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2003

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Recommended Citation

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NOTES ON INTERPOLATION IN THE GENERALIZED

SCHUR CLASS. II. NUDEL′MAN'S PROBLEM

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ABSTRACT. An indefinite generalization of Nudel′man's problem is used in a systematic approach to interpolation theorems for generalized Schur and Nevanlinna functions with interior and boundary data. Besides results on existence criteria for Pick-Nevanlinna and Carathéodory-Fejér interpolation, the method yields new results on generalized interpolation in the sense of Sarason and boundary interpolation, including properties of the finite Hilbert transform relative to weights. The main theorem appeals to the Ball and Helton almost-commutant lifting theorem to provide criteria for the existence of a solution to Nudel′man's problem.

1. INTRODUCTION

In this paper we use a theorem of Ball and Helton [9] to describe a unified approach to a series of classical interpolation problems for the generalized Schur and Nevanlinna classes. These classes contain meromorphic functions having a finite number of poles on a disk or half-plane. The main results of the paper update Chapter 2 of [26], where similar results are derived for holomorphic functions.

In [26], classical interpolation problems are viewed as special cases of Nudel′man's problem: given vectors b and c in a complex vector space $\mathcal{V}$ and an operator A on $\mathcal{V}$ it is required to find a holomorphic function $f(z)$ which is bounded by one on the unit disk such that

$$b = f(A)c$$

with a suitable interpretation of the equation. Particular choices of A, b, and c give various classical problems, such as the Pick-Nevanlinna and Carathéodory-Fejér problems and boundary problems of Loewner type. Conditions for the existence of a solution of Nudel′man's problem are derived in [26, Chapter 2] as an application of the theory of contraction operators on a Hilbert space. This yields existence criteria for classical interpolation problems for Schur and Nevanlinna functions, that is, holomorphic functions which are bounded by one on the unit disk or which have nonnegative imaginary part on the upper half-plane.

Our main tool is an indefinite generalization of Nudel′man's problem, which provides a similar framework for interpolation problems involving the classes $\mathbf{S}_\kappa$ and $\mathbf{N}_\kappa$ of generalized Schur and Nevanlinna functions on the unit disk and upper half-plane. These well-known classes consist of meromorphic functions having a finite number of poles. As in the definite case, we derive existence criteria for the

solvability of the generalized Nudel′man problem in an abstract setting, and the applications to classical problems follow as specializations of this result.

Classical sources for interpolation in the generalized Schur and Nevanlinna classes may be found in [1]. The list below identifies some more recent works that treat these topics and related areas. The list is not a complete bibliography, and the authors regret omissions.

Interpolation of interior data: [1, 3, 10, 14, 15, 17, 18, 20, 24, 28]
Loewner theory and boundary data: [4, 6, 9]
Parametrization of solutions: [1, 9, 17]
Interpolation in the Stieltjes class, including parametrizations: [2, 7]
Moment problems: [11, 12, 22]

Our generalization of Nudel′man's problem and Main Theorem are formulated in Section 2. Sections 3 and 4 contain the applications of the Main Theorem to classical interpolation problems in the disk and half-plane cases, respectively; many of these results parallel the definite case [26]. The proof of the Main Theorem is given in Section 5. This paper can be read independently of [3].

2. MAIN THEOREM

By a **kernel** or **Hermitian form** we mean a complex-valued function $K(\zeta, z) = \overline{K(z, \zeta)}$ on a product set $\Omega \times \Omega$. Such a kernel has κ **negative squares**, in symbols $\mathrm{sq}_- K = \kappa$, if every selfadjoint matrix $(K(\zeta_j, \zeta_i))_{i,j=1}^n$, $\zeta_1, \ldots, \zeta_n \in \Omega$, $n = 1, 2, 3, \ldots$, has at most κ negative eigenvalues, and one such matrix has exactly κ negative eigenvalues (counting multiplicity). When this condition is satisfied with $\kappa = 0$, the kernel is said to be **nonnegative**. A familiar example of a Hermitian form is the inner product $\langle Tf, g \rangle_{\mathfrak{H}}$, where T is a bounded selfadjoint operator on a Hilbert space $\mathfrak{H}$ and f and g are arbitrary vectors in $\mathfrak{H}$. We say that T is **nonnegative** or has κ **negative squares** according as the associated Hermitian form has the same property.

The **Pick class** or **Nevanlinna class** is the set $\mathcal{P}$ of holomorphic functions which have nonnegative imaginary part on the open upper half-plane $\mathbf{C}_+$. The **Schur class** is the set $\mathcal{S}$ of holomorphic functions which are bounded by one on the open unit disk $\mathbf{D}$. Let $\mathbf{C}$ be the complex plane. Following Kreĭn and Langer [22], for any nonnegative integer κ we define the **generalized Nevanlinna class** as the set $\mathbf{N}_\kappa$ of functions $f(z)$ which are holomorphic on some subregion Ω of $\mathbf{C}_+$ such that the kernel $[f(z) - \overline{f(\zeta)}]/(z - \bar{\zeta})$ has κ negative squares on $\Omega \times \Omega$. The **generalized Schur class** is the set $\mathbf{S}_\kappa$ of functions $S(z)$ which are holomorphic on some subregion Ω of $\mathbf{D}$ such that the kernel $[1 - S(z)\overline{S(\zeta)}]/(1 - z\bar{\zeta})$ has κ negative squares on $\Omega \times \Omega$. Functions in the generalized Nevanlinna and Schur classes have analytic continuations to $\mathbf{C}_+$ and $\mathbf{D}$, respectively, excluding at most κ poles, and the corresponding kernels for the extensions also have κ negative squares. We understand that such functions are identified with their meromorphic continuations. With this convention, $\mathbf{N}_0 = \mathcal{P}$ and $\mathbf{S}_0 = \mathcal{S}$. By a theorem of Kreĭn and Langer [21, Theorem 3.2 on p. 382], $\mathbf{S}_\kappa$ coincides with the set of functions

$$(2.1) \qquad S(z) = f(z)/B(z), \qquad B(z) = c \prod_{j=1}^{\kappa} \frac{z - a_j}{1 - z\bar{a}_j},$$

where $f(z)$ is in $\mathbf{S}_0$, $a_1, \ldots, a_\kappa$ are (not necessarily distinct) points in $\mathbf{D}$, $|c| = 1$, and $f(a_j) \neq 0$ for all $j = 1, \ldots, \kappa$. A function $B(z)$ on the unit disk of the preceding form is a **Blaschke product of degree** κ. We call (2.1) a **Kreĭn-Langer factorization** of a function $S \in \mathbf{S}_\kappa$. The factors $f(z)$ and $B(z)$ in a Kreĭn-Langer factorization are determined up to constants of modulus one.

The original inspiration for what is called Nudel′man's problem in [26, p. 22] is Nudel′man [23]. We now consider the following variant.

Nudel′man's Problem for $\mathbf{S}_\kappa$. *Given vectors b, c in a complex vector space $\mathcal{V}$ and a linear operator A on $\mathcal{V}$ into itself, find a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that*

$$(2.2) \qquad f(A)c = B(A)b.$$

We call (A, b, c) the **data** *of the problem.*

Here and in the rest of this section we assume that we are given a complex vector space $\mathcal{V}$ with algebraic dual $\mathcal{V}'$ (no topologies on these spaces are required). If $x' \in \mathcal{V}'$, we write (x, x') for $x'(x)$. If $A : \mathcal{V} \to \mathcal{V}$ is a linear operator, its dual A' is the operator $A' : \mathcal{V}' \to \mathcal{V}'$ such that $(x, A'x') = (Ax, x')$ for all $x \in \mathcal{V}$ and $x' \in \mathcal{V}'$.

Definition 2.1. A set $\mathcal{D} \subseteq \mathcal{V}'$ is **admissible** for given data (A, b, c) if

(i) $\mathcal{D}$ is a linear subspace of $\mathcal{V}'$ which is invariant under A',

(ii) the sums $\sum_{j=0}^{\infty} |(A^j b, x')|^2$ and $\sum_{j=0}^{\infty} |(A^j c, x')|^2$ are finite for all $x' \in \mathcal{D}$, and

(iii) if some x' in $\mathcal{D}$ annihilates $c, Ac, A^2 c, \ldots$, then x' annihilates $b, Ab, A^2 b, \ldots$.

Throughout we write H^2 for the Hardy space on the unit disk.

Definition 2.2. If $\mathcal{D}$ is admissible for (A, b, c), then for any $f(z) = \sum_{j=0}^{\infty} f_j z^j$ and $B(z) = \sum_{j=0}^{\infty} B_j z^j$ in H^2, we interpret (2.2) to mean that

$$(2.3) \qquad \sum_{j=0}^{\infty} f_j (A^j c, x') = \sum_{j=0}^{\infty} B_j (A^j b, x'), \qquad x' \in \mathcal{D}.$$

Thus the meaning of (2.2) depends on the choice of an admissible set $\mathcal{D}$ of linear functionals. It will be seen that in applications natural choices are usually clear.

Main Theorem. *Let (A, b, c) be given data and $\mathcal{D}$ an admissible set. Define a Hermitian form $\mathcal{F}$ on $\mathcal{D} \times \mathcal{D}$ by*

$$(2.4) \qquad \mathcal{F}(x', y') = \sum_{j=0}^{\infty} \left[\left(A^j c, x' \right) \overline{(A^j c, y')} - \left(A^j b, x' \right) \overline{(A^j b, y')} \right], \qquad x', y' \in \mathcal{D}.$$

Let κ be a nonnegative integer.

(1) *If $\mathcal{F}$ has κ negative squares, there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that $f(A)c = B(A)b$.*

(2) *If there is a pair (f, B) as in (1), then $\mathcal{F}$ has κ' negative squares for some $\kappa' \leq \kappa$.*

We show that in the case $\kappa = 0$ this result implies Theorem 2.3 in [26]. The reason there is something to show is that we now require condition (iii) in Definition 2.1 as a hypothesis, and this is not a hypothesis in [26, Theorem 2.3, p. 23]. It is therefore sufficient to show that when $\kappa = 0$, condition (iii) in Definition 2.1 is redundant when the other hypotheses in parts (1) and (2) of the Main Theorem are satisfied.

(1) Assume the hypotheses of part (1) of the Main Theorem with $\kappa = 0$, but not necessarily condition (iii) in Definition 2.1. Then $\mathcal{F}$ is nonnegative, and we have the inequality

$$\sum_{j=0}^{\infty} |(A^j b, x')|^2 \leq \sum_{j=0}^{\infty} |(A^j c, x')|^2, \qquad x' \in \mathcal{D},$$

which implies (iii).

(2) Assume the hypotheses of part (2) of the Main Theorem with $\kappa = 0$, but not necessarily condition (iii) in Definition 2.1. Then B is a constant of absolute value one, and we may take $B \equiv 1$. Thus

$$(2.5) \qquad \sum_{j=0}^{\infty} f_j (A^j c, x') = (b, x'), \qquad x' \in \mathcal{D}.$$

If some x'_0 in $\mathcal{D}$ annihilates $c, Ac, A^2 c, \dots$, then by Definition 2.1(i), for all $k = 0, 1, 2, \dots$, $(A')^k x'_0$ also belongs to $\mathcal{D}$ and annihilates $c, Ac, A^2 c, \dots$, and hence $(b, (A')^k x'_0) = 0$ by (2.5). Thus x'_0 annihilates $b, Ab, A^2 b, \dots$, and we again obtain Definition 2.1(iii).

The role of condition (iii) in Definition 2.1 can be seen in the applications in Section 3. For example, in Theorem 3.1 the condition follows from the assumption that the points $z_1, \dots, z_n$ are distinct; this assumption is redundant in classical Pick-Nevanlinna interpolation in the case $\kappa = 0$.

The proof of the Main Theorem is given in Section 5.

3. CLASSICAL INTERPOLATION PROBLEMS ON THE UNIT DISK

We begin with the prototype for classical interpolation theory, Pick-Nevanlinna interpolation at a finite number of points.

Theorem 3.1. *Let $z_1, \dots, z_n$ be distinct points in $\mathbf{D}$, $w_1, \dots, w_n$ any complex numbers, and let κ be a nonnegative integer. Set*

$$P = \left[\frac{1 - w_j \bar{w}_k}{1 - z_j \bar{z}_k} \right]_{j,k=1}^n .$$

(1) *If P has κ negative eigenvalues, then there is a pair (f, B) with $f \in \mathbf{S}_0$ and B a Blaschke product of degree κ such that $f(z_j) = B(z_j) w_j$ for all $j = 1, \dots, n$.*

(2) *If there is a pair (f, B) as in (1), then P has $\kappa' \leq \kappa$ negative eigenvalues.*

Proof. We apply the Main Theorem with $\mathcal{V} = \mathbf{C}^n$. Identify $\mathcal{V}'$ with $\mathbf{C}^n$ with the pairing $(x, y) = x_1 y_1 + \cdots x_n y_n$ (for any a in $\mathbf{C}^n$, we write a_j for its entries). For the data (A, b, c), choose

$$A = \mathrm{diag}\,\{z_1, \dots, z_n\}, \qquad b = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

and take $\mathcal{D} = \mathcal{V}'$. The verification that $\mathcal{D}$ is admissible is routine. To check property (iii) in Definition 2.1, suppose $x \in \mathcal{D}$ annihilates $c, Ac, A^2 c, \dots$. Then

$$\sum_{p=1}^n z_p^j x_j = 0, \qquad j \geq 0.$$

Since $z_1, \ldots, z_n$ are distinct, $x = 0$ and trivially x annihilates $b, Ab, A^2b, \ldots$. A short calculation shows that the Hermitian form (2.4) is given by

$$\mathcal{F}(x, y) = \sum_{j,k=1}^{n} \frac{1 - w_j \bar{w}_k}{1 - z_j \bar{z}_k} \, x_j \bar{y}_k, \qquad x, y \in \mathcal{D}.$$

Thus sq_$\mathcal{F}$ is equal to the number of negative eigenvalues of P. To see the meaning of the identity $B(A)b = f(A)c$, in (2.3) choose the linear functional induced by the standard unit vector with 1 in the k-th entry and all other entries zero. In this case the condition reduces to $B(z_k)w_k = f(z_k)$. Thus the result is a particular case of the Main Theorem. $\qquad\square$

With a slight variation of method the preceding result can be extended to an arbitrary set of points.

Theorem 3.2. *Let* $S : \Omega \to \mathbf{C}$ *be a function defined on a subset* Ω *of* $\mathbf{D}$, *and let* κ *be a nonnegative integer. Set*

$$K(\zeta, z) = \frac{1 - S(z)\overline{S(\zeta)}}{1 - z\bar{\zeta}}, \qquad \zeta, z \in \Omega.$$

(1) *If* sq_$K = \kappa$, *then there is a pair* (f, B) *with* $f \in \mathbf{S}_0$ *and* B *a Blaschke product of degree* κ *such that* $f(z) = B(z)S(z)$ *for all* $z \in \Omega$.
(2) *If there is a pair* (f, B) *as in* (1), *then* sq_$K = \kappa' \leq \kappa$.

Proof. Let $\mathcal{V}$ be the vector space of all functions $x \colon \Omega \to \mathbf{C}$. For the data (A, b, c), let A be multiplication by the independent variable on $\mathcal{V}$, $b = S$, and $c \equiv 1$. Every function y on Ω with finite support induces a linear functional on $\mathcal{V}$ by the formula $(x, y) = \sum_{z \in \Omega} x(z)y(z)$, and the set $\mathcal{D}$ of all such functionals is admissible. A straightforward application of the Main Theorem as in the proof of Theorem 3.1 yields the result. $\qquad\square$

The next result concerns the interpolation of a finite number of derivatives, that is, it is of the Carathéodory-Fejér type. We use standard notation for matrices and their induced operators on Euclidean spaces. Write 1 for an identity matrix or operator and $*$ for conjugate transpose or adjoint.

Theorem 3.3. *Let* $w(z) = w_0 + w_1 z + \cdots + w_n z^n$ *be a polynomial with complex coefficients, and set*

$$T = \begin{bmatrix} w_0 & w_1 & \cdots & w_n \\ 0 & w_0 & \cdots & w_{n-1} \\ & & \cdots & \\ 0 & 0 & \cdots & w_0 \end{bmatrix}.$$

Let κ *be a nonnegative integer such that* $\kappa \leq n + 1$.

(1) *If* $1 - T^*T$ *has* κ *negative eigenvalues, then there is a pair* (f, B) *with* $f \in \mathbf{S}_0$ *and* B *a Blaschke product of degree* κ *such that* $B(z)w(z) = f(z) + \mathcal{O}(z^{n+1})$.
(2) *If there is a pair* (f, B) *as in* (1), *then* $1 - T^*T$ *has* $\kappa' \leq \kappa$ *negative eigenvalues.*

Corollary 3.4. *Let* $w(z) = w_0 + w_1 z + \cdots + w_n z^n$ *and* T *be as in Theorem* 3.3, *and let* κ *be a nonnegative integer such that* $\kappa \leq n + 1$. *If* $1 - T^*T$ *has* κ *negative eigenvalues, there is a* $\kappa' \leq \kappa$ *and a function* $S(z)$ *in* $\mathbf{S}_{\kappa'}$ *which is holomorphic at the origin and such that* $w(z) = S(z) + \mathcal{O}(z^{n+1-\kappa})$.

Proof of Theorem 3.3. Let $\mathcal{V}$ and $\mathcal{V}'$ be as in the proof of Theorem 3.1 but with $\mathbf{C}^n$ replaced by $\mathbf{C}^{n+1}$. For the data (A, b, c), choose

$$
A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

The set $\mathcal{D} = \mathcal{V}'$ is admissible. If $\mathfrak{H}$ is $\mathbf{C}^{n+1}$ in the Euclidean inner product, then $\mathcal{F}(x, y) = \langle (1 - T^*T)x, y \rangle_{\mathfrak{H}}$, $x, y \in \mathfrak{H}$. In fact, for any $x \in \mathfrak{H}$,

$$
\begin{aligned}
\mathcal{F}(x, x) &= \left(|x_0|^2 + \cdots + |x_n|^2 \right) \\
&\quad - \left(|w_0 x_0 + \cdots + w_n x_n|^2 + |w_0 x_1 \cdots + w_{n-1} x_n|^2 + \cdots + |w_0 x_n|^2 \right) \\
&= \|x\|_{\mathfrak{H}}^2 - \|Tx\|_{\mathfrak{H}}^2 \\
&= \langle (1 - T^*T)x, x \rangle_{\mathfrak{H}}.
\end{aligned}
$$

Thus $\mathrm{sq}_- \mathcal{F}$ is equal to the number of negative eigenvalues of $1 - T^*T$. The equation $f(A)c = B(A)b$ with $f(z) = \sum_{j=0}^{\infty} f_j z^j$ and $B(z) = \sum_{j=0}^{\infty} B_j z^j$ is equivalent to the identities

$$
f_0 = w_0 B_0, \ \ f_1 = w_1 B_0 + w_0 B_1, \ \ \ldots, \ \ f_n = w_n B_0 + w_{n-1} B_1 + \cdots + w_0 B_n,
$$

or $B(z)w(z) = f(z) + \mathcal{O}(z^{n+1})$. The result thus follows from the Main Theorem. $\qquad\square$

Proof of Corollary 3.4. Let (f, B) be a pair as in part (1) of Theorem 3.3. If $B(z)$ has a zero of order r at the origin, $f(z)$ has a zero of order at least r at the origin. Hence $S(z) = f(z)/B(z)$ belongs to $\mathbf{S}_{\kappa'}$ for some $\kappa' \leq \kappa$ and is holomorphic at the origin, and $w(z) = S(z) + \mathcal{O}(z^{n-r+1}) = S(z) + \mathcal{O}(z^{n+1-\kappa})$. $\qquad\square$

A simultaneous generalization of the Pick-Nevanlinna and Carathéodory-Fejér problems can be treated in the same way by choosing A in Jordan form. The calculations are straightforward but somewhat lengthy, and we shall not pursue this direction. For the definite case, see [19] and [26, §2.6].

The Main Theorem also yields a result on generalized interpolation in the sense of Sarason [29]. Let C be an inner function on $\mathbf{D}$, and let $\mathfrak{H}(C) = H^2 \ominus CH^2$. The reproducing kernel for $\mathfrak{H}(C)$ is given by

$$
(3.1) \qquad K_C(w, z) = \frac{1 - C(z)\overline{C(w)}}{1 - z\bar{w}}, \qquad z, w \in \mathbf{D}.
$$

Let S be the shift operator $S \colon h(z) \to zh(z)$ on H^2, and let T be the compression of S to $\mathfrak{H}(C)$, that is,

$$
T = P_{\mathfrak{H}(C)} S|_{\mathfrak{H}(C)},
$$

where $P_{\mathfrak{H}(C)}$ is the projection operator on H^2 with range $\mathfrak{H}(C)$. The space $\mathfrak{H}(C)$ is invariant under S^* and $T^* = S^*|_{\mathfrak{H}(C)}$. Since T is completely nonunitary, for any $\varphi \in H^\infty$ an operator $\varphi(T)$ on $\mathfrak{H}(C)$ is defined by the H^∞-functional calculus (see [31] and [32, p. 114]):

$$
(3.2) \qquad \varphi(T) = s\text{-}\lim_{r \uparrow 1} \varphi(rT).
$$

Equivalently, for this particular situation, $\varphi(T) = P_{\mathfrak{H}(C)} M_\varphi|_{\mathfrak{H}(C)}$, where M_φ is multiplication by φ on H^2. For every $\varphi \in H^\infty$, $\varphi(T)$ commutes with T, and $\varphi(T)$ is a contraction if and only if φ is a Schur function.

Theorem 3.5. *Let C be an inner function on the unit disk, and define T on $\mathfrak{H}(C)$ as above. Let R be a bounded linear operator on $\mathfrak{H}(C)$ such that $TR = RT$.*

(1) *If $1 - RR^*$ has κ negative squares, then there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that*
$$B(T)R = f(T).$$

(2) *If there is a pair (f, B) as in (1), $1 - RR^*$ has κ' negative squares for some $\kappa' \leq \kappa$.*

If R is a contraction, the condition in (1) is satisfied with $\kappa = 0$, and in this case the result reduces to the original theorem of Sarason [29, Theorem 1].

Proof. In the Main Theorem, let $\mathcal{V} = \mathfrak{H}(C)$, $A = T$, $c = K_C(0, \cdot)$, and $b = Rc = RK_C(0, \cdot)$. Let $\mathcal{D}$ be the set of continuous linear functionals on $\mathcal{V} = \mathfrak{H}(C)$; thus $\mathcal{D} = \{x'_k \colon k \in \mathfrak{H}(C)\}$ where for any $k \in \mathfrak{H}(C)$,
$$(h, x'_k) = \langle h, k \rangle_{\mathfrak{H}(C)}, \qquad h \in \mathfrak{H}(C).$$
Then $A' x'_k = x'_{T^* k}$ for any $k \in \mathfrak{H}(C)$, and so condition (i) in Definition 2.1 holds. To verify (ii), notice that for any $k(z) = \sum_0^\infty a_j z^j$ in $\mathfrak{H}(C)$,
$$\sum_{j=0}^\infty |(A^j c, x'_k)|^2 = \sum_{j=0}^\infty |\langle K_C(0, \cdot), T^{*j} k \rangle_{\mathfrak{H}(C)}|^2 = \sum_{j=0}^\infty |a_j|^2 = \|k\|_{H^2}^2 < \infty.$$
If we replace c by b and use the identity $RT = TR$, we obtain
$$\sum_{j=0}^\infty |(A^j b, x'_k)|^2 = \sum_{j=0}^\infty |\langle K_C(0, \cdot), T^{*j} R^* k \rangle_{\mathfrak{H}(C)}|^2 = \|R^* k\|_{H^2}^2 < \infty,$$
and thus condition (ii) in Definition 2.1 holds. Condition (iii) holds trivially because the only $k \in \mathfrak{H}(C)$ such that
$$(A^j c, x'_k) = \langle K_C(0, \cdot), T^{*j} k \rangle_{\mathfrak{H}(C)} = 0$$
for all $j \geq 0$ is $k = 0$. We have shown that $\mathcal{D}$ is admissible. The form (2.4) is given by

$$(3.3) \qquad \mathcal{F}(x'_h, x'_k) = \sum_{j=0}^\infty \left[\langle T^j K_C(0, \cdot), h \rangle_{\mathfrak{H}(C)} \langle k, T^j K_C(0, \cdot) \rangle_{\mathfrak{H}(C)} \right.$$
$$\left. - \langle T^j R K_C(0, \cdot), h \rangle_{\mathfrak{H}(C)} \langle k, T^j R K_C(0, \cdot) \rangle_{\mathfrak{H}(C)} \right]$$
$$= \langle k, h \rangle_{H^2} - \langle R^* k, R^* h \rangle_{H^2}$$
$$= \langle k, h \rangle_{\mathfrak{H}(C)} - \langle R^* k, R^* h \rangle_{\mathfrak{H}(C)}$$
$$= \langle (1 - RR^*) k, h \rangle_{\mathfrak{H}(C)}$$

for any h and k in $\mathfrak{H}(C)$.

(1) Assume that $1 - RR^*$ has κ negative squares. By (3.3), the Hermitian form (2.4) has κ negative squares. Hence by part (1) of the Main Theorem, there is a function $f \in \mathbf{S}_0$ and a Blaschke product B of degree κ such that $B(A)b = f(A)c$ in

the sense of Definition 2.2, that is, if $B(z) = \sum_0^\infty B_j z^j$ and $f(z) = \sum_0^\infty f_j z^j$, then for every $h \in \mathfrak{H}(C)$,

$$\sum_{j=0}^\infty B_j \langle T^j RK_C(0, \cdot), h \rangle_{\mathfrak{H}(C)} = \sum_{j=0}^\infty f_j \langle T^j K_C(0, \cdot), h \rangle_{\mathfrak{H}(C)}.$$

Using Abel summation of these series, we see that

(3.4) $$B(T)RK_C(0, \cdot) = f(T)K_C(0, \cdot).$$

Since R commutes with T, it commutes with $B(T)$ and $f(T)$. Hence $B(T)R$ and $f(T)$ agree on the smallest invariant subspace of T containing $K_C(0, \cdot)$. The latter subspace is all of $\mathfrak{H}(C)$, and thus we obtain $B(T)R = f(T)$.

(2) Assume that a pair (f, B) exists as in (1). Reversing the preceding steps, we see that $B(A)b = f(A)c$, hence by part (2) of the Main Theorem the form (2.4) has κ negative squares. Therefore by (3.3), $1 - RR^*$ has κ' negative squares for some $\kappa' \le \kappa$. $\square$

More generally, let C be a function which is holomorphic and bounded by one on $\mathbf{D}$ (not necessarily an inner function), and let $\mathfrak{H}(C)$ be the Hilbert space with reproducing kernel (3.1) [13, 30]. Then $\mathfrak{H}(C)$ is contained contractively in H^2, that is, the inclusion mapping $E \colon \mathfrak{H}(C) \to H^2$ is a contraction operator (the inclusion is isometric if and only if C is an inner function). There is a contraction operator T on $\mathfrak{H}(C)$ such that $T^* \colon h(z) \to [h(z) - h(0)]/z$ for every $h(z)$ in $\mathfrak{H}(C)$. In view of the difference-quotient inequality

$$\|[h(z) - h(0)]/z\|_{\mathfrak{H}(C)}^2 \le \|h(z)\|_{\mathfrak{H}(C)}^2 - |h(0)|^2,$$

which holds for every element $h(z)$ of $\mathfrak{H}(C)$, T is completely nonunitary. Hence for any $\varphi \in H^\infty$ we may define $\varphi(T)$ by (3.2). The selfadjoint operators $G = E^*E$ on $\mathfrak{H}(C)$ and $D = EE^*$ on H^2 also play a role. They satisfy $0 \le G \le 1$ and $0 \le D \le 1$. It is not hard to see that

$$D = 1 - M_C M_C^*,$$

where M_C is multiplication by C on H^2. Let

$$\mathfrak{M} = H^2 \ominus \ker D.$$

Then $\mathfrak{H}(C)$ coincides with the range of $D^{1/2}$ in the "range norm", that is, the unique norm such that $D^{1/2}$ acts as a partial isometry from H^2 onto $\mathfrak{H}(C)$ with initial space $\mathfrak{M}$. One can use the relation $DE = EG$ to show that G is unitarily equivalent to $D|_{\mathfrak{M}}$ by means of the natural isomorphism $U = D^{1/2}|_{\mathfrak{M}}$ from $\mathfrak{M}$ onto $\mathfrak{H}(C)$.

Theorem 3.6. *Let C be a function which is holomorphic and bounded by one on the unit disk, and let T be the operator on $\mathfrak{H}(C)$ such that $T^* \colon h(z) \to [h(z) - h(0)]/z$ for every $h(z)$ in $\mathfrak{H}(C)$. Let $G = E^*E$ and E be as above. Let R be a bounded linear operator on $\mathfrak{H}(C)$ such that $TR = RT$.*

(1) *If $G - RGR^*$ has κ negative squares, then there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that*

$$B(T)R = f(T).$$

(2) *If there is a pair (f, B) as in (1), $G - RGR^*$ has κ' negative squares for some $\kappa' \le \kappa$.*

Remarks. (1) Let $f \in \mathbf{S}_0$, and define $R = f(T)$ by the H^∞-functional calculus. Set $\tilde{f}(z) = \overline{f(\bar{z})}$. Then $RT = TR$. Since $Ef(T)^* = E\tilde{f}(T^*) = \tilde{f}(S^*)E$, we obtain

$$G - RGR^* = E^*E - f(T)E^*Ef(T)^* = E^*E - E^*\tilde{f}(S^*)^*\tilde{f}(S^*)E \geq 0,$$

because $\tilde{f}(S^*)$ is a contraction.

(2) It is a corollary of the theorem that if R is a bounded operator on $\mathfrak{H}(C)$ such that $RT = TR$ and $G - RGR^* \geq 0$, then $1 - RR^* \geq 0$. Indeed, by the theorem $R = f(T)$ for some $f \in \mathbf{S}_0$, and hence R is a contraction.

Proof of Theorem 3.6. We proceed as in the proof of Theorem 3.5 with the same choice of $\mathcal{V}$, data (A, b, c), and set $\mathcal{D}$ of linear functionals. In exactly the same way, we show that $\mathcal{D}$ is admissible. The calculation of (2.4) is the same except for the last step. Now we have

$$\mathcal{F}(x_h', x_k') = \langle Ek, Eh \rangle_{H^2} - \langle ERk, ERh \rangle_{H^2} = \langle (G - RGR^*)k, h \rangle_{\mathfrak{H}(C)}$$

for all h and k in $\mathfrak{H}(C)$.

(1) If $G - RGR^*$ has κ negative squares, so does (2.4)., and hence by the Main Theorem there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that (2.2) holds. We use Abel summation as in the proof of Theorem 3.5 to show that this implies (3.4), where $B(T)$ and $f(T)$ are defined by the H^∞-functional calculus. The commutivity of T and R then implies that $B(T)R = f(T)$.

(2) Reverse the steps in (1). $\qquad\qquad\square$

We turn now to problems of Loewner type, which concern boundary data. In the scalar case, solutions of interpolation problems of the type that we treat here are unique, and the results that we obtain are characterizations of boundary functions and their restrictions.

Let σ be normalized Lebesgue measure on $\partial \mathbf{D}$: $|u| = 1$. When no confusion can arise, we write L^2, L^∞ for $L^2(\sigma), L^\infty(\sigma)$; we identify H^2 with a subspace of L^2 in the usual way. If Δ is a measurable subset of $\partial \mathbf{D}$, $L^2(\Delta)$ is the subspace of functions in L^2 supported on Δ. The characteristic function of Δ is denoted 1_Δ; if φ is a function on Δ, we view $\varphi 1_\Delta$ as defined on all of $\partial \mathbf{D}$ and equal to zero on the complement of Δ.

We note some preliminary results from function theory.

(i) If $\varphi, \psi \in L^2(\Delta)$, then [26, p. 31]

$$(3.5) \quad \lim_{r \uparrow 1} \int_\Delta \int_\Delta \frac{\varphi(u)\overline{\psi(v)}}{1 - r^2 u\bar{v}} \, d\sigma(u)d\sigma(v)$$

$$= \sum_{j=0}^\infty \left(\int_\Delta u^j \varphi(u) \, d\sigma(u) \right) \overline{\left(\int_\Delta v^j \psi(v) \, d\sigma(v) \right)} = \langle Q_-(\varphi 1_\Delta), \psi 1_\Delta \rangle_{L^2},$$

where Q_- is the orthogonal projection on L^2 whose range is the closed span L_-^2 of all functions u^j, $j \leq 0$. In particular, the limit on the left exists.

(ii) If B is an inner function and M_B is multiplication by B on L^2, then

$$(3.6) \qquad\qquad M_B Q_- M_B^* = Q_- + P_{u\mathfrak{H}(B)},$$

where $\mathfrak{H}(B) = H^2 \ominus BH^2$ and $P_{u\mathfrak{H}(B)}$ is the orthogonal projection on L^2 whose range is $u\mathfrak{H}(B)$. To prove this, write $L^2 = L_-^2 \oplus uBH^2 \oplus u\mathfrak{H}(B)$. Both sides of (3.6) coincide with the identity operator on L_-^2, and both coincide with

the zero operator on uBH^2. It remains to check the actions of each side on elements of L^2 of the form uh with h in $\mathfrak{H}(B)$. This amounts to showing that $BQ_-(\bar{B}uh) = uh$, or equivalently, that $Q_-(\bar{B}uh) = \bar{B}uh$; the last equation holds by the characterization of $\mathfrak{H}(B)$ as the set of elements h of H^2 such that $\bar{B}h \perp H^2$.

(iii) (Kronecker's Theorem) This result characterizes finite-rank Hankel operators and has several formulations, including: *Let $F \in L^\infty$, and assume that the operator H_F on H^2 to $L^2 \ominus H^2$ which is defined as compression of multiplication by F has rank κ. Then there is a Blaschke product B of degree κ such that $BF \in H^\infty$.* For example, see Peller [25, p. 77]. Here $H_F h = P_-(Fh)$, $h \in H^2$, where P_- is the projection onto $L^2 \ominus H^2$, equivalently, $H_F h = \bar{u}Q_-(uFh)$ where Q_- is as above.

Our first boundary result characterizes boundary functions of functions in $\mathbf{S}_\kappa$.

Theorem 3.7. *Let b and c be complex-valued measurable functions on a Borel set $\Delta \subseteq \partial\mathbf{D}$ such that $c \neq 0$ σ-a.e. on Δ. Let $\mathcal{D}$ be the set of functions φ on Δ such that $b\varphi, c\varphi \in L^2(\Delta)$. If κ is a nonnegative integer, then there is a function $S(z)$ in $\mathbf{S}_\kappa$ whose boundary function satisfies $b(u) = S(u)c(u)$ σ-a.e. on Δ if and only if the Hermitian form*

$$(3.7) \qquad L(\varphi, \psi) = \lim_{r \uparrow 1} \int_\Delta \int_\Delta \frac{c(u)\overline{c(v)} - b(u)\overline{b(v)}}{1 - r^2 u\bar{v}} \, \varphi(u)\overline{\psi(v)} \, d\sigma(u)d\sigma(v),$$

$\varphi, \psi \in \mathcal{D}$, has κ negative squares.

Proof. Case 1: $\partial\mathbf{D} \setminus \Delta$ is not a σ-null set.

We apply the Main Theorem with $\mathcal{V} = \{pb + qc : p, q \text{ polynomials}\}$. For the data (A, b, c), let A be multiplication by the independent variable, and let b and c be the given functions. We view $\mathcal{D}$ as a subspace of $\mathcal{V}'$ by identifying each $\varphi \in \mathcal{D}$ with a linear functional on $\mathcal{V}$ by writing

$$(f, \varphi) = \int_\Delta f\varphi \, d\sigma, \qquad f \in \mathcal{V}.$$

Condition (i) in the definition of an admissible family is easily verified, and (ii) holds because the Fourier coefficients of a function in L^2 are square summable. To check (iii), suppose $\varphi \in \mathcal{D}$ annihilates $c, Ac, A^2 c, \ldots$. Then

$$\int_{\partial\mathbf{D}} u^j c\varphi 1_\Delta \, d\sigma = 0, \qquad j = 0, 1, 2, \ldots.$$

Thus $c\varphi 1_\Delta$ belongs to uH^2. Since $\sigma(\partial\mathbf{D} \setminus \Delta) > 0$, $c\varphi 1_\Delta = 0$ σ-a.e. on $\partial\mathbf{D}$ and hence $\varphi = 0$ σ-a.e. on Δ. Therefore φ is the zero element of $\mathcal{D}$ and hence annihilates $b, Ab, A^2 b, \ldots$. Thus $\mathcal{D}$ is admissible.

By (3.5), the Hermitian form (2.4) coincides with (3.7). An argument in [26, p. 31] shows that for any Schur functions f and B, the identity $B(A)b = f(A)c$ is equivalent to the relation $Bb = fc$ σ-a.e. on Δ.

Now suppose $\mathrm{sq}_- L = \kappa$. By part (1) of the Main Theorem, there is a pair (f, B), $f \in \mathbf{S}_0$ and B a Blaschke product of degree κ, such that $f(A)c = B(A)b$ and hence $Bb = fc$ σ-a.e. on Δ. Hence $b = Sc$ σ-a.e. on Δ, where $S = f/B$ is in $\mathbf{S}_{\kappa'}$ for some $\kappa' \leq \kappa$. Let $S = f_1/B_1$ be a Kreĭn-Langer factorization of S, so B_1 is a Blaschke product of degree κ' (which is obtained by cancelling all of the common simple Blaschke factors in f and B). Then $f_1 c = B_1 b$ σ-a.e. on Δ and so

$f_1(A)c = B_1(A)b$. Part (2) of the Main Theorem then gives $\text{sq}_-L \leq \kappa'$. Therefore $\kappa = \kappa'$ and the sufficiency part of the theorem follows.

Conversely, let $b = Sc$ σ-a.e. on Δ where $S \in \mathbf{S}_\kappa$. If $S = f/B$ is a Kreĭn-Langer factorization of S, then $f(A)c = B(A)b$, and by part (2) of the Main Theorem, $\text{sq}_-L = \kappa' \leq \kappa$. By the proof of sufficiency above, $b = S_1c$ σ-a.e. on Δ, where $S_1 \in \mathbf{S}_{\kappa'}$. Since we assume that $c \neq 0$ σ-a.e. on Δ, $S_1 = b/c = S \in \mathbf{S}_\kappa$, and hence L has $\kappa' = \kappa$ negative squares. The necessity part of the theorem follows.

Case 2: $\Delta = \partial\mathbf{D}$.

Given $\varepsilon > 0$, let Δ_ε be the set of points $u \in \partial\mathbf{D}$ with $\varepsilon \leq \arg u \leq 2\pi$. Let $b_\varepsilon, c_\varepsilon$ be the restrictions of b, c to Δ_ε. Let $\mathcal{D}_\varepsilon$ be the set of restrictions of functions in $\mathcal{D}$ to Δ_ε, and define L_ε as the restriction of L to $\mathcal{D}_\varepsilon \times \mathcal{D}_\varepsilon$.

If L has κ negative squares, so does L_ε for all sufficiently small ε. By Case 1, for such ε there is a function $S_\varepsilon(z)$ in $\mathbf{S}_\kappa$ whose boundary function satisfies $b(u) = S_\varepsilon(u)c(u)$ σ-a.e. on Δ_ε. Since functions in $\mathbf{S}_\kappa$ are determined by their boundary values on a set of positive measure, $S_\varepsilon = S$ is independent of ε. By construction, $b(u) = S(u)c(u)$ σ-a.e. on $\partial\mathbf{D}$.

In the other direction, if there is an S in $\mathbf{S}_\kappa$ whose boundary function satisfies $b(u) = S(u)c(u)$ σ-a.e. on $\partial\mathbf{D}$, the same relation holds σ-a.e. on Δ_ε. Again by Case 1, L_ε has κ negative squares for all sufficiently small ε. Hence L has κ negative squares. Thus the result holds in the case $\Delta = \partial\mathbf{D}$. $\qquad\square$

Corollary 3.8. *Let S_0 be a complex-valued measurable function on $\partial\mathbf{D}$, and let κ be a nonnegative integer. Then there is a function $S(z)$ in $\mathbf{S}_\kappa$ whose boundary function satisfies $S(u) = S_0(u)$ σ-a.e. on $\partial\mathbf{D}$ if and only if the Hermitian form*

$$(3.8) \qquad L(\varphi, \psi) = \lim_{r\uparrow 1} \int_{\partial\mathbf{D}} \int_{\partial\mathbf{D}} \frac{1 - S_0(u)\overline{S_0(v)}}{1 - r^2 u\bar{v}} \, \varphi(u)\overline{\psi(v)} \, d\sigma(u)d\sigma(v),$$

$\varphi, S_0\varphi, \psi, S_0\psi \in L^2$, has κ negative squares. When this condition is satisfied, then $|S_0(u)| \leq 1$ σ-a.e. on $\partial\mathbf{D}$.

Proof. In Theorem 3.7 choose $\Delta = \partial\mathbf{D}$, $c = 1$, and $b = S_0$. The last statement follows by the Kreĭn-Langer representation (2.1) of a generalized Schur function. $\qquad\square$

We give a different version of the corollary that assumes $|S_0(u)| \leq 1$ σ-a.e. on $\partial\mathbf{D}$ as a hypothesis and then obtains additional information. Namely, the result remains true if we replace the form $L(\varphi, \psi)$ defined on $L^2 \times L^2$ by (3.8) by its restriction (3.9) to $uH^2 \times uH^2$.

Alternative form of Corollary 3.8 *Let S_0 be a complex-valued measurable function on $\partial\mathbf{D}$ satisfying $|S_0(u)| \leq 1$ σ-a.e. on $\partial\mathbf{D}$, and let κ be a nonnegative integer. Then there is a function $S(z)$ in $\mathbf{S}_\kappa$ whose boundary function satisfies $S(u) = S_0(u)$ σ-a.e. on $\partial\mathbf{D}$ if and only if the Hermitian form*

$$(3.9) \qquad L_+(\varphi, \psi) = \lim_{r\uparrow 1} \int_{\partial\mathbf{D}} \int_{\partial\mathbf{D}} \frac{1 - S_0(u)\overline{S_0(v)}}{1 - r^2 u\bar{v}} \, \varphi(u)\overline{\psi(v)} \, d\sigma(u)d\sigma(v),$$

$\varphi, \psi \in uH^2$, has κ negative squares.

Proof. First assume that $S_0 \in \mathbf{S}_\kappa$, or, more precisely, that $S_0(u) = S(u)$ σ-a.e. for some $S(z)$ in $\mathbf{S}_\kappa$. We show that $\text{sq}_-L_+ = \kappa'$ for some $\kappa' \leq \kappa$. Let $S = f/B$ be a

Kreĭn-Langer factorization. Then by (3.5) and (3.6), for any $\varphi, \psi \in uH^2$,

$$
\begin{aligned}
L_+(\varphi, \psi) &= \langle Q_-\varphi, \psi \rangle_{L^2} - \langle Q_-(\bar{B}f\varphi), \bar{B}f\psi \rangle_{L^2} \\
&= -\langle M_B Q_- M_B^* f\varphi, f\psi \rangle_{L^2} \\
&= -\langle (Q_- + P_{u\mathfrak{H}(B)})f\varphi, f\psi \rangle_{L^2} \\
&= -\langle P_{u\mathfrak{H}(B)}f\varphi, f\psi \rangle_{L^2}.
\end{aligned}
$$

Since $\mathfrak{H}(B)$ has dimension κ, $\mathrm{sq}_- L_+ = \kappa'$ for some $\kappa' \leq \kappa$.

Next suppose that $\mathrm{sq}_- L_+ = \kappa'$ for some κ'. As above, for any $\varphi = uh$ and $\psi = uk$ in uH^2,

$$
L_+(\varphi, \psi) = \langle Q_-\varphi, \psi \rangle_{L^2} - \langle Q_- S_0\varphi, S_0\psi \rangle_{L^2} = -\langle Q_- uS_0 h, uS_0 k \rangle_{L^2}.
$$

Since we assume that $\mathrm{sq}_- L_+ = \kappa'$, the rank of the Hankel operator defined by $H_{S_0} h = \bar{u}Q_-(uS_0 h)$, $h \in H^2$, is κ'. By Kronecker's theorem as stated above, there is a Blaschke product B of degree κ' such that $BS_0 = f$ where $f \in \mathbf{S}_0$ (recall that we assume that $|S_0(u)| \leq 1$ σ-a.e. on $\partial\mathbf{D}$). It follows that $S_0 \in \mathbf{S}_{\kappa''}$ for some $\kappa'' \leq \kappa'$, that is, S_0 is the boundary function of a function in $\mathbf{S}_{\kappa''}$. But by the first part of the proof, we then have $\kappa' = \mathrm{sq}_- L_+ \leq \kappa''$, and so $\kappa'' = \kappa'$ and $S_0 \in \mathbf{S}_{\kappa'}$.

The result follows on combining the two parts of the argument. $\qquad\square$

Example. In most cases, condition (iii) in the definition of an admissible family is trivially satisfied, but there are situations in which it does not hold. An example arises in the proof of Theorem 3.7 when $\Delta = \partial\mathbf{D}$; it is not possible to include this case in the main argument there because condition (iii) for an admissible family may fail. To see this, in the proof of Theorem 3.7 (Case 1) allow $\Delta = \partial\mathbf{D}$, and take $c = 1$ and $b = \bar{u}^\kappa$ for some positive integer κ. An element φ of $\mathcal{D} = L^2$ annihilates $c, Ac, A^2 c, \ldots$ if and only if

$$
(A^j c, \varphi) = \int_{\partial\mathbf{D}} u^j \varphi \, d\sigma = 0, \qquad j \geq 0.
$$

This implies that

$$
(A^j b, \varphi) = \int_{\partial\mathbf{D}} u^{j-\kappa} \varphi \, d\sigma = 0, \qquad j \geq \kappa,
$$

but the last identity can fail for $0 \leq j < \kappa$, that is, condition (iii) in the definition of an admissible family does not hold. Nevertheless, the Hermitian form (3.7) has κ negative squares; indeed by (3.5) for φ, ψ in $\mathcal{D}$,

$$
\begin{aligned}
L(\varphi, \psi) &= \lim_{r\uparrow 1} \int_\Delta \int_\Delta \frac{c(u)\overline{c(v)} - b(u)\overline{b(v)}}{1 - r^2 u\bar{v}} \varphi(u)\overline{\psi(v)} \, d\sigma(u)d\sigma(v) \\
&= -\langle (M_{u^\kappa} Q_- M_{u^\kappa}^* - Q_-)\varphi, \psi \rangle_{L^2},
\end{aligned}
$$

where $M_{u^\kappa} Q_- M_{u^\kappa}^* - Q_-$ is the projection of L^2 onto the span of $1, u, \ldots, u^\kappa$. Of course, even in this case the conclusion of Theorem 3.7 is true as explained in Case 2 of the proof. Indeed, the identity $b(u) = S(u)c(u)$ holds σ-a.e. on $\partial\mathbf{D}$ where $S(z) = 1/z^\kappa$ belongs to $\mathbf{S}_\kappa$.

4. Interior and boundary interpolation on a half-plane

Half-plane results are more simply derived by direct application of the Main Theorem than by change of variables from the disk case. A Blaschke product of degree κ on the upper half-plane $\mathbf{C}_+$ is a function of the form

$$B(z) = c \prod_{j=1}^{\kappa} \frac{z - a_j}{z - \bar{a}_j},$$

where c is a constant of modulus one and $a_1, \ldots, a_\kappa$ are points in $\mathbf{C}_+$ (not necessarily distinct). Every such function can be written $B(z) = B_0((z - i)/(z + i))$, where $B_0(z)$ is a Blaschke product of degree κ on the unit disk, and conversely [27]. We say that a meromorphic function $S(z)$ on $\mathbf{C}_+$ belongs to $\mathbf{S}_\kappa(\mathbf{C}_+)$ if it has the form

$$S(z) = S_0\Big(\frac{z - i}{z + i}\Big),$$

where $S_0(z)$ belongs to $\mathbf{S}_\kappa$ as a function on $\mathbf{D}$. If $S(z)$ belongs to $\mathbf{S}_\kappa(\mathbf{C}_+)$, then

$$(4.1) \qquad\qquad f(z) = i\,\frac{1 + S(z)}{1 - S(z)}$$

defines a function in $\mathbf{N}_\kappa$, and every function in $\mathbf{N}_\kappa$ is obtained in this way; when $\kappa = 0$ we exclude $S(z) \equiv 1$ from this correspondence.

If Δ is a (Lebesgue) measurable subset of $(-\infty, \infty)$ and $\varphi, \psi \in L^2(\Delta)$, then [26, pp. 33–34]

$$(4.2) \quad \lim_{\epsilon \downarrow 0} \frac{i}{2} \int_\Delta \int_\Delta \frac{\varphi(s)\overline{\psi(t)}}{s - t + i\epsilon}\, ds\, dt$$

$$= \sum_{j=0}^{\infty} \left(\int_\Delta \Big(\frac{t - i}{t + i}\Big)^j \frac{1}{t + i}\, \varphi(t)\, dt \right) \overline{\left(\int_\Delta \Big(\frac{t - i}{t + i}\Big)^j \frac{1}{t + i}\, \psi(t)\, dt \right)}$$

$$= \pi\, \langle Q_-(\varphi 1_\Delta), \psi 1_\Delta \rangle_{L^2(-\infty, \infty)}.$$

The meaning of Q_- here is different from that of (3.5). Let H^2 now denote the Hardy class for the upper half-plane [27]. We may alternatively view H^2 as a subspace of $L^2(-\infty, \infty)$ by passing to boundary values. In (4.2), Q_- is the projection of $L^2(-\infty, \infty)$ onto the orthogonal complement of H^2 in $L^2(-\infty, \infty)$, or equivalently onto the set of complex conjugates of functions in H^2, since

$$L^2(-\infty, \infty) = H^2 \oplus \overline{H^2}.$$

The last equality in (4.2) follows from the fact that the functions

$$(4.3) \qquad\qquad \frac{1}{\sqrt{\pi}}\Big(\frac{t - i}{t + i}\Big)^j \frac{1}{t + i}, \qquad j = 0, \pm 1, \pm 2, \ldots,$$

are an orthonormal basis for $L^2(-\infty, \infty)$ and the part of (4.3) for $j \geq 0$ is an orthonormal basis for H^2. In particular, the infinite series in the middle term of (4.2) always converges, and the limit on the left side of (4.2) always exists.

The next result is a half-plane counterpart of Theorem 3.7 and preliminary to boundary theorems for the generalized Nevanlinna class.

Theorem 4.1. *Let b and c be complex-valued measurable functions on a Borel subset Δ of $(-\infty, \infty)$ such that $c \neq 0$ a.e. on Δ. Let $\mathcal{D}$ be the set of measurable functions φ on Δ such that $b\varphi, c\varphi \in L^2(\Delta)$. If κ is a nonnegative integer, then there*

is a function $S(z)$ in $\mathbf{S}_\kappa(\mathbf{C}_+)$ whose boundary function satisfies $b(x) = S(x)c(x)$ a.e. on Δ if and only if the Hermitian form

$$(4.4) \quad L(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \frac{i}{2} \int_\Delta \int_\Delta \frac{c(s)\overline{c(t)} - b(s)\overline{b(t)}}{s - t + i\varepsilon} \varphi(s)\overline{\psi(t)} \, ds \, dt, \qquad \varphi, \psi \in \mathcal{D},$$

has κ negative squares.

Proof. Case 1: $(-\infty, \infty) \setminus \Delta$ is not a Lebesgue null set.

Let $\mathcal{V}$ be the vector space of complex-valued functions on Δ of the form

$$p\left(\frac{x - i}{x + i}\right) b(x) + q\left(\frac{x - i}{x + i}\right) c(x), \qquad p, q \text{ polynomials.}$$

We identify any $\varphi \in \mathcal{D}$ with the linear functional on $\mathcal{V}$ defined by

$$(h, \varphi) = \int_\Delta \frac{h(t)\varphi(t)}{t + i} \, dt, \qquad h \in \mathcal{V}.$$

Define data (A, b, c) by choosing A to be multiplication by $(x-i)/(x+i)$ and letting b and c be the given functions. Arguments as in the proof of Theorem 3.7 show that $\mathcal{D}$ is admissible; the assumption that $(-\infty, \infty) \setminus \Delta$ is not a Lebesgue null set is used to verify condition (iii) for an admissible set.

By (4.2), the Hermitian form (2.4) coincides with (4.4). We show that $f_0(A)c = B_0(A)b$ for a pair (f_0, B_0) of holomorphic functions which are bounded by one on the unit disk $\mathbf{D}$ if and only if $B(x)b(x) = f(x)c(x)$ a.e. on Δ, where f and B are defined on the upper half-plane $\mathbf{C}_+$ by

$$f(z) = f_0\left(\frac{z - i}{z + i}\right) = \sum_0^\infty f_j \left(\frac{z - i}{z + i}\right)^j,$$

$$B(z) = B_0\left(\frac{z - i}{z + i}\right) = \sum_0^\infty B_j \left(\frac{z - i}{z + i}\right)^j.$$

If $f_0(A)c = B_0(A)b$, then for every $\varphi \in \mathcal{D}$,

$$\sum_0^\infty f_j(A^j c, \varphi) = \sum_0^\infty B_j(A^j b, \varphi),$$

that is,

$$\sum_0^\infty f_j \int_\Delta \frac{1}{t + i} \left(\frac{t - i}{t + i}\right)^j c(t)\varphi(t) \, dt = \sum_0^\infty B_j \int_\Delta \frac{1}{t + i} \left(\frac{t - i}{t + i}\right)^j b(t)\varphi(t) \, dt.$$

By the arbitrariness of φ, on summing the series in the Abel sense as in [26, p. 32], we obtain

$$f_0\left(\frac{x - i}{x + i}\right) c(x) = B_0\left(\frac{x - i}{x + i}\right) b(x),$$

and hence $f(x)c(x) = B(x)b(x)$ a.e. on Δ. These steps are reversible, and the assertion follows.

To complete the proof in Case 1, we apply the Main Theorem as in the proof of Theorem 3.7.

Case 2: $\Delta = (-\infty, \infty)$.

We reduce this to Case 1 in the same way as in the proof of Theorem 3.7.   □

An indefinite extension of the Generalized Loewner Theorem [26, p. 34] follows as an immediate consequence of Theorem 4.1.

Theorem 4.2. *Let f_0 be a complex-valued measurable function defined on a Borel subset Δ of $(-\infty, \infty)$ such that $f_0 + i \neq 0$ a.e. on Δ. Let $\mathcal{D}$ be the set of measurable functions φ on Δ such that $\varphi, f_0\varphi \in L^2(\Delta)$. If κ is a nonnegative integer, there exists a function $f \in \mathbf{N}_\kappa$ such that $f = f_0$ a.e. on Δ if and only if the Hermitian form*

$$(4.5) \qquad L(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \int_\Delta \int_\Delta \frac{f_0(s) - \overline{f_0(t)}}{s - t + i\varepsilon} \, \varphi(s)\overline{\psi(t)} \, ds \, dt, \qquad \varphi, \psi \in \mathcal{D},$$

has κ negative squares.

The assumption that $f_0 + i \neq 0$ a.e. on Δ does not restrict the generality: it is automatically satisfied if $f = f_0$ a.e. on Δ for some $f \in \mathbf{N}_\kappa$. For if $f(z)$ and $S(z)$ are related by (4.1), then $|S(x)| \leq 1$ by the Kreĭn-Langer representation (2.1), and hence Im $f_0(x) \geq 0$ a.e. on Δ.

Proof. We apply Theorem 4.1 with $b = f_0 - i$ and $c = f_0 + i$. With this choice of b and c, the set Hermitian forms L in Theorems 4.1 and Theorem 4.2 coincide. By Theorem 4.1, sq_$L = \kappa$ if and only if there is a function $S(z)$ in $\mathbf{S}_\kappa(\mathbf{C}_+)$ such that $b(x) = S(x)c(x)$ a.e. on Δ, that is, $f_0(x) - i = S(x)\big[f_0(x) + i\big]$ or

$$f_0(x) = i \, \frac{1 + S(x)}{1 - S(x)}$$

a.e. on Δ. The result thus follows from the correspondence (4.1) between $\mathbf{S}_\kappa(\mathbf{C}_+)$ and $\mathbf{N}_\kappa$. $\square$

The preceding half-plane results can be recast in a different form using Hilbert transforms. The connection comes from equation (4.2) and the formula [27, p. 113]

$$(4.6) \qquad\qquad\qquad\qquad Q_- = (I + iH)/2$$

for the projection onto the orthogonal complement of H^2 in $L^2(-\infty, \infty)$. Here H denotes the Hilbert transform, which is defined for any φ in $L^2(-\infty, \infty)$ by the principal-value integral

$$(H\varphi)(x) = PV \, \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x} \, dt = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{|t-x|>\epsilon} \frac{\varphi(t)}{t - x} \, dt.$$

The limit exists pointwise a.e. on the real line and also in the metric of $L^2(-\infty, \infty)$. The compression of H to $L^2(\Delta)$ for any measurable subset Δ of $(-\infty, \infty)$ is denoted H_Δ: if $\varphi \in L^2(\Delta)$, then

$$(H_\Delta\varphi)(x) = PV \, \frac{1}{\pi} \int_\Delta \frac{\varphi(t)}{t - x} \, dt = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{|t-x|>\epsilon} \frac{\varphi(t)}{t - x} \, dt \qquad \text{a.e. on } \Delta,$$

where the limit is in the metric of $L^2(\Delta)$. The operators iH and iH_Δ are selfadjoint (in particular, $H_\Delta^* = -H_\Delta$), and iH is also unitary. By the convolution theorem ([26, p. 35], [27, p. 111]), for any $\varphi, \psi \in L^2(\Delta) \cap L^\infty(\Delta)$,

$$H_\Delta\big[\varphi(H_\Delta\psi) + (H_\Delta\varphi)\psi\big] = (H_\Delta\varphi)(H_\Delta\psi) - \varphi\psi \qquad \text{a.e. on } \Delta.$$

Lemma 4.3. *Let f_0 be a complex-valued measurable function defined on a Borel subset Δ of $(-\infty, \infty)$ such that $f_0 + i \neq 0$ a.e. on Δ. Let $\mathcal{D}$ be the set of measurable functions φ on Δ such that $\varphi, f_0\varphi \in L^2(\Delta)$. If κ is a nonnegative integer, there exists a function $f \in \mathbf{N}_\kappa$ such that $f = f_0$ a.e. on Δ if and only if the Hermitian form*

$$(4.7) \qquad L(\varphi, \psi) = \pi \left\langle \left(H_\Delta - iI\right)(f_0\varphi), \psi \right\rangle_{L^2(\Delta)} + \pi \left\langle \varphi, \left(H_\Delta - iI\right)(f_0\psi) \right\rangle_{L^2(\Delta)},$$

$\varphi, \psi \in \mathcal{D}$, has κ negative squares. The conclusion remains true if $\mathcal{D}$ is replaced by the set $\mathcal{E} = \mathcal{D} \cap L^\infty(\Delta)$.

Proof. Combining (4.2) and (4.6), we obtain

$$(4.8) \qquad \lim_{\varepsilon \downarrow 0} \int_\Delta \int_\Delta \frac{\varphi(s)\overline{\psi(t)}}{s - t + i\varepsilon} \, ds \, dt = \pi \left\langle \left(H_\Delta - iI\right)\varphi, \psi \right\rangle_{L^2(\Delta)}, \qquad \varphi, \psi \in L^2(\Delta).$$

It follows that the form L in Theorem 4.2 is given by (4.7), and so the first statement of the lemma follows from Theorem 4.2.

To prove the last statement, it is enough to show that for any φ and ψ in $\mathcal{D}$, there are sequences φ_n and ψ_n in $\mathcal{E}$ such that $L(\varphi_n, \psi_n) \to L(\varphi, \psi)$, since this implies that the number of negative squares of L does not decrease when it is restricted to $\mathcal{E} \times \mathcal{E}$. In fact, let $\varphi_n(t) = \varphi(t)$ or 0 according as $|\varphi(t)| \leq n$ or $|\varphi(t)| > n$, and define $\psi_n(t)$ similarly $(n = 1, 2, \dots)$. Then $\varphi_n, \psi_n \in \mathcal{E}$ for all n, and $\varphi_n \to \varphi$, $\psi_n \to \psi$ in the metric of $L^2(\Delta)$. Hence also $f_0\varphi_n \to f_0\varphi$, $f_0\psi_n \to f_0\psi$ in the metric of $L^2(\Delta)$, and so $L(\varphi_n, \psi_n) \to L(\varphi, \psi)$ because H_Δ is a continuous operator on $L^2(\Delta)$. $\square$

We can convert the limit in (4.5) to a different form when f_0 is real valued.

Theorem 4.4. *Let f_0 be a real-valued measurable function defined on a Borel subset Δ of $(-\infty, \infty)$. Let $\mathcal{D}$ be the set of measurable functions φ on Δ such that $\varphi, f_0\varphi \in L^2(\Delta)$. If κ is a nonnegative integer, there exists a function $f \in \mathbf{N}_\kappa$ such that $f = f_0$ a.e. on Δ if and only if the Hermitian form*

$$(4.9) \qquad L(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \iint_{|t-s| > \epsilon} \frac{f_0(s) - f_0(t)}{s - t} \varphi(s)\overline{\psi(t)} \, ds \, dt, \qquad \varphi, \psi \in \mathcal{D},$$

has κ negative squares

Integration in (4.9) is over the set $(\Delta \times \Delta) \cap \{(s, t) \colon |t - s| > \epsilon\}$.

Proof. As in [26, p. 37], for any φ, ψ in $\mathcal{D}$ we have

$$\lim_{\varepsilon \downarrow 0} \iint_{|t-s| > \epsilon} \frac{f_0(s) - f_0(t)}{s - t} \varphi(s)\overline{\psi(t)} \, ds \, dt$$

$$= \lim_{\varepsilon \downarrow 0} \int_\Delta \left(\int_{|t-s| > \epsilon} \frac{f_0(s)\varphi(s)}{s - t} \, ds \right) \overline{\psi(t)} \, dt$$

$$- \lim_{\varepsilon \downarrow 0} \int_\Delta \left(\int_{|t-s| > \epsilon} \frac{\varphi(s)}{s - t} \, ds \right) f_0(t)\overline{\psi(t)} \, dt$$

$$= \pi \left\langle H_\Delta(f_0\varphi), \psi \right\rangle_{L^2(\Delta)} - \pi \left\langle H_\Delta\varphi, f_0\psi \right\rangle_{L^2(\Delta)}$$

$$= \pi \left\langle H_\Delta(f_0\varphi), \psi \right\rangle_{L^2(\Delta)} + \pi \left\langle \varphi, H_\Delta(f_0\psi) \right\rangle_{L^2(\Delta)}.$$

Since f_0 is real-valued, we can rewrite this in the form (4.7), and therefore the assertion follows from Lemma 4.3. $\square$

In the definite case, Theorem 4.4 has a dual result for functions having pure imaginary values [26, pp. 39–41]. A full generalization of the dual result to the class $\mathbf{N}_\kappa$ is unknown, but we are able to present some partial information. For simplicity, we assume that the given function g_0 and set Δ are bounded.

Theorem 4.5. *Let g_0 be a bounded real-valued measurable function on a bounded Borel subset Δ of $(-\infty, \infty)$. Let κ be a nonnegative integer.*

(1) *There is a function $f \in \mathbf{N}_\kappa$ such that $f = -H_\Delta g_0$ a.e. on Δ if and only if the Hermitian form*

$$(4.10) \qquad L(\varphi, \psi) = \pi \int_\Delta \left[\varphi \bar\psi - (H_\Delta \varphi)(\overline{H_\Delta \psi}) \right] g_0 \, dt, \qquad \varphi, \psi \in L^\infty(\Delta),$$

has κ negative squares.

(2) *If the condition in (1) holds and $g_0 \geq 0$ a.e. on Δ, there is a function $h \in \mathbf{N}_{\kappa'}$, $\kappa' \leq \kappa$, such that $h = ig_0$ a.e. on Δ.*

Proof. (1) We apply Lemma 4.3 with the real-valued function $f_0 = -H_\Delta g_0$. Since g_0 and Δ are bounded, $f_0 \in L^2(\Delta)$ and the set $\mathcal{E}$ in Lemma 4.3 is $\mathcal{E} = L^\infty(\Delta)$. As in [26, p. 39], for any φ and ψ in $L^\infty(\Delta)$, (4.7) is given by

$$\begin{aligned} L(\varphi, \psi) &= \pi \left\langle \left(H_\Delta - iI \right)(f_0\varphi), \psi \right\rangle_{L^2(\Delta)} + \pi \left\langle \varphi, \left(H_\Delta - iI \right)(f_0\psi) \right\rangle_{L^2(\Delta)} \\ &= \pi \left\langle H_\Delta(f_0\varphi), \psi \right\rangle_{L^2(\Delta)} + \pi \left\langle \varphi, H_\Delta(f_0\psi) \right\rangle_{L^2(\Delta)} \\ &= -\pi \left\langle f_0\varphi, H_\Delta\psi \right\rangle_{L^2(\Delta)} - \pi \left\langle H_\Delta\varphi, f_0\psi \right\rangle_{L^2(\Delta)} \\ &= -\pi \int_\Delta \left[f_0\varphi(H_\Delta\bar\psi) + (H_\Delta\varphi)f_0\bar\psi \right] dt \\ &= \pi \int_\Delta \left[\varphi(H_\Delta\bar\psi) + (H_\Delta\varphi)\bar\psi \right] H_\Delta g_0 \, dt \\ &= -\pi \int_\Delta \left\{ H_\Delta \left[\varphi(H_\Delta\bar\psi) + (H_\Delta\varphi)\bar\psi \right] \right\} g_0 \, dt \\ &= \pi \int_\Delta \left[\varphi\bar\psi - (H_\Delta\varphi)(\overline{H_\Delta\psi}) \right] g_0 \, dt. \end{aligned}$$

We have shown that (4.7) coincides with (4.10) on $\mathcal{E} \times \mathcal{E}$, and therefore the result follows from Lemma 4.3.

(2) Assume the condition in (1) holds and $g_0 \geq 0$ a.e. on Δ. Let $f_0 = -H_\Delta g_0$ a.e. on Δ. Then by the first part of the theorem there is a function $f \in \mathbf{N}_\kappa$ such that $f(x + i0) = f_0(x)$ a.e. on Δ. Define $h(z)$ and $k(z)$ in the upper half-plane by

$$h(z) = k(z) + f(z) = \frac{1}{\pi} \int_\Delta \frac{g_0(t)}{t - z} \, dt + f(z).$$

Since $g_0 \geq 0$ a.e. on Δ, $k(z)$ has nonnegative imaginary part on the upper half-plane. Hence the kernel

$$\frac{h(z) - \overline{h(\zeta)}}{z - \bar\zeta} = \frac{k(z) - \overline{k(\zeta)}}{z - \bar\zeta} + \frac{f(z) - \overline{f(\zeta)}}{z - \bar\zeta}$$

has $\kappa' \leq \kappa$ negative squares, that is, $h(z)$ belongs to $\mathbf{N}_{\kappa'}$. The boundary function $k_0(x) = k(x + i0)$ of $k(z)$ can be computed from [26, Theorem A, p. 35]:

$$k_0(x) = i \left[g_0(x) - i(H_\Delta g_0)(x) \right] \qquad \text{a.e. on } \Delta.$$

Therefore $h_0(x) = h(x + i0)$ is given by

$$h_0(x) = k_0(x) + f_0(x) = i\left[g_0(x) - i(H_\Delta g_0)(x)\right] - (H_\Delta g_0)(x) = ig_0(x)$$

a.e. on Δ. $\square$

5. Proof of the main result

We use the Ball and Helton almost-commutant lifting theorem [9]. This result has been extended and refined in [8, 16]. Our statement follows [8, Theorem 1.1], which is slightly more precise than the original. In our applications, it is sufficient to take the underlying Kreĭn spaces to be Hilbert spaces. A Hilbert space isometry $W \in \mathfrak{L}(\mathfrak{G})$ is called an **isometric dilation** of an operator $T \in \mathfrak{L}(\mathfrak{H})$ if $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{K}$ for some Hilbert space $\mathfrak{K}$, and

$$W = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$$

relative to the decomposition $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{K}$. If H is a selfadjoint operator on a Hilbert space, $\mathrm{ind}_- H$ is the dimension of the spectral subspace of H corresponding to the interval $(-\infty, 0)$.

Theorem 5.1. *For each $j = 1, 2$, let $T_j \in \mathfrak{L}(\mathfrak{H}_j)$ be a contraction on the Hilbert space $\mathfrak{H}_j$, let $W_j \in \mathfrak{L}(\mathfrak{G}_j)$ be an isometric dilation of T_j on a Hilbert space $\mathfrak{G}_j$, and let P_j be the projection of $\mathfrak{G}_j$ onto $\mathfrak{H}_j$. Let $C \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ be an operator such that $CT_1 = T_2 C$.*

(1) *If $\mathrm{ind}_- (1 - C^*C) = \kappa$, there exists a pair $(\mathcal{E}, \widetilde{C})$ such that $\mathcal{E}$ is a closed W_1-invariant subspace of $\mathfrak{G}_1$ of codimension equal to κ and $\widetilde{C}$ is a contraction operator on $\mathcal{E}$ into $\mathfrak{G}_2$ satisfying*

$$\widetilde{C}W_1|_{\mathcal{E}} = W_2\widetilde{C} \qquad and \qquad P_2\widetilde{C} = CP_1|_{\mathcal{E}}.$$

(2) *If there is a pair $(\mathcal{E}, \widetilde{C})$ with the properties in (1), then $\mathrm{ind}_- (1 - C^*C) \leq \kappa$.*

In part (1), the subspace $\mathcal{E}$ can be chosen to be any closed W_1-invariant subspace of $\mathfrak{G}_1$ of codimension equal to κ such that $CP_1|_{\mathcal{E}}$ is a contraction [8].

Proof of the Main Theorem. Let S be multiplication by z on H^2. By parts (i) and (ii) of Definition 2.1, the set of functions $h_{x'}(z) = \sum_{j=0}^\infty (A^j c, x')z^j$, $x' \in \mathcal{D}$, is a linear subspace $\mathfrak{H}_c$ of H^2 which is invariant under S^*. By part (iii) of Definition 2.1, the formula

$$X_0 \colon \sum_{j=0}^\infty (A^j c, x')z^j \to \sum_{j=0}^\infty (A^j b, x')z^j$$

defines a linear operator X_0 from $\mathfrak{H}_c$ into H^2.

(1) Assume $\mathcal{F}$ has κ negative squares. We show that this assumption implies that $X_0 = Y_0 + K$, where Y_0 is a contraction and K is a finite-rank operator, and hence X_0 is bounded in the norm of H^2. To do this, let $\mathfrak{X}$ be $\mathfrak{H}_c$ as a vector space equipped with the linear and symmetric inner product

$$\langle h_{x'}, h_{y'} \rangle_{\mathfrak{X}} = \mathcal{F}(x', y'), \qquad x', y' \in \mathcal{D}.$$

Since $\mathcal{F}$ has κ negative squares, $\mathfrak{X}$ contains a κ-dimensional subspace $\mathfrak{N}$ which is the antispace of a Hilbert space, and no $(\kappa + 1)$-dimensional subspace of $\mathfrak{X}$ has this property [5, Lemma 1.1.1′]. Writing

$$\mathfrak{N}^{\perp} = \{h \in \mathfrak{X} \colon \langle h, k \rangle_{\mathfrak{X}} = 0 \text{ for all } k \in \mathfrak{N}\},$$

we claim that

(5.1) $$\mathfrak{X} = \mathfrak{N} + \mathfrak{N}^{\perp} \quad \text{and} \quad \mathfrak{N} \cap \mathfrak{N}^{\perp} = \{0\}.$$

If $h \in \mathfrak{N} \cap \mathfrak{N}^{\perp}$, then $\langle h, h \rangle_{\mathfrak{X}} = 0$; this implies $h = 0$ because $h \in \mathfrak{N}$ and the $\mathfrak{X}$-inner product is definite on $\mathfrak{N}$. To see that $\mathfrak{X} = \mathfrak{N} + \mathfrak{N}^{\perp}$, choose a complete orthogonal set $h_1, \ldots, h_{\kappa}$ in $\mathfrak{N}$ (in the $\mathfrak{X}$-inner product). Then any $h \in \mathfrak{X}$ can be written

$$h = \sum_{j=1}^{\kappa} \frac{\langle h, h_j \rangle_{\mathfrak{X}}}{\langle h_j, h_j \rangle_{\mathfrak{X}}} \, h_j + \left(h - \sum_{j=1}^{\kappa} \frac{\langle h, h_j \rangle_{\mathfrak{X}}}{\langle h_j, h_j \rangle_{\mathfrak{X}}} \, h_j \right).$$

The first term here belongs to $\mathfrak{N}$ and the second is in $\mathfrak{N}^{\perp}$, so $\mathfrak{X} = \mathfrak{N} + \mathfrak{N}^{\perp}$ and the claim is proved. Thus $\mathfrak{N}^{\perp}$ is a subspace of $\mathfrak{X}$ having a finite-dimensional complement in $\mathfrak{X}$. For any $h \in \mathfrak{N}^{\perp}$, by the definitions of $\mathcal{F}$ and the $\mathfrak{X}$-inner product,

(5.2) $$0 \le \langle h, h \rangle_{\mathfrak{X}} = \langle h, h \rangle_{H^2} - \langle X_0 h, X_0 h \rangle_{H^2}.$$

Hence $X_0|_{\mathfrak{N}^{\perp}}$ is a contraction in the norm of H^2. We are now able to construct operators Y_0 and K with the required properties. In fact, using (5.2) we first construct a contraction operator Y_0 on $\mathfrak{H}_c$ such that $Y_0|_{\mathfrak{N}^{\perp}} = X_0|_{\mathfrak{N}^{\perp}}$. Then set $K = X_0 - Y_0$. By construction Y_0 is a contraction, and by (5.1) the range of K is the range of $K|_{\mathfrak{N}}$, which has finite dimension; thus $X_0 = Y_0 + K$ where Y_0 is a contraction and K has finite rank. In particular, X_0 is bounded in the norm of H^2. Let X be the extension by continuity of X_0 to the closure of $\mathfrak{H}_c$ in H^2.

We apply Theorem 5.1 with these choices:

$\mathfrak{H}_1 = H^2$ and $T_1 = S$;
$\mathfrak{H}_2 = \overline{\mathfrak{H}}_c$ (closure in H^2) and $T_2 = E_2^* S E_2$, where $E_2 \colon \mathfrak{H}_2 \to H^2$ is inclusion;
$\mathfrak{G}_1 = \mathfrak{G}_2 = H^2$ and $W_1 = W_2 = S$;
$C = X^*$ (as an operator from H^2 to $\mathfrak{H}_2$).

Trivially, W_1 is an isometric dilation of T_1. We show that W_2 is an isometric dilation of T_2. Since $S^* \mathfrak{H}_c \subseteq \mathfrak{H}_c$, $S^* \mathfrak{H}_2 \subseteq \mathfrak{H}_2$. Hence the matrix decomposition of S^* relative to the decomposition $H^2 = \mathfrak{H}_2 \oplus \mathfrak{H}_2^{\perp}$ has the form

$$S^* = \begin{bmatrix} E_2^* S^* E_2 & * \\ 0 & * \end{bmatrix},$$

which is equivalent to

$$W_2 = S = \begin{bmatrix} E_2^* S E_2 & 0 \\ * & * \end{bmatrix} = \begin{bmatrix} T_2 & 0 \\ * & * \end{bmatrix}.$$

It is not hard to see that $CT_1 = T_2 C$ and $\mathrm{ind}_-\,(1 - CC^*) = \mathrm{ind}_-\,(1 - X^* X) = \kappa$; since always $\mathrm{ind}_-\,(1 - C^* C) = \mathrm{ind}_-\,(1 - CC^*)$, we have $\mathrm{ind}_-\,(1 - C^* C) = \kappa$. By Theorem 5.1(1), there is a pair $(\mathcal{E}, \widetilde{C})$ such that

(i) $\mathcal{E}$ is a closed subspace of H^2 of codimension κ which is invariant under S,
(ii) $\widetilde{C} : \mathcal{E} \to H^2$ is a contraction,
(iii) $P_{\mathfrak{H}_2} \widetilde{C} = X^*|_{\mathcal{E}}$,
(iv) $\widetilde{C} S|_{\mathcal{E}} = S \widetilde{C}$.

For any function $F(z)$ in $\mathbf{S}_0$, write $\widetilde{F}(z) = \overline{F(\bar{z})}$. By (i), $\mathcal{E} = \widetilde{B}H^2$, where $B(z)$ is a Blaschke product of degree κ. For any $\varphi \in \mathbf{S}_0$, let M_φ be the operator multiplication by φ on H^2. We show that $\widetilde{C}M_{\widetilde{B}}$ commutes with S. In fact, for each $h \in H^2$, $\widetilde{B}h \in \mathcal{E}$ and

$$\widetilde{C}M_{\widetilde{B}}Sh = \widetilde{C}S(\widetilde{B}h) \stackrel{(iv)}{=} S\widetilde{C}\widetilde{B}h = S\widetilde{C}M_{\widetilde{B}}h.$$

Since $\widetilde{C}M_{\widetilde{B}}$ is a contraction by (ii), $\widetilde{C}M_{\widetilde{B}} = M_{\tilde{f}}$ for some $f \in \mathbf{S}_0$. For any $x' \in \mathcal{D}$ and $h \in H^2$,

$$\left\langle \sum_{j=0}^{\infty}(A^j c, x')\, z^j, M_{\tilde{f}}h \right\rangle_{H^2} = \left\langle \sum_{j=0}^{\infty}(A^j c, x')\, z^j, \widetilde{C}M_{\widetilde{B}}h \right\rangle_{H^2}$$

$$= \left\langle \sum_{j=0}^{\infty}(A^j c, x')\, z^j, P_{\mathfrak{H}_2}\widetilde{C}M_{\widetilde{B}}h \right\rangle_{H^2} \stackrel{(iii)}{=} \left\langle \sum_{j=0}^{\infty}(A^j c, x')\, z^j, X^* M_{\widetilde{B}}h \right\rangle_{H^2}$$

$$= \left\langle \sum_{j=0}^{\infty}(A^j b, x')\, z^j, \widetilde{B}h \right\rangle_{H^2}.$$

Choosing $h = 1$, we obtain

$$(5.3) \qquad \left\langle \sum_{j=0}^{\infty}(A^j c, x')\, z^j, \tilde{f} \right\rangle_{H^2} = \left\langle \sum_{j=0}^{\infty}(A^j b, x')\, z^j, \widetilde{B} \right\rangle_{H^2}, \qquad x' \in \mathcal{D}.$$

Thus (2.3) and hence (2.2) hold. This completes the proof of part (1) of the Main Theorem.

(2) Assume that (2.2) holds with f in $\mathbf{S}_0$ and B a Blaschke product of degree κ. We show that this assumption implies that the operator X_0 defined at the beginning of the proof is bounded. Define $\tilde{f}$ and $\widetilde{B}$ as in part (1) above. The assumption (2.2) can be rewritten in the form (5.3). Hence for all $x' \in \mathcal{D}$,

$$\left\langle z\tilde{f}, \sum_{j=0}^{\infty}(A^j c, x')\, z^j \right\rangle_{H^2} = \left\langle \tilde{f}, S^*\left(\sum_{j=0}^{\infty}(A^j c, x')\, z^j \right) \right\rangle_{H^2}$$

$$= \left\langle \tilde{f}, \sum_{j=0}^{\infty}(A^j c, A'x')\, z^j \right\rangle_{H^2} \stackrel{(5.3)}{=} \left\langle \widetilde{B}, \sum_{j=0}^{\infty}(A^j b, A'x')\, z^j \right\rangle_{H^2}$$

$$= \left\langle z\widetilde{B}, \sum_{j=0}^{\infty}(A^j b, x')\, z^j \right\rangle_{H^2}.$$

Writing as before $h_{x'}(z) = \sum_{j=0}^{\infty}(A^j c, x')\, z^j$ for all $x' \in \mathcal{D}$, we can extend this relation to

$$(5.4) \qquad \left\langle \tilde{f}h, h_{x'} \right\rangle_{H^2} = \left\langle \widetilde{B}h, X_0 h_{x'} \right\rangle_{H^2}$$

first for every polynomial h and then for every h in H^2. Now the boundedness of X_0 follows from the fact that $\mathcal{E} = \widetilde{B}H^2$ has codimension κ in H^2. Let X be the extension by continuity of X_0 to the closure $\mathfrak{H}_2$ of $\mathfrak{H}_c$ in H^2. Since

$$\mathcal{F}(x', y') = \langle h_{x'}, h_{y'} \rangle_{H^2} - \langle Xh_{x'}, Xh_{y'} \rangle_{H^2}, \qquad x', y' \in \mathcal{D},$$

to show that (2.4) has κ' negative squares for some $\kappa' \leq \kappa$, it is sufficient to show that $\mathrm{ind}_- (1 - X^*X)$ is at most κ. Write

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

relative to the decomposition $H^2 = \mathcal{E} \oplus \mathcal{E}^\perp$. By (5.4), $X^* M_{\widetilde{B}} = P_{\mathfrak{H}_2} M_{\tilde{f}}$ is a contraction. Therefore $X_1^* = X^*|_{\mathcal{E}}$ and X_1 are contractions. Thus

$$1 - X^*X = (1 - X_1^*X_1) - X_2^*X_2,$$

where $1 - X_1^*X_1 \geq 0$ and $-X_2^*X_2$ has rank at most κ. Hence $\mathrm{ind}_- (1 - X^*X) \leq \kappa$ and part (2) of the Main Theorem follows. $\qquad\square$

We restate the Main Theorem in a way which characterizes all solutions of Nudel'man's problem in terms of the solutions of the corresponding lifting problem. In the statement of this result, we identify two pairs (f_1, B_1) and (f_2, B_2) such that $f_1 = \gamma f_2$ and $B_1 = \gamma B_2$ for some constant γ, $|\gamma| = 1$, and we adopt notation as in the proof of the Main Theorem. In particular, S is multiplication by z on H^2. For any $\varphi \in \mathbf{S}_0$, let M_φ be multiplication by φ on H^2. We write $\tilde{\varphi}(z) = \sum_0^\infty \bar{\varphi}_n z^n$ if $\varphi(z) = \sum_0^\infty \varphi_n z^n$.

Main Theorem (Alternative Form). *Let (A, b, c) be given data, $\mathcal{D}$ an admissible set. Define $\mathcal{F}$ on $\mathcal{D} \times \mathcal{D}$ by (2.4), and let κ be a nonnegative integer. Then*

$$\mathrm{sq}_- \mathcal{F} \leq \kappa$$

if and only if there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that $f(A)c = B(A)b$. The set of all such pairs (f, B) is in one-to-one correspondence with the set of all pairs $(\mathcal{E}, \widetilde{C})$ such that

(i) *$\mathcal{E}$ is a closed subspace of H^2 of codimension κ which is invariant under S;*
(ii) *$\widetilde{C} \colon \mathcal{E} \to H^2$ is a contraction operator;*
(iii) *if $P_\mathcal{E}$ is the projection onto $\mathcal{E}$, then for all $x' \in \mathcal{D}$,*

$$\widetilde{C}^* \colon \sum_{j=0}^\infty (A^j c, x') z^j \to P_\mathcal{E}\left\{ \sum_{j=0}^\infty (A^j b, x') z^j \right\};$$

(iv) *$\widetilde{C} S|_{\mathcal{E}} = S\widetilde{C}$.*

The correspondence is determined by the relations $\mathcal{E} = \widetilde{B} H^2$ and $\widetilde{C} M_{\widetilde{B}} = M_{\tilde{f}}$.

Proof. If $\mathrm{sq}_- \mathcal{F} = \kappa' \leq \kappa$, then by part (1) of the main theorem there is a pair (f_1, B_1) where $f_1 \in \mathbf{S}_0$ and B_1 is a Blaschke product of degree κ', such that $f_1(A)c = B_1(A)b$. If $\kappa' = \kappa$, the pair $(f, B) = (f_1, B_1)$ has the required properties. Otherwise write $\kappa = \kappa' + r$ and let $(f(z), B(z)) = (z^r f_1(z), z^r B_1(z))$. Then $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ. If

$$f_1(z) = \sum_{j=0}^\infty \alpha_j z^j \quad \text{and} \quad B_1(z) = \sum_{j=0}^\infty \beta_j z^j,$$

then the coefficients of $f(z)$ consist of r zeros followed by $\alpha_0, \alpha_1, \ldots$, and similarly for $B(z)$. In order to prove that $f(A)c = B(A)b$, we must therefore show that for all $x' \in \mathcal{D}$,

$$\sum_{j=0}^\infty \alpha_j (A^{j+r} c, x') = \sum_{j=0}^\infty \beta_j (A^{j+r} b, x').$$

By the invariance of $\mathcal{D}$ under A', this identity is equivalent to

$$\sum_{j=0}^{\infty} \alpha_j(A^j c, (A')^r x') = \sum_{j=0}^{\infty} \beta_j(A^j b, (A')^r x'),$$

which holds because $f_1(A)c = B_1(A)b$. This proves the necessity part of the first statement. Sufficiency follows from part (2) of the Main Theorem.

Given a pair (f, B) where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ such that $f(A)c = B(A)b$, define $(\mathcal{E}, \widetilde{C})$ by $\mathcal{E} = \widetilde{B}H^2$ and $\widetilde{C}M_{\widetilde{B}} = M_{\widetilde{f}}$. Obviously (i) and (ii) hold, and we easily check (iv). To prove (iii), we use the identity $f(A)c = B(A)b$ to show that

$$\left\langle \sum_{j=0}^{\infty}(A^j c, x')z^j, \tilde{f}h \right\rangle_{H^2} = \left\langle \sum_{j=0}^{\infty}(A^j b, x')z^j, \widetilde{B}h \right\rangle_{H^2}$$

first for monomials $h(z) = z^k$ and hence for all $h \in H^2$. Therefore for all $x' \in \mathcal{D}$ and $h \in H^2$,

$$\left\langle \sum_{j=0}^{\infty}(A^j c, x')z^j, \widetilde{C}\widetilde{B}h \right\rangle_{H^2} = \left\langle \sum_{j=0}^{\infty}(A^j b, x')z^j, \widetilde{B}h \right\rangle_{H^2},$$

and since $\mathcal{E} = \widetilde{B}H^2$, this implies (iii).

Conversely, if $(\mathcal{E}, \widetilde{C})$ is a pair that satisfies (i)–(iv), the proof of the Main Theorem shows that $\mathcal{E} = \widetilde{B}H^2$ and $\widetilde{C}M_{\widetilde{B}} = M_{\widetilde{f}}$, where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that $f(A)c = B(A)b$. The correspondence is one-to-one because the relation $\mathcal{E} = \widetilde{B}H^2$ determines B up to a constant factor of modulus one. $\qquad\square$

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