o$, see (2.4), and $g_{+,1},g_{+,2}\in\mathcal{G}_+$, see (2.1). As poles and zeroes of ψ in $\mathbb{C}_+$ are uniquely determined by g_+, without loss of generality one can write $g_1=g_{o,1}g_+$ and $g_2=g_{o,2}g_+$ for some $g_+\in\mathcal{G}_+$.
Next assume that for $j=1,2$ and some $r\in\mathbb{R}$, $g_{o,j}(s)$ have factors $(s-ir)^{m_j}$ and the corresponding $p_j(s)$ have factors $(s-ir)^{l_j}$, where m_j,l_j are integers (not necessarily positive). Then in $\psi_j(s)$ the respective factors are

$$(s-ir)^{m_j}(s-ir)^{l_j}\left((s-ir)^{m_j}\right)^{\#}=(-1)^{m_j}(s-ir)^{2m_j+l_j}.$$

This implies that: (i) $2m_1+l_1=2m_2+l_2$ and (ii) $m_1-m_2=2k$ for some integer k. Namely, $l_2-l_1=2(m_1-m_2)=4k$. Now recall that on the imaginary axis poles and zeroes of positive functions are simple, see e.g. [8, Theorem 2.2], i.e. $1\geq|l_j|$. This implies that $2\geq|l_2-l_1|$. But, $m_1\neq m_2$ implies that $|l_2-l_1|\geq 4$. Hence, one can now conclude that $m_1 = m_2$, $l_1 = l_2$ and since r was arbitrary (up to a constant) $g_{0,1} = g_{0,2}$, which in turn implies $g_1 = g_2$ (up to a constant) and the proof is complete. $\square$

This partitioning of $\mathcal{GP}$ functions rightfully seems straightforward. In contrast, at the end of Section 6, we show that partitioning of $\mathcal{GB}$, generalized bounded functions (or generalized Schur functions) in the spirit of Observation 3.2, can not be easily mimicked.

For a given $g \in \overline{\mathcal{G}_+}$ we now wish to identify minimal degree functions within $\mathcal{GP}_g$.

Proposition 3.3. *The following is true.*

(i) $g \in \overline{\mathcal{G}_+}$ *can always be factored as* $g(s) = c\prod\limits_{j=1}^{q}\psi_j(s)$ *with* $\psi_j^{\#}(s)$ *positive, see* (2.6)*, and* $c \in \mathbb{C}$.

(ii) *Among all possible factorizations of* $g \in \overline{\mathcal{G}_+}$ *let us choose* $g(s) = g_1(s)g_2(s)$ *so that* $g_1^{\#}(s)$ *is positive and* $\deg(g_1)$ *is maximal.*
Then, $gg_2^{\#}$ *is the minimal degree function in* $\mathcal{GP}_g$.

Proof : Item (i) is immediate from item (i) in Observation 2.4. Specifically, $g \in \overline{\mathcal{G}_+}$ can always be written as $g = \frac{\prod\limits_j (s-z_j)}{\prod\limits_k (s-\pi_k)}$ with $z_j, \pi_k \in \overline{\mathbb{C}_+}$. Note now that $(s-z_j)^{\#} = -(s+z_j^*)$ and $\frac{1}{(s-\pi_k)^{\#}} = \frac{-1}{s+\pi_k^*}$. Namely, up to sign change, this $g^{\#}$ is a product of degree one positive functions.
Item (ii) stems from item (i) and (3.3) noting that $\psi \in \mathcal{GP}_g$ can always be written as,

$$\psi(s) = g(s)p(s)g^{\#}(s) = g_1(s)g_2(s)p(s)g_1^{\#}(s)g_2^{\#}(s),$$

so that $g_1^{\#}(s)$ is positive, see e.g. Example 2.1. Thus, to reduce the degree of the above $\psi(s)$ choose $p(s) = \frac{1}{g_1^{\#}(s)}$ so that

$$\psi(s) = g_1(s)g_2(s)p(s)g_1^{\#}(s)g_2^{\#}(s)_{|_{p=\frac{1}{g_1^{\#}}}} = g_1(s)g_2(s)g_2^{\#}(s) = g(s)g_2^{\#}(s)$$

and the construction is complete. $\square$

The above construction is illustrated in part (a) of Example 3.1.

Within the set $\mathcal{P}$, the Nevanlinna-Pick interpolation problem is classical, see e.g. [14, Chapter 18], [48]. Within $\mathcal{GP}$, variants of this interpolation problem are well studied, see e.g. [3], [10], [5], [18], [29, Section 3] and [4] for generalized Nevanlinna functions and for generalized Schur functions see e.g. [13], [15], [19], and [30] and [2]. Nonetheless from computational point of view the procedure is involved. As an intermediate step, in the following example we illustrate the fact that within the set $\mathcal{GP}_g$, namely when $g \in \overline{\mathcal{G}_+}$ is fixed, the Nevannlin-Pick interpolation problem reduces to the classical version within the set $\mathcal{P}$, which is computationally well established.

Example 3.4. We here illustrate a Nevanlinna-Pick interpolation scheme within the set $\mathcal{GP}_g$. We look for $\psi \in \mathcal{GP}_g$ so that

$$\psi(s)_{|_{s=1}} = 1 \qquad \text{and} \qquad \psi(s)_{|_{s=2}} = 4.$$

(As the associated Pick matrix is $\begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 2 \end{pmatrix}$, its determinant is negative, so there is no $\psi \in \mathcal{P}$).

Take[6] $g(s) = \frac{4}{7-3s}$ i.e. a right half plane pole at $s = \frac{7}{3}$ and a zero at infinity. Thus, $g(s)g^{\#}(s) = \frac{16}{49-9s^2}$. First denote

$$w_1 := g(s)g^{\#}(s)_{|s=1} = \tfrac{2}{5} \qquad \text{and} \qquad w_2 := g(s)g^{\#}(s)_{|s=2} = \tfrac{16}{13} \ .$$

Exploiting the $\mathcal{GP}_g$ structure, see (3.3), we actually seek $p \in \mathcal{P}$ so that

$$p(s)_{|s=1} = \tfrac{1}{w_1} = \tfrac{5}{2} \qquad \text{and} \qquad p(s)_{|s=2} = \tfrac{4}{w_2} = \tfrac{13}{4} \ .$$

But this is a classical Nevanlinna-Pick problem and the resulting Pick matrix is, $\Pi = \begin{pmatrix} \frac{5}{2} & \frac{23}{12} \\ \frac{23}{12} & \frac{13}{8} \end{pmatrix}$.
As this Π is positive definite, there are infinitely many solutions (using Π, they can all be parameterized, see e.g. [14, Chapter 18]).
Take for instance two degree three interpolating functions, $p_a(s) = \frac{3s(s^2+9)}{4(s^2+2)}$ and $p_b(s) = \frac{9}{4}\left(1 + \frac{s^3}{3(s^2+2)}\right)$. The resulting interpolating functions in $\mathcal{GP}_g$ are

$$\begin{array}{rclcl} \psi_a(s) & = & g(s)p_a(s)g(s)^{\#} & = & \frac{12(s^2+9)}{(s^2+2)(49-9s^2)} \ , \\ \psi_b(s) & = & g(s)p_b(s)g(s)^{\#} & = & \frac{12(s^3+3s^2+6)}{(s^2+2)(49-9s^2)} \ . \end{array}$$

Both $\psi_a(s)$ and $\psi_b(s)$ are of degree four.
Recall that the set of interpolating functions is convex. Take for example

$$\psi_c(s) := \tfrac{2}{9}\psi_a(s) + \tfrac{7}{9}\psi_b(s) = g(s) = \frac{4}{7-3s} \ ,$$

to obtain a *unity degree* interpolating function within $\mathcal{GP}_g$. From item (ii) in Proposition 3.3 it follows that in fact this is the minimal degree function within $\mathcal{GP}_g$ and in particular the minimal degree interpolating function. □

4. Odd functions - a sub**cic** of generalized positive functions

As already mentioned, it is easy to see that the set of $\mathcal{GP}$ functions is closed under positive scaling, summation and inversion, i.e. a **cic**. In the previous section we introduced a partitioning of this **cic** into (infinitely many non invertible) convex cones of the form $\mathcal{GP}_g$. We now explore a partitioning of each generalized positive function into even and odd parts. It turns out that the sets of even and odd generalized positive functions are two sub**cic**s of $\mathcal{GP}$.

Abusing the real case terminology, for a given rational function $f(s)$ we shall define the *even* and *odd* parts as

$$f_{\text{even}}(s) := \tfrac{1}{2}\left(f(s) + f^{\#}(s)\right) \qquad f_{\text{odd}}(s) := \tfrac{1}{2}(f(s) - f^{\#}(s)). \tag{4.1}$$

Then, we also define the sets of all even and all odd functions,

$$\mathcal{E}ven := \{\ f(s)\ :\ f = f_{\text{even}}\ \} \qquad \mathcal{O}dd := \{\ f(s)\ :\ f = f_{\text{odd}}\ \}. \tag{4.2}$$

The following observations are almost obvious, they are stated for a comparison in the sequel.

Proposition 4.1. *Let the sets* $\mathcal{E}ven$ *and* $\mathcal{O}dd$ *be as in* (4.1) *and* (4.2).

[6] Recall that we assume g is prescribed.

(i) $\mathcal{E}ven$ and $\mathcal{O}dd$ *are convex invertible cones,* **cic**s *of rational functions.*
(ii) *Let* f, g *be rational functions. If* $g \in \mathcal{E}ven$ *then,*

$$(fg)_{\text{even}} = f_{\text{even}}g \qquad (fg)_{\text{odd}} = f_{\text{odd}}g.$$

Conversely, if $(fg)_{\text{even}} = f_{\text{even}}g$ *and* $f \not\equiv 0$*, then* $g \in \mathcal{E}ven$*.*

Proof (i) Indeed, positive scaling and summation are obvious. As to inversion note that if $f(s) = \frac{n(s)}{d(s)}$, with n, d polynomials, then $f \in \mathcal{E}ven$ is equivalent to $nd^{\#} = n^{\#}d$, which in turn means, $f^{-1} \in \mathcal{E}ven$. The reasoning for $f \in \mathcal{O}dd$ is similar and thus omitted.
(ii) A straightforward calculation shows that having $(fg)_{\text{even}} = f_{\text{even}}g$ is equivalent to $f^{\#}g = f^{\#}g^{\#}$, which in turn for $f \not\equiv 0$ means $g \in \mathcal{E}ven$. □

From item (i) in Proposition 2.2 and item (i) in Proposition 4.1 it respectively follows that $\mathcal{P}$ and $\mathcal{O}dd$ are **cic**s. Recall that a non-empty intersection of **cic**s is a **cic**, see [27, Observation 2.1]. Thus, of particular interest is the sub**cic** of all positive odd rational functions $\mathcal{PO} := \mathcal{P} \cap \mathcal{O}dd$. The real subset of $\mathcal{PO}$ functions are sometimes called "lossless", or "Foster" and they correspond to L-C circuits, see e.g. [12], [17].

$\mathcal{PO}$ functions can be parameterized, see e.g. [17, 5.13], as

$$\mathcal{PO} := \left\{ p(s) = ir_o + a_o s + \sum_{j \geq 1} \frac{a_j}{s - ir_j} \ : \ a_o \geq 0, \ a_j > 0, \ r_j \in \mathbb{R} \right\}. \tag{4.3}$$

Combining Theorem 1.1 together with (4.3) we have the following.

Proposition 4.2. *Let the set of odd functions, $\mathcal{O}dd$, be as in* (4.1) *and* (4.2)*. The following statements are true.*

(i) $\mathcal{O}dd = \mathcal{GP} \cap \mathcal{O}dd$.
(ii) $\psi \in \mathcal{O}dd$ *if and only if* ψ *maps the imaginary axis to itself.*
(iii) *The set* $\mathcal{O}dd$ *is closed (excluding the zero function) under:*
(a) real scaling, (b) addition, (c) inversion (d) composition and (e) the product of an odd number of elements, i.e. $\prod_{j}^{2m+1} \psi_j(s)$ *with* $\psi_j \in \mathcal{O}dd$, $m = 0, 1, \ldots$
(iv) $\psi \in \mathcal{O}dd$ *can always be written as*

$$\psi(s) = g(s) \left(ir_o + a_o s + \sum_{j \geq 1} \frac{a_j}{s - ir_j} \right) g^{\#}(s),$$

with $a_o \geq 0$, $a_j > 0$, $r_j \in \mathbb{R}$ *and* $g \in \overline{\mathcal{G}}_+$ (3.1).

Proof : (i), (ii) Recall that for an arbitrary function f,

$$\left(f^{\#}(s) \right)_{|_{s \in i\mathbb{R}}} = \left(f(s)_{|_{s \in i\mathbb{R}}} \right)^{*}. \tag{4.4}$$

Thus, whenever $f \in \mathcal{O}dd$ it maps $i\mathbb{R}$ to itself and hence it is a $\mathcal{GP}$ function. Next, from (4.1) and (4.4) it follows that f_{even}, the even part of an arbitrary f, maps $i\mathbb{R}$ to

$\mathbb{R}$. Thus, if $f = f_{\text{even}} + f_{\text{odd}}$ maps $i\mathbb{R}$ to $i\mathbb{R}$, it follows that $f_{\text{even}}(s)_{|_{s\in i\mathbb{R}}} \equiv 0$, which in turn means that $f_{\text{even}}(s) \equiv 0$.
Item (iii) follows from item (ii).
Item (iv) follows from item (i) together with (4.3) and Theorem 1.1. $\square$

We now explore the structure of $\mathcal{O}dd$ from a geometric point of view.

Observation 4.3. *For a given set of rational functions* G $(G \neq \{0\})$*, denote by* F *and* H *the following sets,*

$$F := \{g^{-1} \ : \ g \in G\} \qquad\qquad H := \{\ h = g^2 \ : \ g \in G\}$$

$G \subset \mathcal{O}dd$ *if and only if*

$$\text{Re}\left((f(s)h(s))_{|_{s\in i\mathbb{R}}}\right) \equiv 0 \qquad\qquad \forall f \in F \quad \forall h \in H. \tag{4.5}$$

Proof : The claim relies on Proposition 4.2 item (ii). First, if $g_a(s)$ and $g_b(s)$ are odd functions then $\text{Im}\left(g_a^2(s)_{|_{s\in i\mathbb{R}}}\right) \equiv 0$ and $\text{Re}\left(g_b^{-1}(s)_{|_{s\in i\mathbb{R}}}\right) \equiv 0$. Thus, $\text{Re}\left(\left(g_a^2(s)g_b^{-1}(s)\right)_{|_{s\in i\mathbb{R}}}\right) \equiv 0$, i.e. (4.5) is satisfied.

Conversely, if (4.5) is satisfied for all $f \in F$ and all $h \in H$, it in particular holds for $f = g^{-1}$ and $h = g^2$ with the same g. This implies that $fh = g$, which means $g \in \mathcal{O}dd$. $\square$

Note that for a pair of functions f, h one can define a function-valued inner product $< f, h > := \text{Re}\left(f(s)h^{\#}(s)\right)$ (in the sense that $< rf, h >= r < f, h >$ for $r \in \mathbb{R}$). Now, as $(f(s)h(s))_{|_{s\in i\mathbb{R}}} = \left(f(s)h^{\#}(s)\right)_{|_{s\in i\mathbb{R}}}$, equation (4.5) can be written as $< f, h >_{|_{s\in i\mathbb{R}}} \equiv 0$. Namely, one can interpret Observation 4.3 as saying that $0 \not\equiv g \in \mathcal{O}dd$ is equivalent to having the restriction to the imaginary axis of g and g^2, orthogonal in the above inner product.

Based on Observation 3.2 and Proposition 4.2 we can now introduce a *convex* partitioning of all $\mathcal{O}dd$ functions. For a fixed $g \in \overline{\mathcal{G}_+}$ the set $\mathcal{GP}_g$ was defined in (3.3). We now consider the set odd functions in it:

$$\mathcal{O}dd_g := \mathcal{GP}_g \cap \mathcal{O}dd.$$

This subset is given by

$$\mathcal{O}dd_g = \bigcup_{a_o\geq 0, a_j>0, r_j\in\mathbb{R}} g\left(ir_o + a_o s + \sum_{j\geq 1} \frac{a_j}{s - ir_j}\right) g^{\#}. \tag{4.6}$$

Observation 4.4. *Let* $\mathcal{O}dd_g$ *be as in* (4.6)*. Then,*

$$\mathcal{O}dd = \bigcup_{g\in\overline{\mathcal{G}_+}} \mathcal{O}dd_g \ .$$

Recall that in [48] it was shown that (up to possibly compromising the minimal degree interpolating function) without loss of generality the classical Nevanlinna-Pick interpolation may be confined to $\mathcal{P} \cap \mathcal{O}dd$. Similarly, for a fixed $g \in \overline{\mathcal{G}_+}$ interpolation of $\mathcal{GP}_g$ functions may be confined to $\mathcal{O}dd_g$ functions. This is illustrated next.

Example 4.5. In Example 3.4 we studied a variant of Nevanlinna-Pick interpolation problem within the set $\mathcal{GP}_g$ with $g(s)=\frac{4}{7-3s}$, and looked for a function ψ so that

$$\psi(s)_{|_{s=1}}=1 \qquad \text{and} \qquad \psi(s)_{|_{s=2}}=4.$$

It turned out that this was equivalent for a classical Nevanlinna-Pick interpolation problem of searching $p\in\mathcal{P}$ so that

$$p(s)_{|_{s=1}}=\tfrac{5}{2} \qquad \text{and} \qquad p(s)_{|_{s=2}}=\tfrac{13}{4}\ .$$

Following the above analysis, looking for $\psi\in\mathcal{O}dd_g$ is equivalent to restricting the classical Nevanlinna-Pick search for $p\in\mathcal{P}\cap\mathcal{O}dd$. From [48] it follows that this restriction, does not limit the solvability of the problem. In fact, there are still infinitely many solutions. We here mention two solutions $p_a(s)=\frac{3s(s^2+9)}{4(s^2+2)}$ (from Example 3.4) and $p_d(s)=\frac{1}{6}(8s+\frac{7}{s})$. The resulting interpolating functions in $\mathcal{O}dd_g$ are

$$\begin{array}{lllll}\psi_a(s) & = & g(s)p_a(s)g^{\#}(s) & = & \frac{12s(s^2+9)}{(s^2+2)(49-9s^2)},\\ \psi_d(s) & = & g(s)p_d(s)g^{\#}(s) & = & \frac{8(8s^2+7)}{3s(49-9s^2)}\ .\end{array}$$

The function $\psi_a(s)$ is of degree four while $\psi_d(s)$ is of degree three.

Following item (iii)(a) in Proposition 4.2, taking $r\in\mathbb{R}$ as parameter, $r\psi_a(s)+(1-r)\psi_d(s)$ forms a variety of interpolating functions within the same $\mathcal{O}dd_g$.

Finally, a straightforward use of (4.3) reveals that the above ψ_d is a minimal degree interpolation function in $\mathcal{O}dd_g$. □

5. Even generalized positive functions

In Proposition 4.2 we showed that all *odd* functions are generalized positive functions and characterized them. We now characterize $\mathcal{GPE}:=\mathcal{GP}\cap\mathcal{E}ven$, the subset of even functions within $\mathcal{GP}$. We shall use again the convention that $\prod\limits_{1}^{0}=1$.

Proposition 5.1. *The following are equivalent*

(i) $\psi\in\mathcal{GPE}$.

(ii) $\psi(s)=c\cdot\frac{\prod\limits_{j=1}^{m}\left(1-\alpha_j(1+(s-i\beta_j)^2)\right)}{\prod\limits_{k=1}^{n}(1-\gamma_k(1+(s-i\delta_k)^2))}$ *with* $c>0$, $\alpha_j,\gamma_k\in(0,\ 1]$, $\beta_j,\delta_k\in\mathbb{R}$.

(iii) $\psi(s)=g(s)g^{\#}(s)$ *for some rational* $g(s)$.

(iv) $\psi(s)$ *maps* $i\mathbb{R}$ *to* $\overline{\mathbb{R}_+}$.

(v) $\psi\in\mathcal{GP}$ *maps* $i\mathbb{R}$ *to* $\mathbb{R}$.

Proof Any even function maps $i\mathbb{R}$ to $\mathbb{R}$, thus (i) implies (v).
$(v)\iff(iv)$. From (v) it follows that ψ maps $i\mathbb{R}$ to $\overline{\mathbb{C}_+}\cap\mathbb{R}$, thus in fact to $\overline{\mathbb{R}_+}$, so this part is established.
$(iv)\iff(iii)$. From Theorem 1.1 together with with fact that $g(s)\psi(s)g^{\#}(s)_{|_{s\in i\mathbb{R}}}=g(s)\psi(s)\left(g(s)\right)^*$, it follows that $\psi(s)=g(s)p(s)g^{\#}(s)$, where $p\in\mathcal{P}$ maps $i\mathbb{R}$ to $\overline{\mathbb{R}_+}$. Now, up to non-negative scaling, $p(s)\equiv1$ in whole $\mathbb{C}$, is the only function which achieves that.

$(iii) \implies (ii)$. Denote, $g(s)=\tilde{c}\frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{k=1}^{n}(s-p_k)}$ with $\tilde{c}, z_j, p_k \in \mathbb{C}$. Thus,

$$\begin{aligned}
g(s)g(s)^{\#} &= |\tilde{c}|^2\cdot\frac{\prod_{j=1}^{m}\left(-s^2+s(z_j-z_j^*)+|z_j|^2\right)}{\prod_{k=1}^{n}\left(-s^2+s(p_k-p_k^*)+|p_k|^2\right)}\\
&= |\tilde{c}|^2\cdot\frac{\prod_{j=1}^{m}\left((\mathrm{Re}(z_j))^2-(s-i\mathrm{Im}(z_j))^2\right)}{\prod_{k=1}^{n}\left((\mathrm{Re}(p_k))^2-(s-i\mathrm{Im}(p_k))^2\right)}\\
&= |\tilde{c}|^2\cdot\frac{\prod_{j=1}^{m}\left(1+(\mathrm{Re}(z_j))^2-(1+(s-i\mathrm{Im}(z_j))^2)\right)}{\prod_{k=1}^{n}\left(1+(\mathrm{Re}(p_k))^2-(1+(s-i\mathrm{Im}(p_k))^2)\right)}\\
&= |\tilde{c}|^2\cdot\frac{\prod_{j=1}^{m}\left(1+(\mathrm{Re}(z_j))^2\right)}{\prod_{k=1}^{n}\left(1+(\mathrm{Re}(p_k))^2\right)}\cdot\frac{\prod_{j=1}^{m}\left(1-\frac{1}{1+(\mathrm{Re}(z_j))^2}(1+(s-i\mathrm{Im}(z_j))^2)\right)}{\prod_{k=1}^{n}\left(1-\frac{1}{1+(\mathrm{Re}(p_k))^2}(1+(s-i\mathrm{Im}(p_k))^2)\right)},
\end{aligned}$$

Denoting $c:=|\tilde{c}|^2\cdot\frac{\prod_{j=1}^{m}\left(1+(\mathrm{Re}(z_j))^2\right)}{\prod_{k=1}^{n}\left(1+(\mathrm{Re}(p_k))^2\right)}$, $\alpha_j=\frac{1}{1+(\mathrm{Re}(z_j))^2}$, $\beta_j=\mathrm{Im}(z_j)$, $\gamma_k=\frac{1}{1+(\mathrm{Re}(p_k))^2}$ and $\delta_k=\mathrm{Im}(p_k)$ completes the construction.
As trivially $(ii) \implies (i)$, the claim is established. □

From Proposition 5.1 it follows that $\psi(s)$ is in $\mathcal{GPE}$ may be characterized as $\psi=gg^{\#}$. Note however that this factorization is non-unique, namely, one can have $g_1\neq g_2$ and still $g_1g_1^{\#}=g_2g_2^{\#}$. A characterization of all these factorizations is given in [34].
One can now state several properties of $\mathcal{GPE}$, the subset of even functions within $\mathcal{GP}$.

Proposition 5.2. *Let ψ be a rational function. The following statements are true.*

(i) $\psi\in\mathcal{GP}\Longleftrightarrow\psi_{\mathrm{even}}\in\mathcal{GP}$.
(ii) $\mathcal{GPE}$ *is a sub***cic** *of* $\mathcal{GP}$.
(iii) $\mathcal{GP}\cap\mathcal{E}ven$ *is a multiplicative group.*
(iv) $g\in\mathcal{GPE} \iff g\cdot\mathcal{GP}\subset\mathcal{GP}$.
(v) *For arbitrary $\mathcal{GPE}$ functions $g_1(s)g_1^{\#}(s)$, … , $g_m(s)g_m^{\#}(s)$, there always exists $\hat{g}\hat{g}^{\#}\in\mathcal{GPE}$ so that*

$$\sum_{j=1}^{m}g_j(s)g_j^{\#}(s)=\hat{g}(s)\hat{g}^{\#}(s).$$

Moreover, one can always take $\hat{g}\in\overline{\mathcal{G}_+}$.

(vi) *Let* ψ *be the composition function* $\psi(s) := p(g(s))$ *where* $g \in \mathcal{GPE}$ *and* $p \in \mathcal{P}$. *Then,* $\psi \in \mathcal{GP}$. *If in addition* p *leaves the real axis invariant (e.g.* p *is real), then* $\psi \in \mathcal{GPE}$.

Proof (i) This follows from the fact that $\psi \in \mathcal{GP} \Longleftrightarrow \psi^{\#} \in \mathcal{GP}$.
(ii) Recall that the set $\mathcal{GP}$ is a Convex Invertible Cone, the claim is immediate from Proposition 5.1(iv). Alternatively, $\mathcal{GPE}$ is a non-empty intersection of two **cic**s and thus a sub**cic**, [27, Observation 2.1].
(iii) Is immediate from items (ii) or (iv) in Proposition 5.1.
(iv) If f is a generalized positive function, from item (i) we know that $f_{\text{even}} \in \mathcal{GP}$. Now if $g \in \mathcal{GPE}$ from item (ii) it follows that $(f_{\text{even}}g) \in \mathcal{GPE}$. Next, from Proposition 4.1(ii) it follows that $(fg)_{\text{even}} = f_{\text{even}}g$ and hence, $(fg)_{\text{even}} \in \mathcal{GPE}$. Using again item (i) implies, $(fg) \in \mathcal{GP} \cap \mathcal{E}ven$.
For the other direction all we need to show is that if g is a non-even function within $\mathcal{GP}$, one can always find $f \in \mathcal{GP}$ so that $(fg) \notin \mathcal{GP}$. Indeed assume that g is so that $g(s)_{|_{s=i\omega_o}} = a + ib$ with $a \geq 0$ and $0 \neq b \in \mathbb{R}$ for some $\omega_o \in \mathbb{R}$. Then, taking the (odd) $\mathcal{GP}$ function $f = b(\omega_o s + i)$ reveals that $\mathrm{Re}\left(f(s)g(s)_{|_{s=i\omega_o}}\right) = -b^2(1+\omega_o^2)$ and thus $gf \notin \mathcal{GP}$.
(v) Is immediate from item (ii) here together with item (iii) in Proposition 5.1.
(vi) By construction g maps $i\mathbb{R}$ to $\overline{\mathbb{R}_+}$ and in turn p maps $\overline{\mathbb{R}_+}$ to $\overline{\mathbb{C}_+}$, thus, ψ maps $i\mathbb{R}$ to $\overline{\mathbb{C}_+}$. If in addition p maps $\overline{\mathbb{R}_+}$ to $\overline{\mathbb{R}_+}$, ψ maps $i\mathbb{R}$ to $\overline{\mathbb{R}_+}$, so the claim is established. $\square$

The convexity of the set $\mathcal{GP}$ and of its subset of $\mathcal{GPE}$ functions, see item (v) in Proposition 5.2, may be exploited to introduce a straightforward scheme of solving the Nevanlinna-Pick interpolation problem.

Example 5.3. For simplicity we consider the real case.
a. We first look for a real polynomial $gg^{\#} \in \mathcal{GPE}$ so that

$$g(s)g(s)^{\#}{}_{|_{s=\pm 1}} = 1, \qquad g(s)g(s)^{\#}{}_{|_{s=\pm 2}} = 4, \qquad g(s)g(s)^{\#}{}_{|_{s=\pm 3}} = 9. \tag{5.1}$$

(Note that $f(s) = s^2$ is an interpolating polynomial, but only $-f \in \mathcal{GPE}$).
Consider the following real $\mathcal{GPE}$ polynomials,

$$\begin{array}{c} g_1(s)g_1^{\#}(s) = (4-s^2)(9-s^2)(\tfrac{25}{24} - s^2) \\ g_2(s)g_2^{\#}(s) = (1-s^2)(9-s^2)\tfrac{1}{9}\,(\tfrac{8}{5} - s^2) \\ g_3(s)g_3^{\#}(s) = (1-s^2)(4-s^2)\tfrac{1}{360}\,(90-s^2) \end{array}$$

It is easy to verify that,

$$\begin{array}{ccccc} s & = & \pm 1 & \pm 2 & \pm 3 \\ g_1g_1^{\#} & = & 1 & 0 & 0 \\ g_2g_2^{\#} & = & 0 & 4 & 0 \\ g_3g_3^{\#} & = & 0 & 0 & 9 \end{array}$$

Thus, using item (v) in Proposition 5.2, $gg^{\#} = g_1g_1^{\#} + g_2g_2^{\#} + g_3g_3^{\#}$ is a real $\mathcal{GPE}$ polynomial satisfying (5.1).

b. Using part **a**, we now look for a real polynomial $\psi \in \mathcal{GP}$ so that

$$\psi(s)_{|s=1} = 1, \qquad \psi(s)_{|s=2} = 4, \qquad \psi(s)_{|s=3} = 9, \tag{5.2}$$

(with no constraints on $\psi(s)_{|s=-1,-2,-3}$). For $j = 1, 2, 3$ we now construct real $\mathcal{GP}$ polynomials of the form $\psi_j = g_j p_j g_j^{\#}$ with $g_j g_j^{\#}$ from part **a** and $p_j \in \mathcal{P}$ are of the form $\frac{a_j+j}{a_j+s}$ with $a_j > 0$ is so that one of the roots of $g_j g_j^{\#}$ is canceled. Indeed take,

$$p_1(s) = \frac{\frac{5}{2\sqrt{6}}+1}{\frac{5}{2\sqrt{6}}+s}\,, \qquad p_2(s) = \frac{\frac{2\sqrt{2}}{\sqrt{5}}+2}{\frac{2\sqrt{2}}{\sqrt{5}}+s}\,, \qquad p_3(s) = \frac{3\sqrt{10}+3}{3\sqrt{10}+s}\,.$$

Thus, one obtains,

$$\begin{array}{c}\psi_1(s) = (4-s^2)(9-s^2)(\frac{5}{2\sqrt{6}}+1)(\frac{5}{2\sqrt{6}}-s)\\ \psi_2(s) = (1-s^2)(9-s^2)\frac{2}{9}(\frac{\sqrt{2}}{\sqrt{5}}+1)(\frac{2\sqrt{2}}{\sqrt{5}}-s)\\ \psi_3(s) = (1-s^2)(4-s^2)\frac{1}{120}(\sqrt{10}+1)(3\sqrt{10}-s)\end{array}$$

It is easy to verify that,

$$\begin{array}{ccccc} s & = & 1 & 2 & 3\\ \psi_1 & = & 1 & 0 & 0\\ \psi_2 & = & 0 & 4 & 0\\ \psi_3 & = & 0 & 0 & 9. \end{array}$$

Thus, taking $\psi = \psi_1 + \psi_2 + \psi_3$ satisfies (5.2).

Roughly speaking, the simplicity of this scheme of constructing interpolating functions, comes on the expense of high degree. $\square$

As already mentioned, the fact that the set $\mathcal{P}$ is a convex invertible cone **cic**, is classical, see e.g. [17, 5.6] and item (i) in Proposition 2.2. Recall that real $\mathcal{P}$ functions are identified with the driving point impedance of R-L-C electrical circuits [12], [17], [20], [21]. In fact, in the framework of R-L-C electrical circuits the three **cic** operations of positive scaling, summation and inversion have the physical interpretation of transformer ratio, series connection of impedances and impedance/admittance duality, respectively. Moreover, recall that $\mathcal{P}\cap\mathcal{E}ven$ is associated with resistive circuits and $\mathcal{PO}$ with reactive networks (L-C). However, not every network can be realized as a series connection of a resistive and a reactive circuits. Namely, it is only over $\mathcal{GP}$ that the partitioning of a positive function into even and odd parts is always possible.

Recall that in contrast to (4.3), the set $\mathcal{P}\cap\mathcal{E}ven$ is almost empty, i.e. up to positive scaling it consists of a single function, $p(s) \equiv 1$. We now show that if one is interested in even-odd partitioning of functions, the set $\mathcal{GP}$ is closed, while its subset of positive functions is not. Namely, p_{even}, the even part of a positive function p, is either a non-negative constant or not a positive function. One can only guarantee that $p_{\rm even} \in \mathcal{GPE}$. This is illustrated next.

Example 5.4. Consider the positive (real) function: $\psi(s) = \frac{1}{1+s}$ defined in the whole $\mathbb{C}$. Then $p_{\rm even}(s) = \frac{1}{1-s^2}$ and $p_{\rm odd}(s) = \frac{-2s}{1-s^2}$ are so that $p_{\rm even} \in \mathcal{GPE}$ and $p_{\rm odd} \in \mathcal{GP}\cap\mathcal{O}dd$, but neither $p_{\rm even}$ nor $p_{\rm odd}$ are positive. $\square$

We conclude this section by introducing yet another factorization of $\mathcal{GP}$ functions through odd functions.

Observation 5.5. *$\psi \in \mathcal{GP}$ if and only if there exist $f, g \in \mathcal{O}dd$ so that $\psi_{\text{even}} = -f^2$ and $\psi_{\text{odd}} = g$.*

Proof : Since $\psi_{\text{odd}} \in \mathcal{O}dd$, we only need to show that $\psi \in \mathcal{GP}$ if and only if $\psi_{\text{even}} = -f^2$ for some $f \in \mathcal{O}dd$. Now from item (i) in Proposition 5.2 it follows that having $\psi \in \mathcal{GP}$ is equivalent to $\psi_{\text{even}} \in \mathcal{GP}$. From item (iv) in Proposition 5.1 this in turn is equivalent to ψ_{even} mapping $i\mathbb{R}$ to $\overline{\mathbb{R}_+}$. Using item (ii) from Proposition 4.2 completes the proof. $\square$

6. Generalized bounded functions

Recall that a function $f_b(s)$ is called *bounded*, denoted by $f_b \in \mathcal{B}$, (commonly the real case is addressed) if it analytically maps $\mathbb{C}_+$ to the closed unit disk, see e.g. [12, Chapter 7], [17, Section 6.5] and $f_{gb}(s)$ is *generalized bounded* $f_{gb} \in \mathcal{GB}$ if it maps $i\mathbb{R}$ to the closed unit disk, see e.g. [32]. It is well known that through the Cayley transform one can identify positive functions with bounded functions, namely

$$f_b(s) = \frac{1-p(s)}{1+p(s)} \qquad p \in \mathcal{P}, \qquad\qquad f_{gb}(s) = \frac{1-\psi(s)}{1+\psi(s)} \qquad \psi \in \mathcal{GP}. \tag{6.1}$$

Nevertheless, we here focus on the less obvious analogies. In Proposition 6.1 and Corollary 6.3 below we introduce two representations of $\mathcal{GB}$ functions.

Proposition 6.1. *A rational $f_{gb}(s)$ is a generalized bounded function if and only if it is of the form $f_{gb}(s) = f_b(s)/\beta(s)$, where $f_b \in \mathcal{B}$ and where $\beta(s)$ is a finite Blaschke product.*

Proof: We first note that since f_{gb} is bounded on the imaginary axis, all its singularities there are removable. Let $w_1, \ldots, w_\ell$ be the poles of f_{gb} in $\mathbb{C}_+$, and consider the function

$$f_b(s) = f_{gb}(s) \prod_{j=1}^{\ell} \frac{s - w_j}{s + w_j^*} \ .$$

The function f_{gb} is analytic and bounded by 1 in modulus in $\mathbb{C}_+$, as is seen for example by the maximum modulus principle, or by direct inspection. We thus have the result with

$$\beta(s) = \prod_{j=1}^{\ell} \frac{s - w_j}{s + w_j^*} \ .$$

The converse is clear. $\square$

One can characterize generalized bounded functions through the associated kernel.

Corollary 6.2. *A rational $f(s)$ is a generalized bounded function if and only if the kernel*

$$k_f(s, w) = \frac{1 - f(s)f(w)^*}{s + w^*}$$

has a finite number of negative squares in $\mathbb{C}_+ \setminus \{w_1, \ldots, , w_\ell\}$, where the w_j denote the poles of f in $\mathbb{C}_+$.

Proof: Assume that $f = f_{gb} = f_b/\beta$. As proved in a more general context in [7, Theorem 6.6, p. 132], one direction follows from the equality

$$k_{f_{gb}}(s,w) = \frac{1}{\beta(s)} \left\{ k_{f_b}(s,w) - k_\beta(s,w) \right\} \frac{1}{\beta(w)^*} \ ,$$

see for instance the formula on top of page 134 in [7].

The converse is just a particular case of the above mentioned result of Kreĭn Langer [38, Theorem 3.2]. A direct proof for the rational case can also be given, but will be omitted here. □

We now turn to another representation of $f_{gb} \in \mathcal{GB}$. Obviously, (generalized) bounded functions and (generalized) positive functions are related through the Cayley transform (6.1). We now introduce an adapted version of this characterization. To this end, recall (Proposition 5.2) that the set $\mathcal{GP} \cap \mathcal{E}ven$ is characterized by functions of the form $gg^{\#}$. From the above discussion one has the following.

Corollary 6.3. *A rational function $f_{gb}(s)$ is generalized bounded if and only if it admits a representation,*

$$f_{gb}(s) = \Big(g(s)g^{\#}(s) - p(s)\Big)\Big(g(s)g^{\#}(s) + p(s)\Big)^{-1} \ , \tag{6.2}$$

for some $p \in \mathcal{P}$ and some $g(s) \in \overline{G}_+$, (3.1)

Proof Indeed, from (1.2) and (6.1) it follows that $f_{gb} \in \mathcal{GB}$ can be written as,

$$\begin{array}{rcl} f_{gb} &=& (1-\psi)(1+\psi)^{-1} = \left(1 - gpg^{\#}\right)\left(1 + gpg^{\#}\right)^{-1} \\ &=& \left(g\left((g^{\#}g)^{-1} - p\right)g^{\#}\right)\left(g\left((g^{\#}g)^{-1} + p\right)g^{\#}\right)^{-1} \\ &=& \left((g^{\#}g)^{-1} - p\right)\left((g^{\#}g)^{-1} + p\right)^{-1}. \end{array}$$

Now, from Proposition 5.1 and Proposition 5.2 it follows that $\psi \in \mathcal{GP} \cap \mathcal{E}ven$ is equivalent to $\psi(s) = \left(g^{\#}(s)g(s)\right)^{-1}$, so up to inversion, the claim is established. □

It is interesting to compare Corollary 6.3 with f_{gb} in (6.1)

We conclude by pointing out that there is a structural difference between $\mathcal{GP}$ and $\mathcal{GB}$ functions. One may be tempted to try to mimic, in the framework of $\mathcal{GB}$ functions, the convex partitioning of sets of functions of the form $\mathcal{GB}_g$ in the spirit of (3.3) and Observations 3.2 and 4.4, sets where in $\mathbb{C}_+$ the poles and zeroes are fixed. However, unfortunately this is no longer true. This also prevents us from mimicking the interpolation over $\mathcal{GP}_g$ to $\mathcal{GB}_g$ functions.
Indeed, for a given g in (6.2) define the set $\mathcal{GB}$ function,

$$\mathcal{GB}_g := \left\{ \Big(g(s)g^{\#}(s) - p(s)\Big)\Big(g(s)g^{\#}(s) + p(s)\Big)^{-1} \ : \ p \in \mathcal{P} \right\}. \tag{6.3}$$

In the following example we show that in contrast to the set $\mathcal{GP}_g$ in (3.3), within the set $\mathcal{GB}_g$ neither the poles nor the zeroes in $\mathbb{C}_+$ are fixed.

Example 6.4. Fix in (6.3) $g \in \mathcal{G}_+$, see (2.1), namely, $g(s) = \frac{n}{d}$ with $n = \prod_{j=1}^{l}(s - z_j)$ and $d = \prod_{k=1}^{q}(s - \pi_k)$, where $z_1, \ldots, z_l$ and $\pi_1, \ldots, \pi_q$ are prescribed (not necessarily distinct) points in $\mathbb{C}_+$. Let now, $\tilde{n} = \prod_{j=1}^{l}(s - z_j - \epsilon_j)$ and $\tilde{d} = \prod_{k=1}^{q}(s - \pi_k - \delta_k)$ with $\epsilon_j \geq 0$, $\delta_k \geq 0$, $1 >> \sum_{j=1}^{l}\epsilon_j > 0$, $1 >> \sum_{k=1}^{q}\delta_k > 0$. Let $p_1 := \frac{n}{\tilde{n}}$ and $p_2 := \frac{\tilde{d}}{d}$. By construction, each of the functions p_1, p_2 analytically maps $\mathbb{C}_+$ to a neighborhood of the point $+1$ and hence both are positive. Thus both

$$\begin{array}{lcl} f_{gb,1} & := & \big(g(s)g^{\#}(s) - p_1(s)\big)\big(g(s)g^{\#}(s) + p_1(s)\big)^{-1} \\ f_{gb,2} & := & \big(g(s)g^{\#}(s) - p_2(s)\big)\big(g(s)g^{\#}(s) + p_2(s)\big)^{-1} \end{array}$$

are in $\mathcal{GB}_g$. However, in $\mathbb{C}_+$ they do not share the poles nor the zeroes. Indeed, substitution yields $f_{gb,1} = (\tilde{n}g^{\#} - d)(\tilde{n}g^{\#} + d)^{-1}$ and $f_{gb,2} = (ng^{\#} - \tilde{d})(ng^{\#} + \tilde{d})^{-1}$. $\square$

It should be pointed out that the above discussion reflects a property of $\mathcal{GB}_g$ functions, independent of the choice of the representation. Indeed, if the set in (6.3) is substituted by the analogous one, based on Proposition 6.1, a conclusion, similar to that of the above example, is reached.

It should be emphasized that rational generalized Nevanlinna functions, mapping the real axis to the upper half plane, admit a partitioning along the lines of Section 3. In contrast, rational generalized Schur functions mapping the unit circle to the unit disk, share the same difficulty as $\mathcal{GB}$ functions. This suggests that the complicated known scheme for solving Nevanlinna- Pick interpolation problem for generalized Schur functions, see e.g. [13], [15] and [30] and for the single point with derivatives version, [2], can not be simplified along the lines suggested in Example 3.4

7. Future research

In this work, part of ongoing research on $\mathcal{GP}$ functions, we concentrated on exploring structural properties this set. This opens the door for studying various questions and we here mention sample of those. First, in the framework of Nevanlinna-Pick interpolation problem of scalar rational $\mathcal{GP}$ functions.

- Explore the question of minimal degree interpolating functions.
- We conjecture that if the Pick matrix Π has m negative eigenvalues then there exists an interpolating function within a set $\mathcal{GP}_g$ where g has m poles or m zeroes in $\mathbb{C}_+$.
- Parameterize all $\mathcal{GPE}$ interpolating functions.
- Characterize the Nevanlinna-Pick interpolation problems solvable by $\mathcal{O}dd$ and by $\mathcal{GPE}$ functions.

One can then look for generalizations. For example,

- Formalize the extension of the study of $\mathcal{GP}_g$ functions, to the cases of: (i) not necessarily rational, (ii) matrix valued.

- Formalize the extension of the study of the even-odd partitioning of $\mathcal{GP}$ functions, to the cases of: (i) not necessarily rational, (ii) matrix valued.

References

[1] N. I. Akhiezer, *The classical moment problem*, Oliver & Boyd, London 1965.
[2] D. Alpay, T.Ya. Azizov, A. Dijksma H. Langer and G. Wanjala "A basic interpolation problem for generalized Schur functions and coisometric realizations", *Operator Theory: Advances and Applications*,Vol. 143, pp. 39-76, Birkhäuser Verlag, Basel, 2003.
[3] D. Alpay, V. Bolotnikov and A. Dijksma, "On the Nevanlinna-Pick interpolation problem for generalized Stieltjes functions", *Integ. Eq. & Op. Theory*, Vol. 30, pp. 379-408, 1998.
[4] D. Alpay, A. Dijksm, H. Langer, S. Reich and D. Shoikhet, "Boundary interpolation and rigidity for generalized Nevanlinna functions", *Math. Nach.*, Vol. 283, No. 3, pp. 335-364, 2010.
[5] D. Alpay, A. Dijksma, H. Langer and Y. Shondin, "The Schur transformation for generalized Nevanlinna functions: Interpolation and self-adjoint operator realizations", *Compl. anal. oper. theory*, Vol. 1, pp. 169-210, newblock Birkhäuser Verlag, Basel, 2007.
[6] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, *Operator theory: Advances and Applications*, Vol. 96 Birkhäuser Verlag, Basel, 1997.
[7] D. Alpay and H. Dym. "On applications of reproducing kernel spaces to the Schur algorithm and rational J-unitary factorization". In I. Gohberg, editor, *I. Schur methods in operator theory and signal processing*, *Operator Theory: Advances and Applications*, Vol. 18, pp. 89–159. Birkhäuser Verlag, Basel, 1986.
[8] D. Alpay and I. Lewkowicz, *An easy-to-compute factorization of rational generalized positive functions*, *Sys. Cont Lett*, Vol. 59, pp. 517-521, 2010.
[9] D. Alpay and I. Lewkowicz, "The Positive Real Lemma and Construction of all Realizations of Generalized Positive Rational Functions", *Sys. Cont Lett*, Vol. 60, pp. 985-993, 2011.
[10] A. Amirshadyan, and V.A. Derkach, "Interpolation in generalized Nevanlinna and Stieltjes classes", *J. Operator Theory*, Vol. 42, No. 1, pp. 145-188, 1999.
[11] B.D.O. Anderson and J. B. Moore, "Algebraic Structure of Generalized Positive Real Matrices", *SIAM J. Control*, Vol. 6, pp. 615-624, 1968.
[12] B.D.O. Anderson and S. Vongpanitlerd, *Networks Analysis and Synthesis, A Modern Systems Theory Approach*, Prentice-Hall, New Jersey, 1973.
[13] J. A. Ball, "Interpolation problems and Loewner types for meromorphic matrix functions" *Integ. Eq. & Op. Theory*, Vol. 6, pp. 804-840, 1983.
[14] J.A. Ball, I. Gohberg and L. Rodman, *Interpolation of rational matrix functions*, volume 44 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 1990.
[15] J.A. Ball and J.W. Helton, "Interpolation problems of Pick-Nevanlinna and Loewner types for meromorphic matrix functions: parameterization of the set of all solutions", *Integ. Eq. & Op. Theory*, Vol. 9, pp. 155-203, 1986.
[16] B.R. Barmish *New Tools for Robustness of Linear Systems*, Macmillan, 1994.
[17] V. Belevich, *Classical Network Theory*, Holden Day, San-Francisco, 1968.
[18] V. Bolotnikov, "Boundary rigidity for some classes of meromorphic functions", *Oper. Matrices*, Vol. 3 (2009) pp. 283–301.
[19] V. Bolotnikov, "A multi-point degenerate interpolation problem for generalized Schur functions", *Oper. Matrices*, Vol. 4 (2010) pp. 151–191.
[20] O. Brune, *Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency*, Thesis (MIT), 1931.
[21] O. Brune, "Synthesis of a finite two terminal network whose driving point impedance is a prescribed function of frequency", *J. Math. Phys.*, Vol. 10 pp. 191-236, 1931.
[22] W. Cauer, "The realization of impedances of prescribed frequency dependence" (in German), *Archiv für Elektrotechnik*, Vol. 17, pp. 355-388, 1926.

[23] W. Cauer, "Über Funktionen mit positivem Realteil" (in German), *Mathematische Annalen*, Vol 106, pp. 369-394, 1932.

[24] C. Chamfy, "Fonctions méromorphes dans le cercle-unité et leurs séries de Taylor" (in French), *Annales de l'institut Fourier*, Vol. 8, pp. 211-262, 1958.

[25] N. Cohen and I. Lewkowicz, "Convex Invertible Cones and the Lyapunov Equation", *Lin. Alg. & Appl.*, Vol. 250, pp. 105-131, 1997.

[26] N. Cohen and I. Lewkowicz, "Convex Invertible Cones of State Space Systems", *Mathematics of Control Signals and Systems*, Vol. 10, pp. 265-285, 1997.

[27] N. Cohen and I. Lewkowicz, "Convex Invertible Cones and Positive Real Analytic Functions", *Lin. Alg. & Appl.*, Vol. 425, pp. 797-813, 2007.

[28] Ph. Delsarte, Y. Genin and Y. Kamp. "Pseudo-Carathéodory functions and Hermitian Toeplitz Matrices", *Philips J. of Research*, Vol. 41, pp. 1-54, 1986.

[29] M. S. Derevyagin and A.S. Zhedanov, "An operator approach to multipoint Padé approximations", *J. Approx. The*, Vol. 157, pp. 70-88, 2009.

[30] V.A. Derkach and H. Dym, "Bitangential interpolation in generalized Schur class", *Complex Anal. & Op. Theory*, Vol. 4, pp. 761-765, 2010.

[31] V.A. Derkach, S. Hassi and H. de-Snoo, "Operator models associated with Kac subclasses of generalized Nevanlinna functions", *Meth. Funct. Anal. & Topology*, Vol. 5, pp. 65-87, 1999.

[32] B. Dickinson, Ph. Delsarte, Y. Genin and Y. Kamp., "Minimal realization of pseudo positive and pseudo bounded real rational matrices", *IEEE trans. Circ. & Sys*, Vol 32, pp. 603-605, 1985.

[33] A. Dijksma, H. Langer, A. Luger and Yu. Shondin, "A factorization result for generalized Nevanlinna functions of class $\mathcal{N}_\kappa$", *Integ. Eq. & Op. Theory*, Vol. 36, pp. 121-124, 2000.

[34] P.A. Fuhrmann, "On the characterization and parametrization of minimal spectral factors", *J. Math. Systems, Estimation and Control*, Vol. 5, pp. 383-444, 1995.

[35] Y. Genin, "On the index theory of Pseudo-Carathéodory functions. Applications to the linear systems stability problems", *Integ. Eq. & Op. Theory*, Vol. 10, pp. 640-658, 1987.

[36] R.A. Horn and C.R. Johnson, "Topics in Matrix Analysis" Cambridge University Press, 1991.

[37] M.G. Kreĭn, Integral representation of a continuous Hermitian-indefinite function with a finite number of negative squares, *Dokl. Akad. Nauk. SSSR*, 125, pp. 31-34, (1959).

[38] M.G. Kreĭn and H. Langer, "Über die verallgemeinerten resolventen und die charakteristische funktion einen isometrischen operators in raume Π_κ", (in German) *Hilbert operators and operators algebras (Proc. Int. Conf. Tihany, 1970)*, pp. 353-399, North Holland, Amsterdam, 1972. Colloquia Math. Soc Janos Bolyai.

[39] M.G. Kreĭn and H. Langer, "Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhangen. I. Einige Funktionenklassen und ihre Darstellungen", *Math. Nachrichten"*, Vol. 77, pp. 187-236, 1977.

[40] R. Loewy, "An Inertia Theorem for the Lyapunov Equation and the Dimension of a Controllability Subspace", *Lin. Alg. & Appl.*, Vol. 260, pp. 1-7, 1997.

[41] H. Langer and H. Woracek, "Resolvents of symmetric operators and the degenerated Nevanlinna-Pick problem", *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, Vol. 103, pp. 233-261, 1996.

[42] A. Luger, "A factorization of regular generalized regular Nevanlinna functions", *Integ. Eq. & Op. Theory*, Vol. 43, pp. 326-345, 2002.

[43] A. Luger, "About generalized zeros of non-regular generalized Nevanlinna functions", *Integ. Eq. & Op. Theory*, Vol. 45, pp. 461-473, 2003.

[44] A. Luger, "A characterization of generalized poles of generalized Nevanlinna functions", *Math. Nach.*, Vol. 279, pp. 891-910, 2006.

[45] V.M. Popov, *Hyperstability of Control Systems*, Springer Verlag, New-York, 1973.

[46] T. Takagi, "On the algebraic problem related to an analytic theorem of Carathéodory and Fejèr and on an allied theorem of Landau", *Japanese J. of Math.*, Vol. 1, pp. 83-93, 1924.

[47] M.R. Wohlers, *Lumped and Distributed Passive Networks*, Academic Press, New York, 1969.

[48] D. C. Youla and M. Saito, "Interpolation with Positive-Real Functions", *J. Franklin Inst.*, Vol. 284, No. 2: pp. 77-108, 1967.

(DA) Department of mathematics, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel
E-mail address: dany@math.bgu.ac.il

(IL) Department of electrical engineering, Ben-Gurion University of the Negev, P.O.Box 653, Beer-Sheva 84105, Israel
E-mail address: izchak@ee.bgu.ac.il