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A tractable analysis of contagious equilibria†

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Abstract

This paper studies contagious equilibrium in infinitely repeated matching games. The innovation is to identify a key statistic of contagious punishment that, used together with a recursive formulation, generates tractable closed-form expressions for continuation payoffs, off equilibrium. This allows a transparent characterization of the dynamic incentives created by contagious punishment schemes.
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JEL codes: C6, C7

1 Introduction

The studies in [4, 6] have extended the analysis of cooperation in infinitely repeated games from economies with stable partnerships to random matching economies, where relational contracting is unavailable. The central result is that full cooperation can be achieved even if players cannot exploit reciprocity

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mechanisms because agents are anonymous, and can neither communicate with nor can observe others' past behaviors. In these matching economies, cooperation relies on adopting a common strategy (=a social norm) that includes the threat of unforgiving punishment. According to this norm, a player always cooperates unless someone defects, in which case the player switches to punishing by defecting forever.

Studying equilibrium in these economies is analytically cumbersome because, once punishment starts, it spreads at random, through cooperator-defector encounters. This contagious punishment process complicates the characterization of continuation payoffs, which holds the key to establishing whether dynamic incentives exist for players to follow the social norm.

This study contributes to the literature on cooperation and contagious punishments by showing how to attain a tractable closed-form expression of continuation payoffs, off equilibrium. This is done by, first, identifying and characterizing a key statistic of contagious punishment processes, which we call the *contact rate*. This is the rate at which a defector expects to meet cooperators in the continuation game. We then use such a statistic to derive through a recursive formulation tractable closed-form expressions for continuation payoffs off equilibrium, which are simply convex combinations of static payoffs; the convexification factor depends on the number of defectors present in the economy, the discount factor, and the breadth of monitoring.

To see the difference with previous work, note that [4] bases the existence proof on a pointwise analysis of continuation payoffs, i.e., for a specific realization of a matching trajectory. Instead, we follow the approach in [6], which is matrix-theoretic; we augment it by adopting a recursive formulation that allows us to obtain tractable closed-form expressions for continuation payoffs, away from the equilibrium path of play. This has the virtue of making the analysis of contagious equilibrium transparent. In particular, we generalize the expressions for continuation payoffs for all possible beliefs about the number of defectors, whereas the literature typically considers only the case of two defectors. In this manner we can characterize exact bounds on the two parameters that are key to ensuring that cooperation is self-enforcing: the discount factor and the cost sustained to slow down the contagious spread of defections. This is theoretically meaningful it helps us to better understand how changes in the game's parameters affect the incentives to follow contagious punishments and it is also empirically meaningful it helps us to construct laboratory economies based on repeated, random matching games (e.g., see [1, 2]).

We proceeds as follows. Section 2 presents the basic model. Section 3 identifies some basic properties of the typical contagious punishment process, which are then used in Section 4, to recursively derive payoffs as a convex combination of static payoffs. Section 5 shows how this machinery can be used to characterize bounds on parameters that support cooperative equilibrium in repeated matching games with private monitoring. Section 6 extends the analysis to games with (imperfect) public monitoring and public randomization devices. Section 7 concludes.

2 The model

Consider an economy in which anonymous agents are randomly and bilaterally matched in each period to play a stage game. There are $N = 2n \geq 4$ infinitely-lived agents, who have linear preferences and discount the future with common discount factor $\beta \in (0, 1)$. Equivalently, let the economy be of *indefinite* duration where β is the time-invariant probability that, after each period the economy continues for one additional period, and otherwise the economy ends.

In each period $t = 0, 1, \ldots$, an exogenous matching process partitions the population into n pairs. Pairings are random, equally likely, independent over time, and last only one period. Let $o_i(t) \neq i$ be agent i's opponent in period t, where o_i is an involution. Agents cannot observe the identities of others and cannot recognize individuals if they meet them again (= anonymity).

In every period t each agent in $\{i, o_i(t)\}$, for $i = 1, \ldots, N$, faces an identical two-player stage game that consists of simultaneously and independently selecting one action from the set $\{C, D\}$. The possible stage-game payoffs to agent i are $\pi_{DD}, \pi_{CD}, \pi_{DC}$, and π_{CC}, where the first subscript refers to i's action in the four possible outcomes (D, D), (C, D), (D, C), and (C, C). We assume that (C, C) is the socially efficient outcome and that $\pi_{DC} > \pi_{CC}$ and $\pi_{DD} > \pi_{CD}$, i.e., the game is a social dilemma where there are incentives to behave opportunistically. Each agent i observes the actions (but not the identities) of a set of agents denoted $O_i(t, a)$, which includes agent i, i's opponent $o_i(t)$, and $a = 0, \ldots, N-2$ other randomly selected agents. The case $a = 0$ corresponds to private monitoring, which is when $O_i(t, 0) = O_i(t) = \{i, o_i(t)\}$. At the other extreme, $a = N-2$, we have public monitoring. In-between cases capture situations that we dub, with a small abuse in language, "imperfect" public monitoring in which, for instance, players see the actions of those who are spatially close to them but not of everyone in the economy.[1]

Suppose every agent $i = 1, \ldots, N$ adopts the following trigger strategy (e.g., see [4]):

Definition 1. *On $t = 0$, agent i is in state $s = C$ and selects action C. On all $t > 0$, agent i is either in state $s = C$ or $s = D$, and selects action s.*

- *If agent i is in state C in period t, then i switches state on $t+1$ only if*

[1] For example, the matching process randomly partitions the population into pairwise disjoint groups of size $a+2$ in each period. In each group agents play in pairs but observe all the actions taken in their group.

some agent in $O_i(t,a)$ selected D. Otherwise, i remains in state C.

- *State D is absorbing.*

In what follows, we focus on the case when everyone in the population follows the strategy in Definition 1, i.e., we consider a social norm as in [4, 6]. This norm has two components: a rule of desirable behavior (always choose C) and a rule of punishment (always choose D) selected only if a departure from desirable behavior is observed. For this reason, we will call an agent who is in state C a "cooperator," and a "defector" otherwise.

The central feature of grim play is that *any* defection starts an irreversible contagious punishment process that eventually leads to an environment in which everyone is a defector. Depending on the parameter a, punishment may spread in the economy either by means of direct contact with a defector or indirectly, by observing a defection outside of the agent's match. Such unforgiving decentralized punishment scheme forms the basis of cooperation because it removes the incentive to behave opportunistically when agents are sufficiently patient. In the next section we discuss the properties of decentralized punishment, and in the section that follows we show that such properties hold the key to a tractable formulation of out-of-equilibrium payoffs.

3 Properties of decentralized punishment

Consider the start of a generic period. Suppose the population is partitioned into $k = 1, \ldots, N$ defectors and $N - k$ cooperators. Let

$$\sigma_k := \tfrac{N-k}{N-1} \quad \text{with} \quad \sigma_k \in \sigma \quad = \quad (\sigma_1, \sigma_2, \ldots, \sigma_{N-1}, 0)^{\mathsf{T}}$$

define the probability that on this date defector i meets a cooperator.[2] It should be clear that $0 = \sigma_N < \sigma_{k'} < \sigma_k < \sigma_1 = 1$ for $2 \leq k < k' \leq N - 1$. Let

$$k \in \kappa := (1, \ldots, N)^{\mathsf{T}}$$

denote the *state* of the economy on a generic date and define the $N-$dimensional column vector e_k with 1 in the k^{th} position and 0 everywhere else.

Theorem 1. *Suppose there are $k = 1, \ldots, N$ defectors and that each agent i observes the actions of agents in $O_i(t, a)$ for $a = 0, \ldots, N-2$. The probability distribution of defectors evolves over the span of $t \geq 1$ periods according to $e_k^{\mathsf{T}} Q^t$ where Q is an $N \times N$ transition matrix with elements $Q_{kk'}$ and mean $\mu_k(t) := e_k^{\mathsf{T}} Q^t \kappa$ satisfying:*

1. *$Q_{kk'} = 0$ for $k' < k$;*
2. *$Q_{kk} < 1 = Q_{NN}$ for all $k = 1, \ldots, N-1$;*
3. *$\mu_{k+1}(t) \geq \mu_k(t)$, for all $k = 1, \ldots, N-1$ and all $t \geq 1$;*
4. *$\mu_k(t+1) \geq \mu_k(t) \geq k$, for all $k = 1, \ldots, N-1$ and all $t \geq 1$;*
5. *$\mu_k(t)$ is non-decreasing in a.*

Proof. In Appendix □

[2] T = transpose. With a slight abuse in notation, we use $y_j \in y := (y_1, \ldots, y_N)$ to denote a generic element of vector y.

When everyone follows the strategy in Definition 1, the upper-triangular matrix

$$Q := \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \dots & Q_{1,N-1} & Q_{1N} \\ 0 & Q_{22} & Q_{23} & \dots & Q_{2,N-1} & Q_{2N} \\ 0 & 0 & Q_{33} & \dots & Q_{3,N-1} & Q_{3N} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q_{N-1,N-1} & Q_{N-1,N} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

describes how contagious punishment spreads from period to period, i.e., how the economy transitions from a state with k to k' defectors.

Decentralized punishment has five main properties. Punishment is *irreversible* (Property 1) and *contagious* (Property 2): If someone defects today, then next period there can only be more defectors than today, and can never be less. Clearly, the number of additional defectors depends on the monitoring process.[3] Punishment is *decentralized* (Property 3) and *unforgiving* (Property 4): Because defection is an absorbing state, the number of defectors expected on any date is greater if we start with more defectors, and can only increase over time. Finally, the number of defectors expected on any date is non-

[3]It follows that $Q_{kk'}$ may or may not depend on the outcome of the random matching process. For instance, consider private monitoring, $O_i = \{i, o_i\}$ for all $i = 1, \dots, N$. Here the number of additional defectors depends only on the random matching process because a cooperator j may switch to being a defector during t only by "direct contagion", i.e., if $o_j(t)$ is a defector; consequently, defectors can at most double from period to period, i.e., we have $Q_{kk'} = 0$ for $k' > \min(2k, N)$. Instead, the matching realization does not matter under public monitoring, i.e., when $O_i = \{1, \dots, N\}$ for all $i = 1, \dots, N$. The reader interested in closed-form transition matrices for $a = 0$ can consult [5] or the Supporting Materials of this paper, where we also study the case of $a > 0$ and of noise in transitions.

decreasing in the number of actions that agents can observe in the economy (Property 5).

We also define the probability $\eta_{kk'}$ that a defector, say, agent i, meets a cooperator *conditional* on an outcome being realized that raises the number of defectors from k to k'. Clearly, $\eta_{kk'} \leq \eta_{k-1,k'}$ for all $k = 2, \ldots, k'$ with $\eta_{kk} = 0$; this is due to random matching, which implies that meeting a cooperator today is less likely when there are more defectors.

The *unconditional* probability that agent i meets a cooperator when there are k defectors is:

$$\sigma_k = \sum_{k'=k}^{N} Q_{kk'} \eta_{kk'}. \tag{1}$$

A single defection eventually leads to 100% defections in the economy, an absorbing state that is reached in finite time. To compute the expected number of periods to full defection which is helpful in experimental contexts (e.g., see [1]) define τ_k as the average number of periods required to have N defectors in the economy when we start with $k = 1, \ldots, N$ defectors. We have

$$\begin{aligned} \tau_k &= 1 + \sum_{k'=k}^{N} Q_{kk'} \tau_{k'} \qquad \text{for} \quad k = 1, \ldots, N-1 \\ \tau_N &= 0. \end{aligned}$$

With probability Q_{kk} the number of defectors does not further increase over the course of one period; hence, the following period we expect once again τ_k

periods before we have N defectors in the economy. With probability $Q_{kk'}$ there are $k' - k$ new defectors by the end of the period, hence, tomorrow we expect $\tau_{k'}$ periods before we have N defectors in the economy. Clearly, $\tau_N = 0$.

Let Q_0 denote the matrix obtained when the last row of Q is a vector of zeros. The elements of vector $\tau := (\tau_1, \ldots, \tau_N)^\intercal$ are solutions to the system of equations

$$\tau = \mathbf{1}_0 + Q_0 \cdot \tau \qquad \Rightarrow \qquad (I - Q_0) \cdot \tau = \mathbf{1}_0,$$

where I is the identity matrix and $\mathbf{1}_0$ is the N-dimensional unit vector whose N^{th} component is zero. Since $I - Q_0$ is upper-triangular with non-zero diagonal elements, then $I - Q_0$ is invertible, and $\tau = (I - Q_0)^{-1} \cdot \mathbf{1}_0$ is the unique solution.

4 Continuation payoffs: a recursive approach

Start by recognizing that the equilibrium payoff to any agent is

$$v_0 = \frac{\pi_{CC}}{1 - \beta}.$$

Now consider out-of-equilibrium situations in which there are $k \geq 1$ defectors at the start of a period and fix one defector, say, agent i.[4] If decentralized punishment is characterized by matrix Q as in Theorem 1, then using standard

[4]If $k = 1$ this corresponds to a situation in which one agent moves out of equilibrium.

recursive methods the payoff to defector i is

$$v_k = \sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}\pi_{DC} + (1-\eta_{kk'})\pi_{DD} + \beta v_{k'}]. \tag{2}$$

Using (1), letting $v := (v_1, \ldots, v_N)^\mathsf{T}$ and using $e_k^\mathsf{T} Q v \equiv \sum_{k'=k}^{N} Q_{kk'} v_{k'}$ we have

$$v_k = \sigma_k \pi_{DC} + (1-\sigma_k)\pi_{DD} + \beta e_k^\mathsf{T} Q v.$$

The remainder of this section is devoted to proving the following result:

Theorem 2. *Let there be $k \geq 1$ defectors. The payoff to any defector satisfies*

$$v_k = \frac{1}{1-\beta}[\phi_k \pi_{DC} + (1-\phi_k)\pi_{DD}] \tag{3}$$

where

$$\phi_k := (1-\beta)e_k^\mathsf{T}(I-\beta Q)^{-1}\sigma,$$

with $0 = \phi_N < \phi_{k+1} < \phi_k < \sigma_k$, $\lim_{\beta \to 1^-} \phi_k = 0$ and $\lim_{\beta \to 1^-} \frac{\phi_k}{1-\beta} < \infty$. Moreover, ϕ_k is non-increasing in a, $\phi_k - \phi_{k+1}$ is non-increasing in k and in a, and $\frac{\phi_k - \phi_{k+1}}{1-\beta}$ is non-decreasing in β.

The message is that the continuation payoff to a defector is a convex combination of the static payoff from meeting a cooperator and another defector. The function ϕ_k is the rate at which a defector expects to meet cooperators in the continuation game, as we will show in the remainder of this section.

To prove this result, let there be $k \geq 1$ defectors at the start of a period; fix one, say, agent i. Here σ_k is the probability that on this date defector i meets a cooperator. Now consider *future* possible encounters between defector i and cooperators. The probability that defector i meets a cooperator $t \geq 1$

periods from now depends on how many defectors there will be on that date. To calculate this number, without loss in generality let $t = 0$ be the present date (when we have k defectors). The probability to have k' defectors on date $t = 1$ is $Q_{kk'}$, on date $t = 2$ is $Q^2_{kk'} \equiv \sum_{h=1}^{N} Q_{kh}Q_{hk'}$, and it is $Q^t_{kk'}$ on date $t \geq 1$, i.e., it is cell (k, k') of matrix Q^t because Q is a transition matrix.

Consider period $t = 0$; $\sum_{k'=1}^{N} Q_{kk'}\sigma_{k'}$ is the probability that on $t = 1$ defector i will meet a cooperator. The probability that initial defector i meets a cooperator $t \geq 0$ periods from now is

$$\sum_{k'=1}^{N} Q^t_{kk'}\sigma_{k'} = e_k^{\intercal} Q^t \sigma < 1.$$

To verify that $e_k^{\intercal} Q^t \sigma < 1$ note that $\sum_{k'=1}^{N} Q^t_{kk'} = 1$ and $\sigma_{k'} < 1$ for all $k' > 1$.

Given that there are $k \geq 1$ defectors at the start of a period, we now wish to calculate the expected number of cooperators that any of these defectors will meet in the continuation game (their future lifetime). To do so, suppose for a moment that the economy is infinitely-lived, i.e., $\beta = 1$. Fix defector i among the k current defectors. The expected number of cooperators that i will meet in the continuation game is

$$e_k^{\intercal}\sigma + \sum_{t=1}^{\infty} e_k^{\intercal} Q^t \sigma.$$

When $\beta \in (0, 1)$, this number is calculated by adding the continuation probability β, i.e., we have

$$\sigma_k + \beta \sum_{k'=1}^{N} Q_{kk'}\sigma_{k'} + \beta^2 \sum_{k'=1}^{N} Q^2_{kk'}\sigma_{k'} + \dots \; = \; e_k^\intercal \sigma + \sum_{t=1}^{\infty} \beta^t e_k^\intercal Q^t \sigma = e_k^\intercal (I - \beta Q)^{-1} \sigma.$$

Lemma 1. *The sum* $e_k^\intercal \sigma + \sum_{t=1}^{\infty} \beta^t e_k^\intercal Q^t \sigma$ *converges for all* $\beta \in [0, 1]$ *and is bounded above by* $(1-\beta)^{-1}$*.*

Proof. In Appendix □

The expected number of cooperators that a defector meets in the continuation game is finite (even if the horizon is infinite), because each current cooperator can be met only once. This is due to the contagious punishment process: when a defector meets a cooperator, the cooperator switches to defection.[5] Because cooperators may meet defectors in every period, the number of cooperators is likely to fall over time, hence eventually no-one cooperates.

Now notice that if $\beta < 1$ and a defector meets one cooperator in each period, then the number of cooperators met in the continuation game corresponds exactly to the expected duration of the economy, which is $(1-\beta)^{-1}$.[6]

We therefore normalize the expected number of cooperators that a defector meets in the continuation game by the expected duration of the economy.

[5] Clearly, this expected number is bounded above by $N - k$ because a defector cannot meet more than $N - k$ cooperators since each cooperator encountered becomes a defector. This would still be true if defector-cooperator matches generated one new defector with a probability less than one, i.e., if we added some noise to the decentralized punishment process.

[6] Here $(1-\beta)\beta^t$ is the probability that the economy lasts exactly $t + 1$ periods, so the expected duration of the economy is $(1-\beta)\sum_{t=1}^{\infty} \beta^{t-1} t = (1-\beta)^{-1}$.

Given Q, σ, and β, this gives us a ratio that we call the *contact rate*:

$$\phi_k = (1-\beta)e_k^\intercal (I-\beta Q)^{-1}\sigma \quad \text{for} \quad \beta \in (0,1).$$

One can interpret this as the rate at which defector i expects to meet co-operators in a period, in the continuation game, when there are currently $k \geq 1$ defectors. If there are few defectors, then agent i meets cooperators at a high rate, and ϕ_k is close to one; otherwise, it is close to zero. Because $e_k^\intercal \sigma + \sum_{t=1}^{\infty} \beta^t e_k^\intercal Q^t \sigma$ is finite for all β, it is immediate that

$$\lim_{\beta \to 1^-} \phi_k = 0 \quad \text{and} \quad \lim_{\beta \to 1^-} (1-\beta)^{-1}\phi_k < \infty. \tag{4}$$

In short, the contact rate reaches zero as the expected duration of the continuation game approaches infinity because, in the long run, the economy will be entirely populated by defectors.

Clearly, $\phi_N = 0$ because matrix Q is upper-triangular and $\sigma_N = 0$; intuitively, defection is an absorbing state, so once we have N defectors it is impossible to meet cooperators in the continuation game. We have:

Lemma 2. *For all $k = 1, \ldots, N-1$ we have (i) $0 = \phi_N < \phi_{k+1} < \phi_k < \sigma_k$, (ii) ϕ_k is non-increasing in a, (iii) $\phi_k - \phi_{k+1}$ is non-increasing in k and in a (iv) $\dfrac{\phi_k - \phi_{k+1}}{1-\beta}$ is non-decreasing in β.*

Proof. In Appendix □

To complete the proof of Theorem 2 note that since Q is upper triangular, matrix $(I - \beta Q)$ is invertible, hence we use the definition of v_k to obtain

$$v = \sigma\pi_{DC} + (\mathbf{1} - \sigma)\pi_{DD} + \beta Q v \qquad \Rightarrow \qquad v = (I - \beta Q)^{-1}[\sigma\pi_{DC} + (\mathbf{1} - \sigma)\pi_{DD}]$$

where $v_k \in v$ satisfies

$$\begin{aligned} v_k &= e_k^{\mathsf{T}}(I - \beta Q)^{-1}[\sigma\pi_{DC} + (\mathbf{1} - \sigma)\pi_{DD}] \\ &= \frac{1}{1-\beta}[\phi_k\pi_{DC} + (1 - \phi_k)\pi_{DD}]. \end{aligned}$$

The defector's payoff is a convex combination of two possible static payoffs: the one from meeting a cooperator and the one from meeting a defector. The contact rate ϕ_k serves as the convexification parameter. The virtue of this approach is that it makes analysis of equilibrium quite tractable. We provide an example of what we mean, next.

5 Cooperation with private monitoring

Here we demonstrate the tractability of our approach by revisiting the proof of a well-known existence result for the repeated prisoners' dilemma game with private monitoring studied in [4, 6]. Let the static payoffs to an agent i be: $\pi_{DD} = d, \pi_{CD} = d - l, \pi_{DC} = d + g$, and $\pi_{CC} = c$, where $g > 0$, $c > d$, $l \geq 0$.[7] There is private monitoring, so $O_i(t) = \{i, o_i(t)\}$ for all agents i in each period t. That is, contagious punishment only spreads through direct

[7]The payoff formulation is a bit more general than in [4, 6], where the normalization $d = 0$ is assumed. Note also that neither [6] nor [4] assume $2c > c + g + d - l$, which—as noted in [7]—is sometimes part of the definition of Prisoners' Dilemma

contagion. Conjecture that everyone follows the strategy in Definition 1.

The study in [6] proves that if l is sufficiently large, then there exists a $\beta^* \in (0,1)$ such that for all $\beta \in [\beta^*, 1)$ the outcome where $k = 0$ in each period can be supported as a sequential equilibrium.

5.1 Existence of cooperative equilibrium

Start by recognizing that, from Theorem 2, the payoff to a defector when there are $k \geq 1$ defectors at the start of a generic period is

$$v_k = \frac{1}{1-\beta}[\phi_k(c+g) + (1-\phi_k)d].$$

Consider one-shot deviations from the strategy in Definition 1. In equilibrium an agent must choose C and not D. This holds if $v_0 - v_1 \geq 0$, because v_1 defines the payoff to someone who defects in equilibrium. From (3) we have

$$v_0 - v_1 = \frac{1}{1-\beta}[c - d - \phi_1(c+g-d)]. \tag{5}$$

From Theorem 2 we have $\lim_{\beta \to 1^-} \frac{\phi_1}{1-\beta} < \infty$. By continuity there exists a value $\beta^* \in (0,1)$ such that $v_0 - v_1 \geq 0$ for all $\beta \in [\beta^*, 1)$.

Out of equilibrium, let agent i be one of $k \geq 2$ defectors. Agent i chooses

D (as specified in Definition 1), whenever

$$\begin{aligned}&\sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}(c+\beta v_{k'-1})+(1-\eta_{kk'})(d-l+\beta v_{k'})]\\&\quad\leq \sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}(c+g)+(1-\eta_{kk'})d+\beta v_{k'}]\end{aligned} \tag{6}$$

The right hand side is simply the payoff v_k (recall that $\sum_{k'=k}^{N} Q_{kk'}\eta_{kk'} = \sigma_k$). The left-hand side reports the payoff to defector i when he does not punish: he deviates by choosing C today, reverting to playing D forever, tomorrow. It is derived using the recursive formulation of continuation payoffs in (2). With probability $Q_{kk'}$ the economy should transition to a state with k' defectors, i.e., there are $k'-k$ mixed matches. The (conditional) probability that defector i is in one of such mixed-matches is $\eta_{kk'}$. Because defector i deviates only today by choosing C instead of D, his current opponent does not become a defector due to private monitoring the opponent only observes i's action. Hence, with probability $Q_{kk'}\eta_{kk'}$ agent i's continuation payoff is $v_{k'-1}$ and not $v_{k'}$. This adjustment does not apply if agent i is not in a mixed match (with probability $1-\eta_{kk'}$), as in this case his deviation C cannot reduce the number of future defectors from k' to $k'-1$.

Rewrite the above inequality as

$$\beta \sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}(v_{k'-1}-v_{k'}) \leq \sigma_k g + (1-\sigma_k)l. \tag{7}$$

From equation (3) in Theorem 2, for all $k \geq 2$ we have

$$v_{k-1} - v_k = \frac{1}{1-\beta}(\phi_{k-1} - \phi_k)(c + g - d) < \infty \quad \text{for all} \quad \beta \in (0,1) \tag{8}$$

because $\lim_{\beta \to 1^-} (1-\beta)^{-1}\phi_k < \infty$. Given any $\beta \in (0,1)$, deviating off equilibrium is suboptimal if l is sufficiently large, and the existence proof is completed.

5.2 Characterization of bounds on parameters

The procedure developed in the previous sections has useful applications. It allows us to find exact bounds for the parameters that sustain the contagious equilibrium and to characterize them as functions of the model's parameters. Here, we characterize the lower bounds on β and l.

The discount factor

Here we derive the exact lower bound for β, such that deviating in equilibrium is never optimal, and characterize it in terms of the cost of cooperation in equilibrium. To do so, it is convenient to normalize the payoff parameter g by $c - d$, which can be interpreted as the surplus from cooperation relative to defection. The variable $\gamma := \dfrac{g}{c-d}$ roughly speaking captures the (opportunity) cost of cooperation in equilibrium, as it measures the gain from defecting relative to the surplus from cooperating.

Proposition 1. *In the repeated Prisoners' Dilemma with private monitoring,* *the lower bound for β, such that deviating in equilibrium is never optimal is*

$$\beta^* := \phi_1^{-1}\Big(\frac{1}{1+\gamma}\Big),$$

a strictly increasing and concave function of γ.

Proof. In Appendix □

FIG. 1 APPROXIMATELY HERE

Figures 1 illustrates the mapping between the bound β^*, the size of the economy N, and the cost of cooperation in equilibrium γ. The minimal discount factor that is necessary to support cooperation in equilibrium grows as the opportunistic incentives increase, and as the economy grows larger.

The cost of slowing down contagion

Now we determine an exact lower bound for l, such that deviating out of equilibrium is never optimal for a defector, for any belief on the number of defectors k. We can think of l as the cost of cooperation off equilibrium.

Proposition 2. *In the repeated Prisoners' Dilemma with private monitoring, if*

$$l \geq l(\beta, k) := \frac{1}{1-\sigma_k}\left\{(c+g-d)\sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}\frac{\beta(\phi_{k'-1}-\phi_{k'})}{1-\beta} - \sigma_k g\right\},$$

then off equilibrium punishment is optimal under the belief that there are $k = 2, \ldots, N$ defectors. In particular, $l(\beta, k)$ is non-decreasing in β, it is decreasing in k and, if

$$l \geq l^* := (N-2)(c-d),$$

then off-equilibrium punishment is optimal for any $\beta \in (0, 1)$ *and* k.

Proof. In Appendix □

Figures 2 helps us to understand how the cost of slowing down contagion and the discount factor must co-vary to maintain the incentive to punish.

FIG. 2 APPROXIMATELY HERE

It plots the mapping between the minimum value of l, the beliefs about the number of defectors off the equilibrium path of play, and the discount factor β. The minimum value l needed to support punishment off equilibrium is non-decreasing in β and non-increasing in k.

6 Extensions

This section explores two issues that emerge from the previous discussion. The cost of cooperating off-equilibrium, i.e., the lower bound for the parameter l, falls as k increases (Figure 2). This suggests that the possibility to see actions outside of a match would strengthen the incentives to cooperate and to punish because contagious punishment would spread more rapidly (Theorem 1, result 5). Proposition 2 has also made explicit the finding in [6] that as economies get progressively large the incentives to carry out punishments can be maintained only if the cost from cooperating off equilibrium grows large.

Adding public randomization devices, as in [4], may offer a way to resolve such a shortcoming.[8]

6.1 Imperfect public monitoring

Here we provide counterparts for Propositions 1-2 under (imperfect) public monitoring, which in this paper has been defined as a situation in which agents observe the actions of $a = 1, \ldots, N-2$ anonymous agents outside of their match, as discussed in Section 2.

The functional forms for off-equilibrium payoffs do not vary; the central difference is that transition matrix and contact rate now depend on a. Therefore, we write $Q(a)$ and $\phi_k(a)$ for $a > 0$ and $\phi_k(0) \equiv \phi_k$ and $Q(0) \equiv Q$.[9] Now each player expects that a individuals   in addition to his opponent   observe his action in a period (see the proof of Theorem 1). Consequently, greater a supports faster contagion, off-equilibrium, and a lower contact rate with cooperators. Naturally, this strengthens the incentive to cooperate, in equilibrium, so the lower bound on the discount factor falls when actions can be observed outside a match. Formally, we have a version of Proposition 1

Corollary 1 (Imperfect Public Monitoring and β)**.** *Consider the repeated Prisoners' Dilemma with $a = 1, \ldots, N-2$. The lower bound for β, such that deviating in equilibrium is suboptimal is*

$$\beta(a) := \phi_1^{-1}(a)\left(\frac{1}{1+\gamma}\right) \leq \beta^*.$$

[8] We thank an anonymous referee for suggesting these extensions.

[9] An explicit expression for $Q(a)$ is derived in the Supplementary Materials.

Proof of Corollary 1. See Appendix $\square$

To study deviations off-equilibrium we must generalize expression (6) for $a > 0$. Suppose agent i is one of k defectors, and deviates choosing C instead of D. Such deviation is suboptimal if

$$\sum_{k'=k}^{N} Q_{kk'}(a)\{\eta_{kk'}(a)c + (1 - \eta_{kk'}(a))(d - l) + \beta \sum_{j=0}^{k'-k} \alpha_{kk'}(j; a)v_{k'-j}\}$$
$$\leq \sum_{k'=k}^{N} Q_{kk'}(a)[\eta_{kk'}(a)(c + g) + (1 - \eta_{kk'}(a))d + \beta v_{k'}].$$

Given a and conditional on $k' - k$ new defectors, $\eta_{kk'}(a)$ is the probability that defector i meets a cooperator, while $\alpha_{kk'}(j; a)$ is the probability that $j = 0, \ldots, k' - k$ cooperators see the action of i and of no other defector.[10] Hence, if defector i plays C, then j cooperators do not switch to punishing.

The right-hand side is the payoff v_k, as before. The left-hand side reports the payoff to defector i when he cooperates, instead of punishing as he should. Agent $o_i(t)$ may be a cooperator or not, which impacts i's the period payoff (either c or $d-l$). The continuation payoff depends on how many cooperators see the action of no other defector but defector i. Since $\sigma_k = \sum_{k'=k}^{N} Q_{kk'}(a)\eta_{kk'}(a)$, for $k \leq k'$ the inequality above yields

$$\beta \sum_{k'=k}^{N} Q_{kk'}(a)(\hat{v}_{k'} - v_{k'}) \leq \sigma_k g + (1 - \sigma_k)l,$$

[10]For $a = 0$ we have $\alpha_{kk'}(1; 0) = \eta_{kk'}(0) \equiv \eta_{kk'}$, $\alpha_{kk'}(0; 0) = 1 - \eta_{kk'}$, and $\alpha_{kk'}(j; 0) = 0$ for all $j = 2, \ldots, k' - k$, and we get back (6).

where the expectation $\hat{v}_{k'} := \sum_{j=0}^{k'-k} \alpha_{kk'}(j;a)v_{k'-j}$ is taken over the possible number of cooperators j who see the action of i (directly or indirectly) and of *no other* defector. If i cooperates, these j agents keep cooperating, so we have $v_{k'-j}$. We obtain a version of Proposition 2.

Corollary 2 (Imperfect Public Monitoring and l)**.** *Consider the repeated Prisoners' Dilemma with $a = 1, \ldots, N-2$. Proposition 2 holds by replacing $l(\beta, k)$ with*

$$l(\beta, k; a) := \frac{1}{1-\sigma_k}\left\{(c+g-d)\sum_{k'=k}^{N} Q_{kk'}(a)\frac{\beta(\hat{\phi}_{k'}(a) - \phi_{k'}(a))}{1-\beta} - \sigma_k g\right\},$$

a non-increasing function of a, with $\hat{\phi}_{k'}(a) := \sum_{j=0}^{k'-k} \alpha_{kk'}(j;a)\phi_{k'-j}(a)$.

Proof of Corollary 2. In Appendix. □

A defector will not deviate (by cooperating) if $l \geq l(\beta, k; a)$, and since $l(\beta, k; a)$ is non-increasing in a, the cost of cooperation can be smaller when players observe more actions in the economy. Intuitively, off-equilibrium a defector is less capable to slow down contagion because cooperators are more likely to observe defections somewhere else in the economy.

To sum up, if players can observe actions outside their match, then the incentive to move off-equilibrium decreases since a deviation quickly generates larger numbers of defectors compared to private monitoring. The incentive to not punish also decreases because contagion spreads through indirect observation of defections. The message is that cooperative equilibrium is easier to sustain when players can observe some actions outside of their match.

6.2 Reverting to cooperation

Consider the case in which defection is not an absorbing state, following [4]. To simplify the discussion, assume private monitoring; the results go through when this is not so. Suppose a public randomization device is available. At the start of each date, the device randomly selects and makes public a number $\tilde{q}_t \in [0, 1]$ with uniform probability. Defectors switch state if $\tilde{q}_t$ is sufficiently high, say, higher than $q \in (0, 1)$; everyone else remains in their state. Consequently, the strategy in Definition 1 is modified as follows:[11] at the start of a period t, a cooperator who observes a defection starts punishing in $t+1$ only if $\tilde{q}_{t+1} < q$; a defector reverts back to cooperation in $t+1$ only if $\tilde{q}_{t+1} \geq q$. In sum, out of equilibrium, the economy can revert back to full cooperation, with probability $1-q$. We show that in this case the incentives to cooperate (in equilibrium) decrease, while the incentives to punish (off-equilibrium) increase.

The continuation payoff in equilibrium is still v_0. Suppose that off-equilibrium there are $k \geq 1$ defectors at the start of some period, and fix one, say, agent i. Since decentralized punishment is still characterized by matrix Q, the off-

[11] See Supplementary Materials for formal definitions and details of this section's analysis.

equilibrium payoff to agent i for $k = 1, \ldots, N$ is

$$w_k = \frac{1}{1-\beta q}[\phi_k(\beta q)\pi_{DC} + (1-\phi_k(\beta q))\pi_{DD} + \beta(1-q)v_0],$$

where $\phi_k(\beta q) = (1-\beta q)e_k^\intercal(I - \beta q Q)^{-1}\sigma$, to emphasize the difference with $\phi_k \equiv \phi_k(\beta)$ (no random device). As $q \to 1$, we have $w_k \to v_k$ for all $k \geq 1$.

In equilibrium a generic agent must choose C and not D, which holds if

$$v_0 - w_1 = \frac{1}{1-\beta q}[c - d - \phi_1(\beta q)(c+g-d)] \geq 0.$$

There exists a value $q\beta = \beta^* \in (0,1)$ satisfying $c - d = \phi_1(\beta q)(c+g-d)$ (Proposition 1). Hence, $v_0 - w_1 \geq 0$ for all $q\beta \in [\beta^*, 1)$; defecting is suboptimal in equilibrium for all $\beta \in \left[\frac{\beta^*}{q}, 1\right)$. Therefore, the availability of public randomization devices makes it harder to sustain cooperation in equilibrium. Intuitively, the possibility to revert to cooperation after a defection is akin to introducing the possibility of renegotiation, which raises off-equilibrium payoffs, hence strengthens the incentive to defect.

Off-equilibrium, punishment is incentive-compatible if

$$q\beta \sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}(w_{k'-1} - w_{k'}) \leq \sigma_k g + (1-\sigma_k)l.$$

which is simply expression (7) with the adjustment for the randomization q. So, cooperating off-equilibrium is suboptimal for any β and l, if q is sufficiently small; and it is suboptimal for any β and q, when l is sufficiently large.

The inequality above holds for $l \geq l(q\beta, k)$, defined in Proposition 2. Since $l(x, k)$ is a non-decreasing function of $x \in (0, 1)$ (Proposition 2), $l(q\beta, k) \leq l(\beta, k)$ for all $q \in (0, 1]$, and for all $k \geq 1$. With a public randomization device, punishment is more easily sustained off the equilibrium path, even if the cost from cooperating is small, and even if the population is large.

7 Final remarks

We have studied contagious equilibrium in infinitely repeated games where players are randomly matched in pairs, in each period, to play a game. The methodological innovation is to identify a key statistic of contagious punishment that, together with a recursive formulation, generates tractable closed-form expressions for continuation payoffs, out of equilibrium. A virtue of this approach is that it makes the analysis of contagious equilibrium transparent, allows us to generalize the expressions for continuation payoffs for all beliefs about the number of defectors, and gives us a way to characterize exact bounds on the parameters that are key to ensuring that cooperation is self-enforcing. An application of the analysis developed in this study for something other than a Prisoners' Dilemma game is found in [3], which studies sequential equilibrium with and without monetary exchange in random matching economies in which agents play a helping game, repeatedly.

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Appendix

Proof of Theorem 1. The existence of the transition matrix Q immediately follows from the indefinite repetition of a game in which everyone adopts the strategy in Definition 1. In particular, notice that $\sum_{k'=1}^{N} Q_{kk'} = 1$ for all $k = 1, \ldots, N$. The properties of Q are derived from the features of the strategy in Definition 1.
Property 1: It directly follows from the random matching assumption. For $1 \leq k < N$, there is a positive probability that some cooperator meets a defector; hence, there is a positive probability that *some* cooperator switches to playing D forever.
Property 2: It hinges on the fact that defection is an absorbing state for an agent.
Property 3: Let $\kappa := (1, \ldots, N)^\intercal$ be the vector of all possible defectors in the economy. Let there be $k \geq 1$ defectors at the start of some date. The average number of defectors in the economy after $t \geq 1$ periods is

$$\mu_k(t) = e_k^\intercal Q^t \kappa = \sum_{k'=1}^{N} Q_{kk'}^t k' = \sum_{k'=k}^{N} Q_{kk'}^t k' \geq k,$$

We have

$$\mu_k(t+1) - \mu_k(t) = e_k^\intercal Q^t (Q - I)\kappa \geq 0$$

because each element of vector $(Q - I)\kappa$ is non-negative. This is so because each row j of vector $Q\kappa$ gives us $\mu_j(1)$, while each row j of vector $I\kappa$ gives us j, and we know from the previous result that $\mu_j(1) \geq j$. Intuitively, defection is an absorbing state, so, if we have k defectors, then the expected number of defectors can only increase over time above the initial number k.
Property 4: Consider two economies differentiated according to their initial number of defectors, k and $k+1$. Let agent h be starting as a cooperator in the $k-$economy and as a defector in the $(k+1)-$economy. Recall that the matching process is independent of k.
Let $\mathcal{C}(t)$ be the set of cooperators at date t when the economy starts with k defectors. We have

$$|\mathcal{C}(0)| = N - k \quad \text{and} \quad \mathcal{C}(t+1) \subseteq \mathcal{C}(t)$$

Consider $h \in \mathcal{C}(0)$, and denote by $\mathcal{D}_h(t)$ the set of new defectors generated by making h a initial defector, instead of a cooperator. Clearly $\mathcal{D}_h(0) = \{h\}$ and, by properties 1 and 2, we have that $|\mathcal{D}_h(t)| > 1$ with positive probability in all $t \geq 1$. Now let $\mathbb{E}|\mathcal{D}_h(t)|$ denote the expected number of additional defectors that exist on date t as a consequence of agent h being an initial defector in the $(k+1)-$economy. We have

$$\mu_{k+1}(t) - \mu_k(t) = \mathbb{E}|\mathcal{D}_h(t)|$$

because $\mu_k(t)$ is the expected cardinality of the set of defectors present on date t, given k initial defectors.

Property 5: Consider two economies differentiated according to the parameter $a = 0, \ldots, N-2$, i.e., the number of agents outside of a player's match, whose actions are observed in a period. These a agents are randomly selected with a uniform probability, iid across agents. Denote by $Q_{kk'}(a)$ the elements of the transition probability, to make explicit their dependence from the number a of observations made outside of a match. Let there be $k \geq 1$ defectors at the start of some date. The expected number of defectors in the economy after $t \geq 1$ periods is

$$\mu_k(t; a) = e_k^\top Q^t(a)\kappa = \sum_{k'=k}^{N} Q_{kk'}^t(a)k',$$

We want to prove

$$\mu_k(t; a+1) \geq \mu_k(t; a), \quad \text{for all } a.$$

Suppose agent i defects on some date. The deviation is observed by the opponent o_i on that date (direct contagion) and possibly by $r = 0, 1, \ldots, N-2$ agents outside of the match $\{i, o_i\}$ (indirect contagion). A generic agent $h \notin \{i, o_i\}$ observes the action of i with probability $p(a) = \dfrac{a}{N-2}$, i.e., $i \in O_h$ with probability $p(a)$ in that period. This probability neither depends on the identity of h nor on the period. Hence, in each period the probability $P(r; a)$ that $r = 0, \ldots, N-2$ agents who have not met agent i observe his action is

$$P(r; a) = \binom{N-2}{r} p(a)^r (1 - p(a))^{N-2-r}.$$

Hence, we expect that o_i and $\sum_{r=0}^{N-2} P(r; a)r = a$ others observe the defection of i; so, the expected number of defectors $\mu_k(t; a)$ is non-decreasing in a.

$\square$

Proof of Lemma 1. When $\beta \in [0, 1)$ we have

$$e_k^\mathsf{T}\sigma + \sum_{t=1}^{\infty} \beta^t e_k^\mathsf{T} Q^t \sigma < e_k^\mathsf{T}\mathbf{1} + \sum_{t=1}^{\infty} \beta^t e_k^\mathsf{T} Q^t \mathbf{1} = (1-\beta)^{-1}.$$

where $\mathbf{1}$ is an $N \times 1$ unit vector. The inequality follows from Theorem 1, which proves that $e_k^\mathsf{T} Q^t \mathbf{1} = 1$ for all t because Q^t is a transition matrix.

When $\beta = 1$, recall that $\sigma_N = 0$ and Q is upper-triangular. Hence we can write

$$\sum_{t=0}^{\infty} e_k^\mathsf{T} Q^t \sigma = e_k^\mathsf{T} \sum_{t=0}^{\infty} Q_0^t \sigma = e_k^\mathsf{T}(I - Q_0)^{-1}\sigma < \infty,$$

where Q_0 is matrix Q where row N is all zeros. Clearly, $\sum_{t=0}^{\infty} Q_0^t$ converges since all diagonal elements in Q_0 are less than one (Theorem 1). $\square$

Proof of Lemma 2. Let $k = 1, \ldots, N-1$.

Proving $\phi_k \le \sigma_k$: Recall that $Q_{kk'}^t = 0$ for $k' < k$ and $\sigma_k > \sigma_{k'}$ for $k' > k$. Hence,

$$\begin{aligned} e_k^\mathsf{T}(I - \beta Q)^{-1}\sigma &= \sigma_k + \sum_{t=1}^{\infty} \beta^t \sum_{k'=1}^{N} Q_{kk'}^t \sigma_{k'} = \sigma_k + \sum_{t=1}^{\infty} \beta^t \sum_{k'=k}^{N} Q_{kk'}^t \sigma_{k'} \\ &< \sigma_k + \sigma_k \sum_{t=1}^{\infty} \beta^t \sum_{k'=k}^{N} Q_{kk'}^t = \sigma_k + \sigma_k \beta (1-\beta)^{-1} \\ &= \sigma_k (1-\beta)^{-1}. \end{aligned}$$

Proving $\phi_k > \phi_{k+1}$: We need

$$\begin{aligned} \phi_k &= (1-\beta)^{-1}\sigma_k + (1-\beta)^{-1} \sum_{t=1}^{\infty} \beta^t \sum_{k'=k}^{N} Q_{kk'}^t \sigma_{k'} \\ &> (1-\beta)^{-1}\sigma_{k+1} + (1-\beta)^{-1} \sum_{t=1}^{\infty} \beta^t \sum_{k'=k+1}^{N} Q_{k+1,k'}^t \sigma_{k'} = \phi_{k+1}, \end{aligned}$$

which always holds if $\sum_{k'=k}^{N} Q_{kk'}^t \sigma_{k'} \ge \sum_{k'=k+1}^{N} Q_{k+1,k'}^t \sigma_{k'}$, because $\sigma_k > \sigma_{k+1}$. Note that

$$
\begin{aligned}
\sum_{k'=k}^{N} Q^t_{kk'}\sigma_{k'} &= \sum_{k'=k}^{N} Q^t_{kk'}\frac{N-k'}{N-1} = \frac{N}{N-1} - \frac{1}{N-1}\sum_{k'=k}^{N} Q^t_{kk'}k' \\
&\geq \frac{N}{N-1} - \frac{1}{N-1}\sum_{k'=k+1}^{N} Q^t_{k+1,k'}k' = \sum_{k'=k+1}^{N} Q^t_{k+1,k'}\frac{N-k'}{N-1} \\
&= \sum_{k'=k+1}^{N} Q^t_{k+1,k'}\sigma_{k'}
\end{aligned}
$$

since $\mu_{k+1}(t) = \sum_{k'=k+1}^{N} Q^t_{k+1,k'}k' \geq \mu_k(t) = \sum_{k'=k}^{N} Q^t_{kk'}k'$ by Theorem 1.

Proving ϕ_k is non-increasing in a: We make explicit the dependence of matrix Q from a, using the notation $Q(a)$. We wish to prove that for each $k = 1, \ldots, N$ and for each $a = 0, 1, \ldots, N-2$

$$\phi_k(a+1) \leq \phi_k(a). \tag{9}$$

Recall that

$$\phi_k(a) = (1-\beta)e_k^{\mathsf{T}}(I - \beta Q(a))^{-1}\sigma,$$

where

$$e_k^{\mathsf{T}}(I - \beta Q(a))^{-1}\sigma = \sigma_k + \beta\sum_{k'=1}^{N} Q_{kk'}(a)\sigma_{k'} + \beta^2\sum_{k'=1}^{N} Q^2_{kk'}(a)\sigma_{k'} + \ldots$$

So, it is sufficient to show that

$$
\begin{aligned}
\beta\sum_{k'=1}^{N} Q_{kk'}(a+1)\sigma_{k'} &+ \beta^2\sum_{k'=1}^{N} Q^2_{kk'}(a+1)\sigma_{k'} + \ldots \\
&\leq \beta\sum_{k'=1}^{N} Q_{kk'}(a)\sigma_{k'} + \beta^2\sum_{k'=1}^{N} Q^2_{kk'}(a)\sigma_{k'} + \ldots
\end{aligned}
$$

We exploit Property 5 of Theorem 1, using the notation

$$\mu_k(t;a) := e_k^{\mathsf{T}} Q^t(a)\kappa = \sum_{k'=k}^{N} Q^t_{kk'}(a)k'.$$

For each $t = 1, 2, \ldots$

$$\begin{aligned}
\sum_{k'=1}^{N} Q^t_{kk'}(a+1)\sigma_{k'} &= \sum_{k'=1}^{N} Q^t_{kk'}(a+1)\frac{N-k'}{N-1} \\
&= \frac{N}{N-1} - \frac{1}{N-1}\sum_{k'=k}^{N} Q^t_{kk'}(a+1)k' \\
&\leq \frac{N}{N-1} - \frac{1}{N-1}\sum_{k'=k}^{N} Q^t_{kk'}(a)k' = \sum_{k'=1}^{N} Q^t_{kk'}(a)\sigma_{k'}.
\end{aligned}$$

Proving that $\phi_k - \phi_{k+1}$ is non-increasing in k: Consider an economy with $k = 1, \ldots, N-1$ initial defectors, and fix one of them, say, agent i. Let 0 denote the initial date and let $\mathcal{C}(t)$ denote the set of cooperators at the start of period $t \geq 0$ when we have fixed k initial defectors. Clearly $i \notin \mathcal{C}(0)$ and, since defection is an absorbing state, we have $\mathcal{C}(t+1) \subseteq \mathcal{C}(t)$ for all t.

Now, suppose that we start with $k+1$ initial defectors; to do so, we move one cooperator, called agent $h \neq i$, from $\mathcal{C}(0)$ to the complementary set of k initial defectors $\mathcal{C}^C(0)$. We wish to track the set of *additional defectors* $\mathcal{D}_h(t)$ that exist in period t, as a (direct or indirect) consequence of making agent h an initial defector, instead of an initial cooperator. This can be found by recursively defining the set of cooperators who, on some date t, have switched to defection *only* as the result of seeing the actions of h or of any defectors created as a result of h's initial defection.

We have

$$\begin{aligned}
\mathcal{D}_h(0) &= \{h\} \\
\mathcal{D}_h(t) &= \Big\{ j \in \Big(\mathcal{C}^C(t) \cap \mathcal{C}(t-1)\Big) \cup \mathcal{D}_h(t-1) | \\
&\qquad O_j(t-1) \subseteq \Big(\mathcal{C}(t-1) \sqcup \mathcal{D}_h(t-1)\Big)\Big\}
\end{aligned}$$

The element $j \in \mathcal{C}^C(t) \cap \mathcal{C}(t-1)$ captures the requirement that agent j can be a new defector: he cooperates in $t-1$ and starts to defect in t. But agent j can also be an old defector, infected by agent h, i.e., $j \in \mathcal{D}_h(t-1)$. The component $O_j(t-1) \subseteq \Big(\mathcal{C}(t-1) \sqcup \mathcal{D}_h(t-1)\Big)$ captures the requirement that if j observes defections, then these defections must come from agents "infected" by agent h. This means that, if j was a cooperator in $t-1$, then j starts to defect in t *exclusively* as a consequence of seeing the action of some defector in $\mathcal{D}_h(t-1)$. Instead, if $j \in \mathcal{D}_h(t-1)$, then j should not meet a defector outside of $\mathcal{D}_h(t-1)$ in period $t-1$; i.e., agent j would remain a cooperator in t if agent h did not start to defect in period 0. Clearly $\mathcal{D}_h(t)$ depends on the number of defectors k' at the start of date t, because $|\mathcal{C}^C(t)| = k'$.

Using the definition of ϕ_k we have

$$\phi_k - \phi_{k+1} = (1-\beta)\sum_{t=0}^{\infty}\beta^t \sum_{k'=k}^{N} Q^t_{kk'} \Pr[o_i(t) \in \mathcal{D}_h(t)|k'].$$

Recall that $\dfrac{\phi_{k+1}}{1-\beta}$ is the expected number of cooperators that a defector encounters (over his lifetime), when the economy starts with $k+1$ defectors, one of which is agent h. If agent h were not a defector in period 0, then the agents in $\mathcal{D}_h(t,a)$ would be cooperators in t. Therefore, $\dfrac{\phi_k - \phi_{k+1}}{1-\beta}$ is the expected number of *additional* cooperators that a defector encounters (over his lifetime), if agent h were a cooperator instead of being a defector on date 0 (i.e., if we started with k instead of $k+1$ defectors). So, suppose agents i and h are defectors on the initial date 0. $\Pr[o_i(t) \in \mathcal{D}_h(t)|k']$ is the probability that, t periods forward, agent i meets either h or any of the cooperators "infected" by h. Clearly, this probability is conditional on the number of additional defectors $k'-k$ added over the course of t periods, since $\mathcal{D}_h(t)$ is contained in the set of all defectors added over the periods $1,\ldots,t$.

Now, fix another agent $l \neq h$ and define the set $\mathcal{G}(t) = \mathcal{D}_h(t) \cup \mathcal{D}_l(t)$ We have

$$\phi_k - \phi_{k+2} = (1-\beta)\sum_{t=0}^{\infty}\beta^t \sum_{k'=k}^{N} Q^t_{kk'} \Pr[o_i(t) \in \mathcal{G}(t)|k']$$

where

$$\begin{aligned}\Pr[o_i(t) \in \mathcal{G}(t)|k'] \;&=\; \Pr[o_i(t) \in \mathcal{D}_h(t)|k'] + \Pr[o_i(t) \in \mathcal{D}_l(t)|k'] \\ &\quad - \Pr[o_i(t) \in \mathcal{D}_h(t) \cap \mathcal{D}_l(t)|k']\end{aligned}$$

Since the random matching process is independent of agent's identities, we have

$$\Pr[o_i(t) \in \mathcal{D}_h(t)|k'] = \Pr[o_i(t) \in \mathcal{D}_l(t)|k']$$

Hence,

$$\begin{aligned}
\phi_{k+1} - \phi_{k+2} &= (\phi_k - \phi_{k+2}) - (\phi_k - \phi_{k+1}) \\
&= (1-\beta)\sum_{t=0}^{\infty}\beta^t \sum_{k'=k}^{N} Q^t_{kk'} \Pr[o_i(t) \in \mathcal{D}_h(t)|k'] \\
&\quad -(1-\beta)\sum_{t=0}^{\infty}\beta^t \sum_{k'=k}^{N} Q^t_{kk'} \Pr[o_i(t) \in \mathcal{D}_h(t) \cap \mathcal{D}_l(t)|k'] \\
&\leq \phi_k - \phi_{k+1}
\end{aligned}$$

Proving that $\phi_k - \phi_{k+1}$ is non-increasing in a: Again, we make explicit the dependence of matrix Q from a, using the notation $Q(a)$.

We wish to show that

$$\phi_k - \phi_{k+1} = (1-\beta)\sum_{t=0}^{\infty}\beta^t \sum_{k'=k}^{N} Q^t_{kk'}(a) \Pr[o_i(t) \in \mathcal{D}_h(t,a)|k']\}$$

is non-increasing in a for all k and all a. Here, set $\mathcal{D}_h(t,a)$ comprises all agents who become defectors in some period $0,\ldots,t$ *only* as a consequence of agent h being one of $k+1$ initial defectors and that a observations can be made outside a match.

Defining $d_t(a; k'-k) := |\mathcal{D}_h(t,a)|$ when there are k' defectors on date t and k defectors on date 0, we have $\Pr\left(o_i(t) \in \mathcal{D}_h(t,a)|k'\right) = \frac{d_t(a;k'-k)}{N-1}$. In what follows we omit the argument $k'-k$ when understood.

To prove that $\phi_k - \phi_{k+1}$ is non-increasing in a, we proceed by induction. Fix a, and fix a set of k defectors in period 0, and the number of defectors in period $t \geq 1$, i.e., k'. Now fix a trajectory of matching and of observations, that takes us from k defectors in period 0 to k' defectors in period t, given the observations a that can be done outside of a match. Use the definition of $\mathcal{D}_h(t,a)$ in the proof of Lemma 2, where a is made explicit.

For the initial step, we prove that if $t = 1$, then $d_1(a) \geq d_1(a+1)$. Recall that $\mathcal{D}_h(0,a) = \mathcal{D}_h(0,a+1) = \mathcal{D}_h(0) = \{h\}$ and $\mathcal{C}(0,a) = \mathcal{C}(0,a+1) = \mathcal{C}(0)$ by assumption since the set of k initial defectors (hence of initial cooperators) is fixed. Consequently,

$$\begin{aligned}
d_1(a) &= \left|\left\{j \in \left(\mathcal{C}^C(1,a) \cap \mathcal{C}(0)\right) \sqcup \mathcal{D}_h(0) | \; O_j(0,a) \subseteq \left(\mathcal{C}(0) \sqcup \mathcal{D}_h(0)\right)\right\}\right| \\
&\geq \left|\left\{j \in \left(\mathcal{C}^C(1,a+1) \cap \mathcal{C}(0)\right) \sqcup \mathcal{D}_h(0)| \right.\right. \\
&\qquad \left.\left. O_j(0,a+1) \subseteq \left(\mathcal{C}(0) \sqcup \mathcal{D}_h(0)\right)\right\}\right| = d_1(a+1)
\end{aligned}$$

because we have $|\mathcal{C}^C(1,a)| = |\mathcal{C}^C(1,a+1)| = k'$ and $O_j(0,a) \subseteq O_j(0,a+1)$.

For the induction step, assume $d_{t-1}(a) \geq d_{t-1}(a+1)$ for some $t > 2$. We want to prove that $d_t(a) \geq d_t(a+1)$. Recall that $|\mathcal{C}^C(t,a)| = |\mathcal{C}^C(t,a+1)| = k'$ by assumption. We have

$$\begin{aligned}
d_t(a) &= \left|\left\{j \in \left(\mathcal{C}^C(t,a) \cap \mathcal{C}(t-1,a)\right) \sqcup \mathcal{D}_h(t-1,a)| \right.\right. \\
&\qquad \left.\left. O_j(t-1,a) \subseteq \left(\mathcal{C}(t-1,a) \sqcup \mathcal{D}_h(t-1,a)\right)\right\}\right| \\
&\geq \left|\left\{j \in \left(\mathcal{C}^C(t,a+1) \cap \mathcal{C}(t-1,a+1)\right) \sqcup \mathcal{D}_h(t-1,a+1)| \right.\right. \\
&\qquad \left.\left. O_j(t-1,a+1) \subseteq \left(\mathcal{C}(t-1,a+1) \sqcup \mathcal{D}_h(t-1,a+1)\right)\right\}\right| \\
&= d_t(a+1)
\end{aligned}$$

because

- $d_{t-1}(a) \equiv |\mathcal{D}_h(t-1,a)| \geq |\mathcal{D}_h(t-1,a+1)| \equiv d_{t-1}(a+1)$ by the induction hypothesis.
- $|\mathcal{C}(t-1,a+1)| \leq |\mathcal{C}(t-1,a)|$, by the properties of the contagious process reported in Theorem 1 (the number of defectors cannot be lower if agents can make more observations).
- $O_j(t-1,a) \subseteq O_j(t-1,a+1)$

Hence, given an initial set of k defectors, and given a number of defectors $|\mathcal{C}^C(t,a)| = |\mathcal{C}^C(t,a')| = k'$ in period t for some $a' > a$, the size of $\mathcal{D}_h(t,a)$ is non-increasing in a for all t. Noticing that $\Pr[o_i(t) \in \mathcal{D}_h(t,a)|k'] = \dfrac{d_t(a)}{N-1}$ we conclude that $\phi_k - \phi_{k+1}$ is non-increasing in a.

Finally, it is immediate that $\dfrac{\phi_k - \phi_{k+1}}{1-\beta}$ is non-decreasing in β. □

Proof of Proposition 1. Using (7) and the definition of γ observe that β^* is a solution to the implicit function

$$\phi_1 - \frac{1}{1+\gamma} = 0$$

where ϕ_1 is a function of β as indicated in Theorem 2. We will prove that ϕ_1 is a strictly monotone, decreasing function of β and, consequently, the function ϕ_1 is invertible so

$$\beta^* := \phi_1^{-1}\Big(\frac{1}{1+\gamma}\Big),$$

in which case

$$\begin{aligned}\frac{\partial \beta^*}{\partial \gamma} &= -\frac{1}{\phi_1'}\frac{1}{(1+\gamma)^2}\Big|_{\beta=\beta^*} > 0,\\ \frac{\partial^2 \beta^*}{\partial \gamma^2} &= \frac{1}{\phi_1'}\left[\frac{2}{(1+\gamma)^3} - \phi_1''\left(\frac{1}{\phi_1'(1+\gamma)^2}\right)^2\right]\Big|_{\beta=\beta^*} < 0.\end{aligned}$$

To prove that ϕ_k is a decreasing function of β for all $k = 1, \dots, N-1$, use the expression for ϕ_k. We have

$$\begin{aligned}\phi_k' &:= \frac{\partial \phi_k}{\partial \beta} = -e_k^{\intercal}(I-\beta Q)^{-1}[I-(1-\beta)Q(I-\beta Q)^{-1}]\sigma < 0.\\ \phi_k'' &:= \frac{\partial^2 \phi_k}{\partial \beta^2} = -2e_k^{\intercal}(I-\beta Q)^{-1}Q(I-\beta Q)^{-1}[I-(1-\beta)Q(I-\beta Q)^{-1}]\sigma < 0.\end{aligned}$$

The negative sign of the derivatives follow from noting (from Lemma 1) that the vector $(I-\beta Q)^{-1}\sigma < (1-\beta)^{-1}\mathbf{1}$, where $\mathbf{1}$ is an $N \times 1$ unit vector. Since Q is a transition matrix, then $Q(I-\beta Q)^{-1}\sigma < Q(1-\beta)^{-1}\mathbf{1}$ and so $[I-(1-\beta)Q(I-\beta Q)^{-1}]\sigma \geq 0$ is a non-zero vector because $0 = \sigma_N < \sigma_{k'} < \sigma_k < \sigma_1 \leq 1$ by definition of σ_k. □

Proof of Proposition 2. Using (8), inequality (7) is rearranged as

$$\frac{\beta}{1-\beta}\sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}(c+g-d)(\phi_{k'-1}-\phi_{k'}) \leq \sigma_k g + (1-\sigma_k)l,$$

which holds if

$$l \geq l(\beta, k) := \frac{1}{1-\sigma_k} \left\{ \sum_{k'=k}^{N} Q_{kk'} \eta_{kk'} \left[(c+g-d) \times \frac{\beta(\phi_{k'-1} - \phi_{k'})}{1-\beta} - g \right] \right\}.$$

where we have used that $\sigma_k = \sum_{k'=k}^{N} Q_{kk'} \eta_{kk'}$.

We know that $\frac{\phi_{k'-1} - \phi_{k'}}{1-\beta}$ is non-decreasing in β (Theorem 2). Therefore $l(\beta, k)$ is non-decreasing in β.

Now we argue that $l(\beta, k)$ is decreasing in k. To see this, notice that $\sum_{k'=k}^{N} Q_{kk'} \eta_{kk'} a_{k'} = \sigma_k$ when $a_{k'} = 1$ for all $k' \geq k$. Hence, $\sum_{k'=k}^{N} Q_{kk'} \eta_{kk'} a_{k'}$ is decreasing in k when $a_{k'} = 1$ for all $k' \geq k$, and therefore also when $a_{k'}$ is a decreasing sequence. Clearly,

$$a_{k'} = \frac{(c+g-d)\beta(\phi_{k'-1} - \phi_{k'})}{1-\beta} - g$$

is a decreasing sequence because $\frac{\phi_{k'-1} - \phi_{k'}}{1-\beta}$ is decreasing in k' (Theorem 2) . Consequently, $l(\beta, k) > l(\beta, k+1)$ for all $k = 2, \ldots, N$.

We can find an upper bound for $l(\beta, k)$ by noticing, from Theorem 2, that

$$\frac{\beta}{1-\beta}(\phi_{k-1} - \phi_k) \leq \frac{\beta}{1-\beta}(\phi_1 - \phi_2) < \infty \qquad \text{for each } k = 2, \ldots, N.$$

To find $\frac{\beta}{1-\beta}(\phi_1 - \phi_2)$ use the recursive equation $v_1 = c + g + \beta v_2$, which substituting v_1 and v_2 from (3) is written as

$$\frac{1}{1-\beta}[\phi_1(c+g) + (1-\phi_1)d] = c + g + \frac{\beta}{1-\beta}[\phi_2(c+g) + (1-\phi_2)d],$$

or, equivalently,

$$\frac{\beta}{1-\beta}(\phi_1 - \phi_2) = 1 - \phi_1. \tag{10}$$

Since $\frac{\beta}{1-\beta}(\phi_{k-1} - \phi_k)$ is non-decreasing in β (Theorem 2), we have

$$\sup_{\beta \in (0,1)} \frac{\beta}{1-\beta}(\phi_1 - \phi_2) = \lim_{\beta \to 1^-} \frac{\beta}{1-\beta}(\phi_1 - \phi_2) = 1 - \lim_{\beta \to 1^-} \phi_1 = 1, \tag{11}$$

where we used (10) to get the second equality and the last equality follows from $\lim_{\beta \to 1^-} \phi_k = 0$ for all $k \geq 1$ (Theorem 2). From (11) we have

$$\begin{aligned} l(\beta, k) &\leq \frac{1}{1-\sigma_k} \sum_{k'=k}^{N} Q_{kk'} \eta_{kk'} \left[(c+g-d) \times 1 - g \right] \\ &= \frac{\sigma_k}{1-\sigma_k}(c-d) = \frac{N-k}{k-1}(c-d) \\ &\leq l^* := (N-2)(c-d). \end{aligned}$$

Hence $l \geq l^*$ is sufficient for the optimality of off equilibrium punishment. □

Proof of Corollary 1. As before, deviating in equilibrium is suboptimal if $v_0 - v_1 \geq 0$. Following the same procedure used in the earlier proof, using (7), we have

$$v_0 - v_1 = \frac{1}{1-\beta}[c - d - \phi_1(a)(c+g-d)] = 0$$

for a value $\beta(a) \in (0,1)$. Recall that, when $a = 0$, $\beta^* \equiv \beta(0)$ satisfies $v_0 - v_1 = 0$. Since $\phi_1(a) \leq \phi_1(0)$ (Lemma 2) and $\phi_k(a)$ is decreasing in β for every a, we have that $\beta(a) \leq \beta^*$. □

Proof of Corollary 2. By Theorem 2 $\lim_{\beta \to 1^-} (1-\beta)^{-1} \phi_k(a) < \infty$ for all k. Hence, deviating off-equilibrium is suboptimal when l is sufficiently large.

We now characterize $l(\beta, k; a)$, starting by proving that it is non-decreasing in β. Note that

$$\frac{\hat{\phi}_{k'}(a) - \phi_{k'}(a)}{1-\beta} \equiv \frac{1}{1-\beta} \sum_{j=0}^{k'-k} \alpha_{kk'}(j;a)[\phi_{k'-j}(a) - \phi_{k'}(a)].$$

We also have the telescoping sum (omitting a when understood)

$$\frac{\phi_{k'-j} - \phi_{k'}}{1-\beta} = \frac{\phi_{k'-j} - \phi_{k'-j+1}}{1-\beta} + \frac{\phi_{k'-j+1} - \phi_{k'-j+2}}{1-\beta} + \ldots + \frac{\phi_{k'-1} - \phi_{k'}}{1-\beta},$$

whose terms of the right-hand side are non-negative and non-decreasing in β

(Theorem 2). It follows that

$$\frac{\beta(\hat{\phi}_{k'}(a) - \phi_{k'}(a))}{1-\beta}$$

is non-decreasing in β. Consequently, $l(\beta, k; a)$ is non-decreasing in β.

To prove that $l(\beta, k)$ is decreasing in k, rewrite it as

$$\begin{aligned} l(\beta, k; a) = &\frac{1}{1-\sigma_k} \sum_{k'=k}^{N} Q_{kk'}(a)\eta_{kk'}(a) \left[(c+g-d)\frac{\beta(\hat{\phi}_{k'}(a) - \phi_{k'}(a))}{1-\beta} - g \right] \\ &+ \frac{1}{1-\sigma_k} \sum_{k'=k}^{N} Q_{kk'}(a)(1-\eta_{kk'}(a))(c+g-d)\frac{\beta(\hat{\phi}_{k'}(a) - \phi_{k'}(a))}{1-\beta} \end{aligned}$$

Using the telescoping sum above, it follows that the first term is decreasing in k. The second term, instead, can be rewritten as

$$(c+g-d)\frac{\beta}{1-\beta}\frac{\sum_{k'=k}^{N} Q_{kk'}(a)(1-\eta_{kk'}(a))(\hat{\phi}_{k'}(a) - \phi_{k'}(a))}{\sum_{k'=k}^{N} Q_{kk'}(a)(1-\eta_{kk'}(a))}$$

where we have exploited the fact that

$$1 - \sigma_k = \sum_{k'=k}^{N} Q_{kk'}(a)(1-\eta_{kk'}).$$

Since $\hat{\phi}_{k'}(a) - \phi_{k'}(a)$ is decreasing in $k' \geq k$, it follows that the second term of the expression above is also decreasing in k. Consequently, $l(\beta, k; a)$ is decreasing in k.

To prove that $l(\beta, k)$ is non-increasing in a it is sufficient to show that

$$\hat{\phi}_{k'}(a) - \phi_{k'}(a)$$

is decreasing in a. Given that $\hat{\phi}_{k'}(a) = \sum_{j=0}^{k'-k} \alpha_{kk'}(j;a)\phi_{k'-j}(a)$, this follows from $\phi_k(a) - \phi_{k+1}(a)$ being non increasing in a (Lemma 2). Hence, the cooperation cost $l(\beta, k; a)$ is non-increasing in the public monitoring parameter a.

□

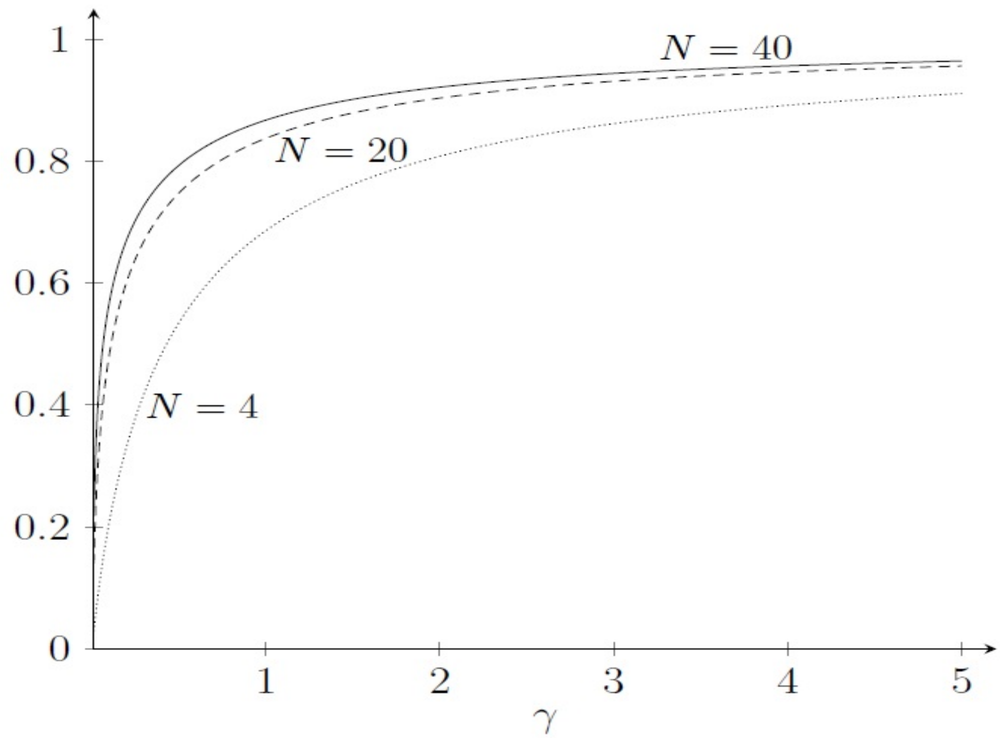

Figure 1: The lower bound for β

Notes: The figure plots the function β^* for $N = 4, 20, 40$ as γ varies from 0 to 5 by fixing $(c, d) = (1, 0)$ and varying g from 0 to 5

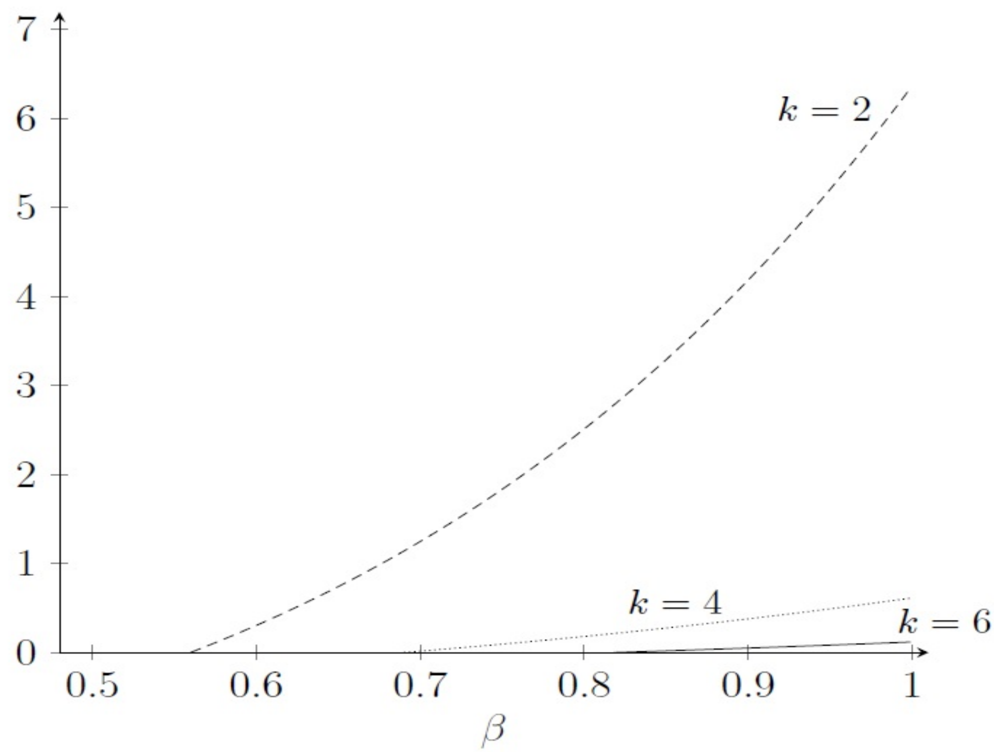

Figure 2: The lower bound for l

Notes: The figure plots the function $\max(0, l(\beta, k))$ for $N = 20$, $k = 2, 4, 6$, and $\beta \in (\beta^*, 1)$. We have fixed $(c, d, g) = (1, 0, 0.1)$, hence $\beta^* = 0.48$