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# Regular Functions on the Space of Cayley Numbers

Graziano Gentili  
*University of Florence*

Daniele C. Struppa  
*Chapman University, [struppa@chapman.edu](mailto:struppa@chapman.edu)*

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# Regular Functions on the Space of Cayley Numbers

## **Comments**

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## REGULAR FUNCTIONS ON THE SPACE OF CAYLEY NUMBERS

GRAZIANO GENTILI AND DANIELE C. STRUPPA

**ABSTRACT.** In this paper we present a new definition of regularity on the space  $\mathbf{O}$  of Cayley numbers (often referred to as octonions), based on a Gateaux-like notion of derivative. We study the main properties of regular functions, and we develop the basic elements of a function theory on  $\mathbf{O}$ . Particular attention is given to the structure of the zero sets of such functions.

**1. Introduction.** Let  $\mathbf{O}$  denote the nonassociative, alternative, division algebra of real Cayley numbers (also known as octonions). We refer the reader to the excellent survey [2] for a thorough discussion of the importance and interest of this object. A simple way to describe the construction of this algebra is to consider a basis  $\mathcal{E} = \{e_0 = 1, e_1, \dots, e_6, e_7\}$  of  $\mathbf{R}^8$  and relations

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} e_\gamma, \quad \alpha, \beta, \gamma = 1, 2, \dots, 7,$$

where  $\delta_{\alpha\beta}$  is the Kronecker delta, and  $\psi_{\alpha\beta\gamma}$  is totally antisymmetric in  $\alpha, \beta, \gamma$ , nonzero and equal to 1 on the seven combinations in the following set

$$\sigma = \{(1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 4, 7), (2, 5, 7), (1, 6, 7), (5, 3, 6)\},$$

so that every element in  $\mathbf{O}$  can be written as  $w = x_0 + \sum_{k=1}^7 x_k e_k$ . One can then define in a natural fashion its conjugate  $\bar{w} = x_0 - \sum_{k=1}^7 x_k e_k$ , and its square norm  $|w|^2 = w\bar{w} = \sum_{k=0}^7 x_k^2$ .

The basic elements of  $\mathbf{O}$  can be written (see e.g. [16]) as

$$e_0 = 1, e_1, e_2, e_1 e_2, e_4, e_1 e_4, e_2 e_4, (e_1 e_2) e_4.$$

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One can use this representation, together with the property of the algebra being alternative (i.e., the subalgebra generated by any two elements is associative), to construct the multiplication table for  $\mathbf{O}$  with which one can verify all the subsequent computations. In particular, one sees that  $(1, e_1, e_2, e_1e_2)$  form a basis for a subalgebra of  $\mathbf{O}$  isomorphic to the algebra  $\mathbf{H}$  of quaternions. It is easy to prove the following technically important decomposition.

**Proposition 1.1.** *A generic element of  $\mathbf{O}$  can be written as*

$$w = \sum_{k=0}^7 x_k e_k = (x_0 + x_1 e_1) + (x_2 + x_3 e_1) e_2 + [(x_4 + x_5 e_1) + (x_6 + x_7 e_1) e_2] e_4.$$

This proposition shows that every Cayley number can be thought of as four complex numbers (each one in  $\mathbf{C} = \mathbf{R} + \mathbf{R}e_1$ ) or as two quaternions (each one in  $\mathbf{C} + \mathbf{C}e_2$ ). We have therefore the decomposition

$$(1) \quad \begin{aligned} \mathbf{O} &= (\mathbf{R} + \mathbf{R}e_1) + (\mathbf{R} + \mathbf{R}e_1)e_2 + [(\mathbf{R} + \mathbf{R}e_1) + (\mathbf{R} + \mathbf{R}e_1)e_2]e_4 \\ &= \mathbf{C} + \mathbf{C}e_2 + (\mathbf{C} + \mathbf{C}e_2)e_4 = \mathbf{H} + \mathbf{H}e_4. \end{aligned}$$

Since differential operators in the spirit of Dirac can be defined on functions on  $\mathbf{O}$ , in the last few years many papers have appeared to study regular functions on  $\mathbf{O}$ , defined as null-solutions of such operators. We refer the reader to [5, 7, 14–16, 18] for more details on these functions. The study of null-solutions of Dirac-like operators is akin to attempting to extend the theory of Fueter-regular functions on quaternions (see for example [8, 9] as well as [4]) to the case of Cayley numbers.

Quite recently, the authors have offered an alternative definition and theory of regularity for functions of quaternionic variables, inspired by an idea of Cullen [6]. This alternative theory is intriguing because it allows the study of power series with quaternionic coefficients, which is otherwise excluded when the Fueter approach is followed. A description of this theory can be found in [11, 12], but the interested reader can find the most recent developments in [3, 10].

In this paper we show how the ideas contained in [12] can be used to construct a similar theory for functions defined on Cayley numbers. We refer to [13] for some interesting extensions and applications of our results.

Let us denote by  $\mathbf{S}$  the unit sphere of purely imaginary Cayley numbers, i.e.,  $\mathbf{S} = \{w = \sum_{k=1}^7 x_k e_k \text{ such that } \sum_{k=1}^7 x_k^2 = 1\}$ . Notice that if  $I \in \mathbf{S}$ , then  $I^2 = -1$ ; for this reason the elements of  $\mathbf{S}$  are called imaginary units.

**Definition 1.2.** Let  $\Omega$  be a domain in  $\mathbf{O}$ . A real differentiable function  $f : \Omega \rightarrow \mathbf{O}$  is said to be regular if, for every  $I \in \mathbf{S}$ , its restriction  $f_I$  to the complex line  $L_I = \mathbf{R} + \mathbf{R}I$  passing through the origin and containing 1 and  $I$  is holomorphic on  $\Omega \cap L_I$ . With an abuse of notation, we will also call regular those functions defined on an open subset of a quaternionic subspace  $\mathcal{H}$  of  $\mathbf{O}$ , and whose restriction to  $L_I$  is holomorphic for every  $I \in \mathbf{S} \cap \mathcal{H}$ .

*Remark 1.3.* The requirement that  $f : \Omega \rightarrow \mathbf{O}$  is regular is equivalent to require that, for every  $I$  in  $\mathbf{S}$ ,

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + \partial \partial y \right) f_I(x + yI) = 0,$$

on  $\Omega \cap L_I$ .

*Remark 1.4.* Notice that, just like in [12], this definition of regularity can be interpreted in the spirit of the Gateaux derivative.

We can define a notion of  $I$ -derivative as follows:

**Definition 1.5.** Let  $\Omega$  be a domain in  $\mathbf{O}$ , and let  $f : \Omega \rightarrow \mathbf{O}$  be a real differentiable function. For any  $I \in \mathbf{S}$  and any point  $w = x + yI$  in  $\Omega$  ( $x$  and  $y$  are real numbers here) we define the  $I$ -derivative of  $f$  in  $w$  by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI).$$

In this paper we prove that regular functions can be expressed in power series of the form

$$\sum_{n=0}^{+\infty} w^n a_n,$$

with  $a_n \in \mathbf{O}$ . Note that such power series are well defined because the algebra  $\mathbf{O}$  is alternative and therefore power-associative (see for example [2]).

The crux of the paper is Section 2, where we study the complex geometry of vectors in  $\mathbf{O}$  (Proposition 2.5), we show how to build suitable bases for  $\mathbf{O}$  (Proposition 2.6), and we use these bases to represent regular functions on  $\mathbf{O}$  as four-tuples of holomorphic functions. This result (Lemma 2.7) will then allow us to mimic the proofs in [12] to build a theory for these regular functions, including the identity principle, the maximum modulus principle, the Cauchy representation formula and corresponding estimates, the Liouville theorem, the Morera theorem, the Schwarz lemma and the biregularity of the unit ball to the eight-dimensional analog of the Siegel right-half plane. Finally, we are able to describe the zeros of regular functions, with a result inspired by [17]. In particular, we show that, for a class of regular functions, the zero set consists of the union of isolated points and isolated six-spheres.

**2. Power series and series expansions for regular functions of Cayley numbers.** In order to study polynomials and power series in  $w \in \mathbf{O}$ , we first note that the basic polynomial  $w^n a$ , with  $a$  an octonion, is regular according to Definition 1.2. Since the sum of regular functions is clearly regular, we immediately have that polynomials with octonionic coefficients on the right are regular. Classical arguments (see, e.g., [1]) yield the analog of the Abel's theorem.

**Theorem 2.1.** *For every power series  $\sum_{n=0}^{\infty} w^n a_n$  there exists a number  $R$ ,  $0 \leq R \leq \infty$ , called the radius of convergence, such that the series converges absolutely for every  $w$  with  $|w| < R$  and uniformly for every  $w$  with  $|w| \leq \rho < R$ . Moreover, if  $|w| > R$ , the series is divergent.*

Since convergence of power series is uniform on compact sets, it turns out that power series are regular in their domain of convergence. Note that every power series is also real analytic.

The first important consequence of our definition of regularity is that, for regular functions, we can introduce a notion of derivative.

**Definition 2.2.** Let  $\Omega$  be a domain in  $\mathbf{O}$ , and let  $f : \Omega \rightarrow \mathbf{O}$  be a regular function. The derivative of  $f$ ,  $\partial f$ , is defined as follows:

$$\partial(f)(w) = \begin{cases} \partial_I(f)(w) & \text{if } w = x + yI \text{ with } y \neq 0 \\ \frac{\partial f}{\partial x}(x) & \text{if } w = x \text{ is real.} \end{cases}$$

This definition of derivative is well posed because it is applied only to regular functions.

Let  $f$  be a regular function. Since for every  $I$  in  $\mathbf{S}$  it is  $\bar{\partial}_I(\partial(f)) = \partial(\bar{\partial}_I(f)) = 0$  we obtain that the derivative of a regular function is still regular.

Note also that the derivative of a power series can be done term by term because of the uniform convergence, so that

$$\partial \left( \sum_{n=0}^{\infty} w^n a_n \right) = \sum_{n=1}^{\infty} w^{n-1} n a_n,$$

has the same radius of convergence of the original series.

In what follows, we will always restrict our attention to functions which are regular on an open ball  $B(0, R)$  centered in the origin and of radius  $R$ .

In order for us to study regular functions, we will need a simple representation of the restriction of a regular function as four holomorphic functions. To do so, we need a few preliminary results on the set  $\mathbf{S}$ .

First we consider two elements  $I = \sum_{k=1}^7 x_k e_k$  and  $J = \sum_{k=1}^7 y_k e_k$  in  $\mathbf{S}$ . Construct the  $3 \times 7$  matrix

$$M = M(I, J) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{pmatrix},$$

and let  $M(r, s, t)$  be the  $3 \times 3$  minor of  $M$  formed by columns  $r, s$ , and  $t$ . Then we define the vector product  $I \times J$  as:

$$I \times J = \sum_{(r,s,t) \in \sigma} \det(M(r, s, t)).$$

Note that  $I \times J$  is always orthogonal to both  $I$  and  $J$  just as in the classical case of three dimensions.

**Proposition 2.3.** *Let  $I$  and  $J$  be two elements in  $\mathbf{S}$ , and let  $\langle I, J \rangle$  denote their Euclidean scalar product. Then their product  $IJ$  can be computed through the following formula:*

$$IJ = -\langle I, J \rangle + I \times J.$$

*Proof.* The result follows immediately from the direct computation of the product  $IJ$ .  $\square$

The previous proposition shows, in particular, that the product of two orthogonal elements of  $\mathbf{S}$  is purely imaginary (in particular it lies in  $\mathbf{S}$  as well, as we will show in a moment). In fact, there is an interesting consequence of this result, which allows a speedy computation for mixed products of vectors in  $\mathbf{S}$ . Let  $I = \sum_{k=1}^7 x_k e_k$ ,  $J = \sum_{k=1}^7 y_k e_k$  and  $K = \sum_{k=1}^7 z_k e_k$  be three vectors in  $\mathbf{S}$ . We define the  $3 \times 7$  matrix

$$N = N(I, J, K) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \end{pmatrix},$$

and let  $N(r, s, t)$  be the  $3 \times 3$  minor of  $N$  formed by columns  $r, s$  and  $t$ . Since the determinant of each such minor changes its sign if two rows are exchanged, and in view of the previous proposition, applied in the case of orthogonal vectors, we immediately have

**Proposition 2.4.** *If  $I$  is orthogonal both to  $J$  and  $K$ , we have that*

$$\langle IJ, K \rangle = -\langle IK, J \rangle = \sum_{(r,s,t) \in \sigma} \det(N(r, s, t)).$$



We will use these facts to build orthogonal bases in  $\mathbf{S}$ .

**Proposition 2.5.** *Let  $I_1$  and  $I_2$  be two orthogonal elements in  $\mathbf{S}$ , and let  $I_3 = I_1 I_2$ . Then:*

1.  $I_1 I_2 = -I_2 I_1$  is an element of  $\mathbf{S}$ ,
2.  $I_3$  is orthogonal to both  $I_1$  and  $I_2$ ,
3.  $I_2 I_3 = -I_3 I_2 = I_1$  and  $I_3 I_1 = -I_1 I_3 = I_2$ .

*In particular the basis  $(1, I_1, I_2, I_3)$  spans a subalgebra  $\mathcal{H}$  of  $\mathbf{O}$  isomorphic to  $\mathbf{H}$ .*

*Proof.* We will prove the three statements independently.

1. First note that since  $I_1$  and  $I_2$  are orthogonal,

$$I_1 I_2 = I_1 \times I_2 = -I_2 \times I_1 = -I_2 I_1.$$

To prove that  $|I_3| = 1$ , we note that  $I_1 I_2 = I_1 \times I_2$  is orthogonal to  $I_1$ . Therefore,

$$\langle I_1 I_2, I_1 I_2 \rangle = -\langle I_1 (I_1 I_2), I_2 \rangle = \langle I_2, I_2 \rangle = 1.$$

2. This follows immediately from the orthogonality of  $I_1$  and  $I_2$ .
3. We will prove just the first equality, since the technique is always the same.

$$I_2 I_3 = I_2 (I_1 I_2) = -I_2 (I_2 I_1) = I_1. \quad \square$$

We now note that  $I_1, I_2$  and  $I_3$  generate a 2-sphere inside the six-dimensional sphere  $\mathbf{S}$ . There is therefore room for four additional orthogonal vectors.

**Proposition 2.6.** *Let  $I_4 \in \mathbf{S}$  be orthogonal to  $I_1, I_2$  and  $I_3$ . Then*

1. *The vectors  $I_4, I_5 = I_1 I_4, I_6 = I_2 I_4$  and  $I_7 = I_3 I_4$  are orthonormal and lie in  $\mathcal{H}^\perp$ .*

2. *The vectors  $1, I_1, \dots, I_7$  are a basis  $\mathcal{B}$  for the algebra  $\mathbf{O}$ . Moreover, it is possible to choose  $I_1, I_2$  and  $I_4$  oriented in such a way that the resulting basis  $\mathcal{B}$  has the same values for the coefficients  $\psi_{\alpha,\beta,\gamma}$  given for the original basis  $\mathcal{E}$ , and therefore has the same multiplication table.*

*Proof.* We will prove the two statements independently.

1. We need to show three kinds of perpendicularity: that  $I_r$  is perpendicular to  $I_s$  with  $r = 1, 2, 3$  and  $s = 5, 6, 7$ , that  $I_4$  is perpendicular to  $I_r$  with  $r, s = 5, 6, 7$ , and that  $I_r$  is perpendicular to  $I_s$  with  $r, s = 5, 6, 7$ ,  $r \neq s$ . We also need to show that  $|I_r|^2 = 1$  for  $r = 5, 6, 7$ . As to the perpendicularity, we will use the previous proposition to show that  $I_5$  is perpendicular to  $I_2, I_4$  and  $I_6$ , since all the other verifications are essentially the same. In fact,

$$\begin{aligned} \langle I_5, I_2 \rangle &= \langle I_1 I_4, I_2 \rangle = -\langle I_1 I_2, I_4 \rangle = \langle I_3, I_4 \rangle = 0, \\ \langle I_5, I_4 \rangle &= \langle I_1 I_4, I_4 \rangle = 0, \end{aligned}$$

and since  $I_4$  and  $I_2 I_4$  are perpendicular, and  $\mathbf{O}$  is alternative,

$$\langle I_5, I_6 \rangle = \langle I_1 I_4, I_2 I_4 \rangle = -\langle I_4 I_1, I_2 I_4 \rangle = \langle I_4 I_2 I_4, I_1 \rangle = \langle I_2, I_1 \rangle = 0.$$

To prove normality, we simply show that  $|I_5|^2 = 1$ . Indeed

$$|I_5|^2 = \langle I_1 I_4, I_1 I_4 \rangle = -\langle I_1 I_1 I_4, I_4 \rangle = \langle I_4, I_4 \rangle = 1.$$

2. To begin with, we write the multiplication table as much as we can by using the definitions of the elements of the new base. As shown below, this gives the following table, where a few multiplications cannot yet be determined.

	$I_1$	$I_2$	$I_3$	$I_4$	$I_5$	$I_6$	$I_7$
$I_1$	-1	$I_3$	$-I_2$	$I_5$	$-I_4$		
$I_2$	$-I_3$	-1	$I_1$	$I_6$		$-I_4$	
$I_3$	$I_2$	$-I_1$	-1	$I_7$			$-I_4$
$I_4$	$-I_5$	$-I_6$	$-I_7$	-1	$I_1$	$I_2$	$I_3$
$I_5$	$I_4$			$-I_1$	-1		
$I_6$		$I_4$		$-I_2$		-1	
$I_7$			$I_4$	$-I_3$			-1

To attempt the completion of this table, we note that it is possible to determine that all the products  $I_r I_s$  are, up to a sign, other elements

of  $\mathcal{B}$ . In fact, for example, for any  $r \neq 1, 6, 7$ , one has (using the table above)

$$\langle I_1 I_7, I_r \rangle = -\langle I_1 I_r, I_7 \rangle = \pm \langle I_t, I_7 \rangle = 0,$$

since  $t \neq 7$ . Similarly, it is obvious to see that  $\langle I_1 I_7, I_1 \rangle = \langle I_1 I_7, I_7 \rangle = 0$ . This implies that  $I_1 I_7$  must be a multiple of  $I_6$ . Since  $|I_1 I_7| = 1$ , it is obvious that  $I_1 I_7 = \pm I_6$ . Note however that we have no indication as to which sign one needs to use to complete the multiplication table. Indeed, different choices of  $I_1, I_2, I_4$  will yield different signs. We keep this in mind and assume, at this point, that we have the negative sign, so that we assume  $I_1 I_7 = -I_6$  to be consistent with the table associated to the initial choice of  $\psi_{\alpha, \beta, \gamma}$ . Using this choice, we can easily add more entries to the table and obtain

	$I_1$	$I_2$	$I_3$	$I_4$	$I_5$	$I_6$	$I_7$
$I_1$	-1	$I_3$	$-I_2$	$I_5$	$-I_4$	$I_7$	$-I_6$
$I_2$	$-I_3$	-1	$I_1$	$I_6$		$-I_4$	
$I_3$	$I_2$	$-I_1$	-1	$I_7$			$-I_4$
$I_4$	$-I_5$	$-I_6$	$-I_7$	-1	$I_1$	$I_2$	$I_3$
$I_5$	$I_4$			$-I_1$	-1		
$I_6$	$-I_7$	$I_4$		$-I_2$		-1	
$I_7$	$I_6$		$I_4$	$-I_3$			-1

We can now argue as before and determine that  $I_2 I_7 = \pm I_5$  and, for consistency with the choice of  $\psi_{\alpha, \beta, \gamma}$ , we select  $I_2 I_7 = -I_5$ . Once again, this choice allows us to further complete the table and get

	$I_1$	$I_2$	$I_3$	$I_4$	$I_5$	$I_6$	$I_7$
$I_1$	-1	$I_3$	$-I_2$	$I_5$	$-I_4$	$I_7$	$-I_6$
$I_2$	$-I_3$	-1	$I_1$	$I_6$	$I_7$	$-I_4$	$-I_5$
$I_3$	$I_2$	$-I_1$	-1	$I_7$	$-I_6$	$I_5$	$-I_4$
$I_4$	$-I_5$	$-I_6$	$-I_7$	-1	$I_1$	$I_2$	$I_3$
$I_5$	$I_4$	$-I_7$	$I_6$	$-I_1$	-1		
$I_6$	$-I_7$	$I_4$	$-I_5$	$-I_2$		-1	
$I_7$	$I_6$	$I_5$	$I_4$	$-I_3$			-1

We need to remark that at this point we are using the so-called Moufang identities (see, e.g., [16 and references therein]). Specifically, we use the fact that for any three octonions  $u, v, x$ , one has  $(uvu)x = u(v(ux))$ . In our case, for example, we can compute  $I_3I_6$  by observing that

$$I_7 = I_3I_4 = -(I_2I_2I_3)(I_4) = (I_2I_3I_2)I_4 = I_2(I_3(I_2I_4)) = I_2(I_3I_6),$$

and therefore that

$$I_3I_6 = -I_2I_7 = I_5.$$

We can finally complete the multiplication table with one more choice, namely the sign of  $I_5I_7$  which we take as  $I_2$ . The full table is now:

	$I_1$	$I_2$	$I_3$	$I_4$	$I_5$	$I_6$	$I_7$
$I_1$	$-1$	$I_3$	$-I_2$	$I_5$	$-I_4$	$I_7$	$-I_6$
$I_2$	$-I_3$	$-1$	$I_1$	$I_6$	$I_7$	$-I_4$	$-I_5$
$I_3$	$I_2$	$-I_1$	$-1$	$I_7$	$-I_6$	$I_5$	$-I_4$
$I_4$	$-I_5$	$-I_6$	$-I_7$	$-1$	$I_1$	$I_2$	$I_3$
$I_5$	$I_4$	$-I_7$	$I_6$	$-I_1$	$-1$	$-I_3$	$I_2$
$I_6$	$-I_7$	$I_4$	$-I_5$	$-I_2$	$I_3$	$-1$	$I_1$
$I_7$	$I_6$	$I_5$	$I_4$	$-I_3$	$-I_2$	$-I_1$	$-1$

To conclude the proof we simply note that the table we have constructed depends on three arbitrary choices of signs. However, we have enough freedom to make such choices, because we can choose the signs of  $I_1, I_2$  and  $I_4$  in  $2^3$  different ways. In the proof we have arbitrarily selected signs for  $I_1I_7 = I_1(I_3I_4)$ ,  $I_2I_7 = I_2(I_3I_4)$  and  $I_5I_7 = (I_1I_4)(I_3I_4)$ . It is immediate to verify that all possible cases are covered.  $\square$

The result we have just proved shows that we can use  $I_1, \dots, I_7$ , as a basis for  $\mathbf{S}$ ; moreover, given any element  $I_1$  in  $\mathbf{S}$ , we can always construct such a basis (though not in a unique way, as the basis will ultimately depend on the choices of  $I_2$  and  $I_4$ ).

The following splitting lemma plays a key role throughout the paper.

**Lemma 2.7.** *If  $f$  is a regular function on  $B = B(0, R)$ , then for every  $I_1 \in \mathbf{S}$ , we can find  $I_2$  and  $I_4$  in  $\mathbf{S}$  such that there are four holomorphic functions  $F_1, F_2, G_1, G_2$  from  $B \cap L_{I_1}$  to  $L_{I_1}$  such that, for any  $z = x + yI_1$ , it is*

$$f_{I_1}(z) = F_1(z) + F_2(z)I_2 + (G_1(z) + G_2(z)I_2)I_4.$$

*Proof.* With the vectors  $I_1, I_2$  and  $I_4$ , we can proceed as in Proposition 2.6 and construct a basis  $1, I_1, \dots, I_7$  for  $\mathbf{O}$ , whose table of multiplication is also given in the proof of Proposition 2.6. We can write  $f_{I_1}(x + yI_1) = f(x + yI_1)$  as

$$f = f_0 + I_1f_1 + I_2f_2 + I_3f_3 + I_4f_4 + I_5f_5 + I_6f_6 + I_7f_7.$$

Since  $f$  is regular, we know that

$$\left(\frac{\partial}{\partial x} + I_1\frac{\partial}{\partial y}\right)f_{I_1}(x + yI_1) = 0,$$

i.e.,

$$\frac{\partial f_0}{\partial x} + I_1\frac{\partial f_1}{\partial x} + \dots + I_7\frac{\partial f_7}{\partial x} + I_1\left(\frac{\partial f_0}{\partial y} + I_1\frac{\partial f_1}{\partial y} + \dots + I_7\frac{\partial f_7}{\partial y}\right) = 0.$$

The expression above yields (taking advantage of the properties of the imaginary units) that the functions  $f_0 + I_1f_1 = F_1$ ,  $f_2 + I_1f_3 = F_2$ ,  $f_4 + I_1f_5 = G_1$  and  $f_6 + I_1f_7 = G_2$  all satisfy the standard Cauchy-Riemann system and therefore they are holomorphic. This concludes the proof.  $\square$

The following corollary will be useful in the sequel.

**Corollary 2.8.** *If  $f$  is a regular function on  $B = B(0, R)$ , then for every  $I_1 \in \mathbf{S}$ , we can find  $I_2$  and  $I_4$  in  $\mathbf{S}$ , such that if  $\mathcal{H}$  is the subspace of  $\mathbf{O}$  generated by  $(1, I_1, I_2, I_1I_2)$ , then there are two functions  $F : B \cap \mathcal{H} \rightarrow \mathcal{H}$  and  $G : B \cap \mathcal{H} \rightarrow \mathcal{H}I_4$ , regular on  $B \cap \mathcal{H}$  as in Definition 1.2 such that for any  $q \in B \cap \mathcal{H}$ , it is*

$$f(q) = F(q) + G(q).$$

*Proof.* One clearly sets  $F = F_1 + F_2I_2$  and  $G = (G_1 + G_2I_2)I_4$  on the complex plane  $L_{I_1}$ . The function  $F$  is holomorphic where defined and quaternionic valued; thus, by [12], it uniquely extends to a regular function on  $\mathcal{H} \cap B$ . The function  $G$ , on the other hand, is holomorphic with values in  $\mathcal{H}I_4$ . An immediate variation of the argument in [12] shows also that  $G$  uniquely extends to a regular function on  $\mathcal{H} \cap B$ . This concludes the proof.  $\square$

*Remark 2.9.* Note that, because of the lack of associativity in  $\mathbf{O}$ , the function  $G_1 + G_2I_2$  is not, by itself, regular.

*Remark 2.10.* This last result shows that one can represent regular functions on  $\mathbf{O}$  either as pairs of regular functions, or as four-tuples of holomorphic functions, consistently with decomposition (1).

Given that the functions  $F_i$  and  $G_i$  ( $i = 1, 2$ ) are holomorphic on the plane  $\mathbf{R} + \mathbf{R}I_1$ , and since the derivative of regular functions is defined in the sense of Gateaux, and only involves complex planes, the proof of the following proposition can be found in [12].

**Proposition 2.11.** *Let  $f : B \rightarrow \mathbf{O}$  be a regular function. Then it is  $C^\infty$  and, moreover, for any  $n \in \mathbf{N}$ , its derivative  $\partial^n f : B \rightarrow \mathbf{O}$  is regular and it is*

$$\partial^n f(x + yI_1) = \frac{\partial^n f}{\partial x^n}(x + yI_1).$$

It is now possible to deduce the following important result.

**Theorem 2.12.** *If  $f : B \rightarrow \mathbf{O}$  is regular, then it has a series expansion of the form*

$$f(w) = \sum_{n=0}^{\infty} w^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0).$$

*Proof.* Consider, in the complex plane  $L_{I_1}$ , the disc  $\Delta$  centered in the origin and with radius  $a > 0$ , where  $a < R$ . Then we can use

the representation from the splitting Lemma 2.7 to find an integral representation for  $f_{I_1}$  inside  $\Delta$ . Specifically, using the fact that  $F_1, F_2, G_1$  and  $G_2$  are holomorphic, we obtain that for any  $z$  in  $\Delta$  we have:

$$\begin{aligned}
 f_{I_1}(z) &= \frac{1}{2\pi I_1} \int_{\partial\Delta} \frac{F_1(\zeta)}{\zeta - z} d\zeta \\
 &\quad + \left( \frac{1}{2\pi I_1} \int_{\partial\Delta} \frac{F_2(\zeta)}{\zeta - z} d\zeta \right) I_2 \\
 &\quad + \left[ \frac{1}{2\pi I_1} \int_{\partial\Delta} \frac{G_1(\zeta)}{\zeta - z} d\zeta + \left( \frac{1}{2\pi I_1} \int_{\partial\Delta} \frac{G_2(\zeta)}{\zeta - z} d\zeta \right) I_2 \right] I_4.
 \end{aligned}$$

Each of these four integrals may now be transformed into a power series as in classical complex analysis, and the rest of the proof therefore follows as in [12].  $\square$

**3. Identity principle and Cauchy integral formulas.** We begin this section with the proof of the identity principle.

**Theorem 3.1.** *Let  $f : B \rightarrow \mathbf{O}$  be a regular function. Denote by  $Z_f = \{w \in B : f(w) = 0\}$  the zero-set of  $f$ . If there exists an  $I \in \mathbf{S}$  such that  $L_I \cap Z_f$  has an accumulation point, then  $f \equiv 0$  on  $B$ .*

*Proof.* By choosing the basis  $(1, I_1 = I, I_2, \dots, I_7)$ , Corollary 2.8 allows us to write on  $L_I \cap B$

$$f(x + yI) = F(x + yI) + G(x + yI).$$

Now, we can conclude the theorem by applying the identity principle for regular functions of a quaternionic variable.  $\square$

As a consequence, we obtain

**Corollary 3.2.** *Let  $f$  and  $g$  be regular functions on the ball  $B$ . If there exists an  $I \in \mathbf{S}$  such that  $f \equiv g$  on  $L_I \cap B$ , then  $f \equiv g$  everywhere on  $B$ .*

The next few results are proved with the techniques used in [12], taking into account Corollary 2.8.

**Proposition 3.3.** *If  $f : B \rightarrow \mathbf{O}$  is a regular function, and if  $I \in \mathbf{S}$ , then  $f_I : L_I \cap B \rightarrow \mathbf{O}$  has the mean value property.*

**Theorem 3.4.** *Let  $f : B \rightarrow \mathbf{O}$  be a regular function. If  $|f|$  has a relative maximum at a point  $a \in B$ , then  $f$  is constant on  $B$ .*

In order to state the analog, for regular functions, of the Cauchy representation formula we set, for  $w \in B$ ,

$$I_w = \begin{cases} \frac{\operatorname{Im}(w)}{|\operatorname{Im}(w)|} \in \mathbf{S} & \text{if } \operatorname{Im}(w) \neq 0 \\ \text{any element of } S & \text{otherwise.} \end{cases}$$

**Theorem 3.5.** *Let  $f : B \rightarrow \mathbf{O}$  be a regular function, and let  $w \in B$ . Then*

$$f(w) = \frac{1}{2\pi I_w} \int_{\partial \Delta_w(0,r)} \frac{d\zeta}{\zeta - w} f(\zeta)$$

where  $\zeta \in L_{I_w} \cap B$ , and where  $r > 0$  is such that

$$\overline{\Delta_w(0,r)} = \{x + yI_w : x^2 + y^2 \leq r^2\}$$

is contained in  $B$  and contains  $w$ .

*Proof.* The result follows from the splitting lemma, or its corollary.  $\square$

As a consequence we obtain:

**Theorem 3.6** (Cauchy estimates). *Let  $f : B(0, R) \rightarrow \mathbf{O}$  be a regular function, let  $r < R$ ,  $I \in \mathbf{S}$  and  $\partial \Delta_I(0, r) = \{(x + yI) : x^2 + y^2 = r^2\}$ . If  $M_I = \max\{|f(w)| : w \in \partial \Delta_I(0, r)\}$  and if  $M = \inf\{M_I : I \in \mathbf{S}\}$ , then*

$$\frac{1}{n!} \left| \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{M}{r^n}, \quad n \geq 0.$$

The power series representation, and the arguments from the theory of one complex variable as they appear in [12] allow us to state the following two results (analog to the Liouville and the Morera theorems).



**Theorem 3.7.** *Let  $f : \mathbf{O} \rightarrow \mathbf{O}$  be an entire regular map, i.e., a regular map defined and regular everywhere on  $\mathbf{O}$ . If  $f$  is bounded, i.e., there exists a positive number  $M$  such that  $|f(w)| \leq M$  on all of  $\mathbf{O}$ , then  $f$  is constant.*

**Theorem 3.8.** *Let  $f : B \rightarrow \mathbf{O}$  be a differentiable function. If, for every  $I \in \mathbf{S}$ , the differential form  $f(z) dz$ ,  $z = x + yI$ ,  $x, y \in \mathbf{R}$ , defined on  $L_I \cap B$  is closed, then the function  $f$  is regular.*

We conclude this section with a couple of results on the geometry of the open unit ball  $B = \{w \in \mathbf{O} : |w| < 1\}$  in  $\mathbf{O}$ . First we note that, with the modifications pointed out in [12], the Schwarz lemma holds due to the power series representation of regular functions.

**Theorem 3.9.** *Let  $f : B \rightarrow B$ ,  $f(w) = \sum_{n=1}^{+\infty} w^n a_n$ , be a regular function such that  $f(0) = 0$ . Then, for every  $w \in B$ ,*

$$|f(w)| \leq |w|$$

and

$$|\partial f(0)| \leq 1.$$

Moreover, equality holds in the formulas above, at a point  $w \neq 0$ , if and only if  $f(w) = wu$  for some  $u \in \mathbf{O}$ ,  $|u| = 1$ .

If we define  $\mathbf{O}^+ = \{w = x_0 + x_1 e_1 + \dots + x_7 e_7 \in \mathbf{O} : x_0 > 0\}$ , and if we set  $\psi(w) = (1 - w)^{-1}(1 + w)$ , we can prove

**Theorem 3.10.** *The octonionic right half space  $\mathbf{O}^+$  is diffeomorphic to the open unit ball  $B$  via the biregular transformation  $\psi$ .*

**4. Zeroes of power series.** In this last section, we study the zero sets of octonionic power series. We begin with a result whose proof follows the one given for Theorem 5.1 in [12].

**Theorem 4.1.** *Let  $\sum_{n=0}^{+\infty} w^n a_n$  be a given octonionic power series with radius of convergence  $R$ . Suppose that there exist  $x_0, y_0 \in \mathbf{R}$  and*

$I, J \in \mathbf{S}$  with  $I \neq J$  such that

$$(2) \quad \sum_{n=0}^{+\infty} (x_0 + y_0 I)^n a_n = 0$$

and

$$(3) \quad \sum_{n=0}^{+\infty} (x_0 + y_0 J)^n a_n = 0.$$

Then for all  $L \in \mathbf{S}$  we have

$$\sum_{n=0}^{+\infty} (x_0 + y_0 L)^n a_n = 0.$$

The next few results describe the nature of the zero sets of octonionic power series. For the sake of clarity, we will say that an octonion  $w_0 = x_0 + y_0 I$  is a *spherical zero* for a regular function  $f$  if every point of the six-sphere  $x_0 + y_0 \mathbf{S}$  is a zero for  $f$ .

**Proposition 4.2.** *If  $f$  has a series representation  $f(w) = \sum w^n a_n$  with real coefficients  $a_n$ , then every real zero  $x_0$  is isolated, and if  $x_0 + y_0 I$  is a nonreal zero (i.e.,  $y_0 \neq 0$ ) then it is a spherical zero. In particular, if  $f \not\equiv 0$ , the zero set of  $f$  consists of isolated zeroes (lying on  $\mathbf{R}$ ) or isolated six-spheres.*

**Proposition 4.3.** *Let  $f$  be a regular function on a ball  $B$  centered in the origin, and suppose that there exists an imaginary unit  $I$  in  $\mathbf{S}$  such that  $f(L_I) \subset L_I$ . If there exists an imaginary unit  $J$  in  $\mathbf{S}$  such that  $J \notin L_I$  and  $f(x_0 + y_0 J) = 0$ , then  $f(x_0 + y_0 L) = 0$  for all  $L \in \mathbf{S}$ . In particular, if  $f \not\equiv 0$ , the zero set of  $f$  consists of isolated zeroes (lying on  $B \cap L_I$ ) or isolated six-spheres in  $B$ .*

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DIPARTIMENTO DI MATEMATICA “U. DINI,” UNIVERSITA’ DI FIRENZE, VIALE MORGAGNI, 67 A FIRENZE, ITALY

**Email address:** gentili@math.unifi.it

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CHAPMAN UNIVERSITY, ORANGE, CA 92866

**Email address:** struppa@chapman.edu